On Vertex-Ply and Empty-Ply Proximity Drawings

25th Int. Symp. on Graph Drawing and Network Visualization

September 25-27 2017, Northeastern University, Boston, USA

Patrizio Angelini[1] Steven Chaplick[2] Felice De Luca[3] Jiri Fiala[4]
Jaroslav Hanĉl Jr.[4] Niklas Heinsohn[1] Michael Kaufmann[1]
Stephen Kobourov[5] Jan Kratochvíl[4] Pavel Valtr[4]

[1]Wilhelm-Schickard-Institut für Informatik, Universität Tübingen, Germany

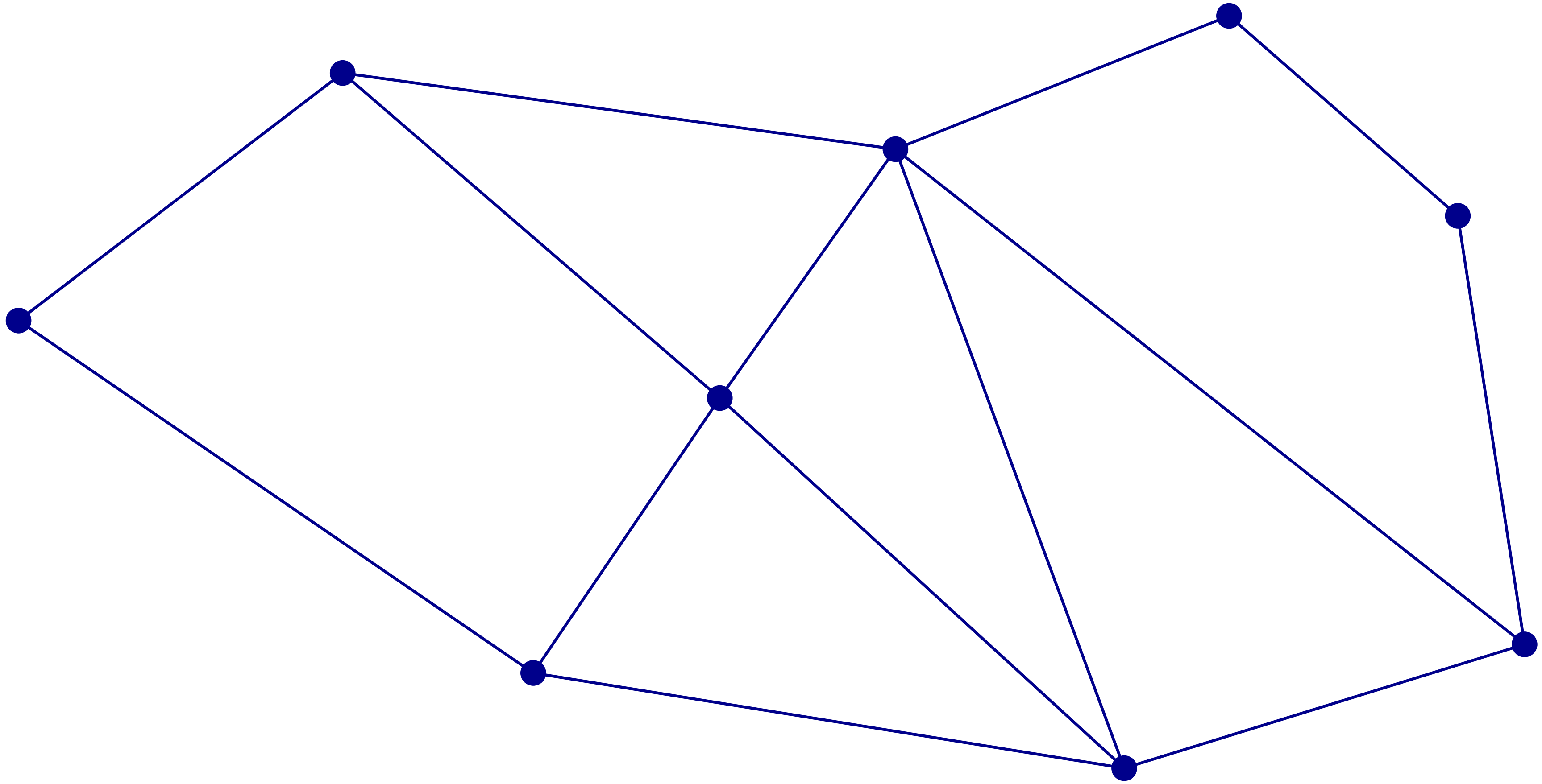
[2]Lehrstuhl für Informatik I, Universität Würzburg, Germany

[3]Universitá degli Studi di Perugia, Perugia, Italy

[4]Department of Applied Mathematics, Charles University (KAM), Czech Republic

[5]Department of Computer Science, University of Arizona, Tucson, USA

Ply of a straight-line drawing



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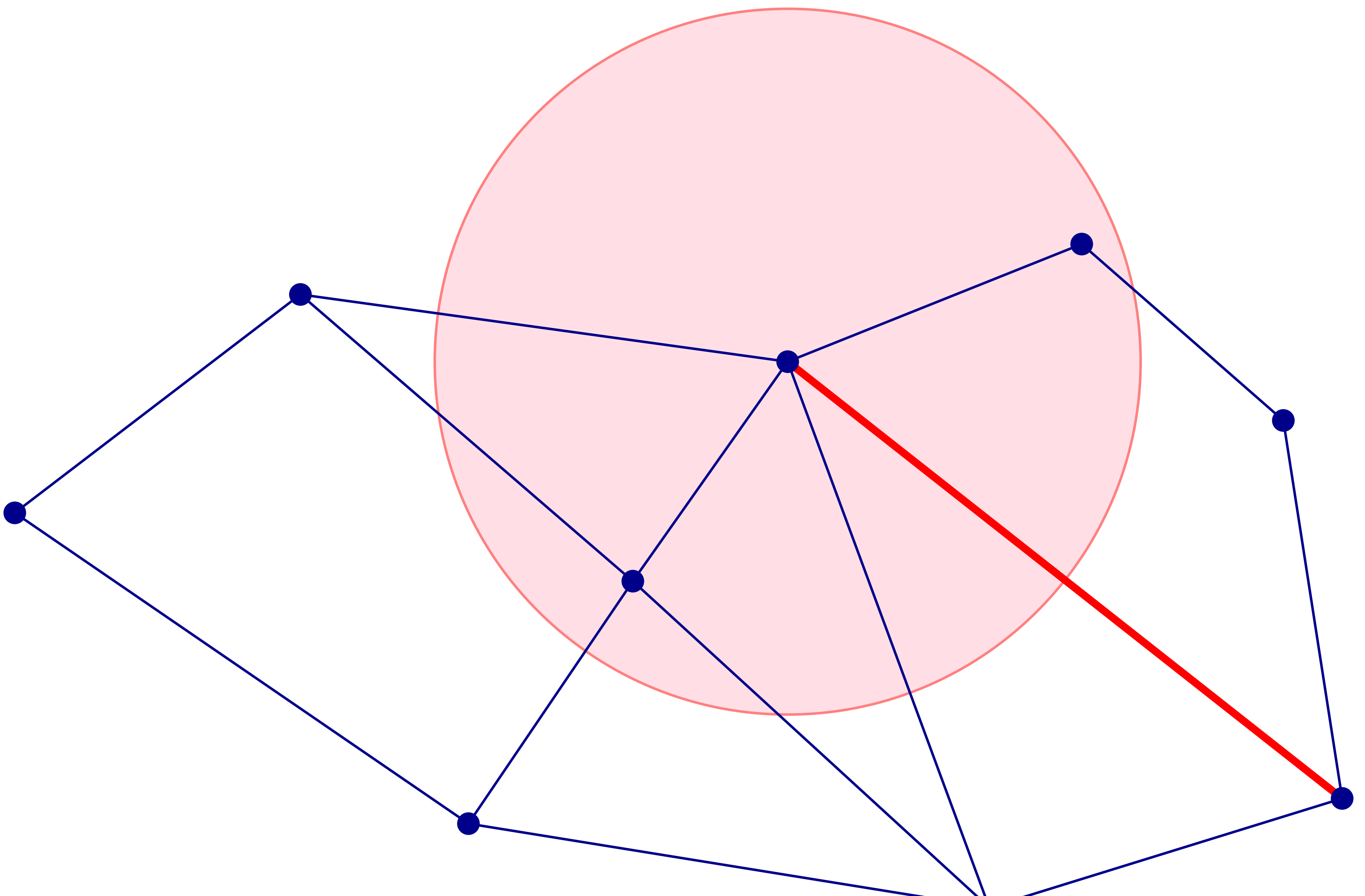
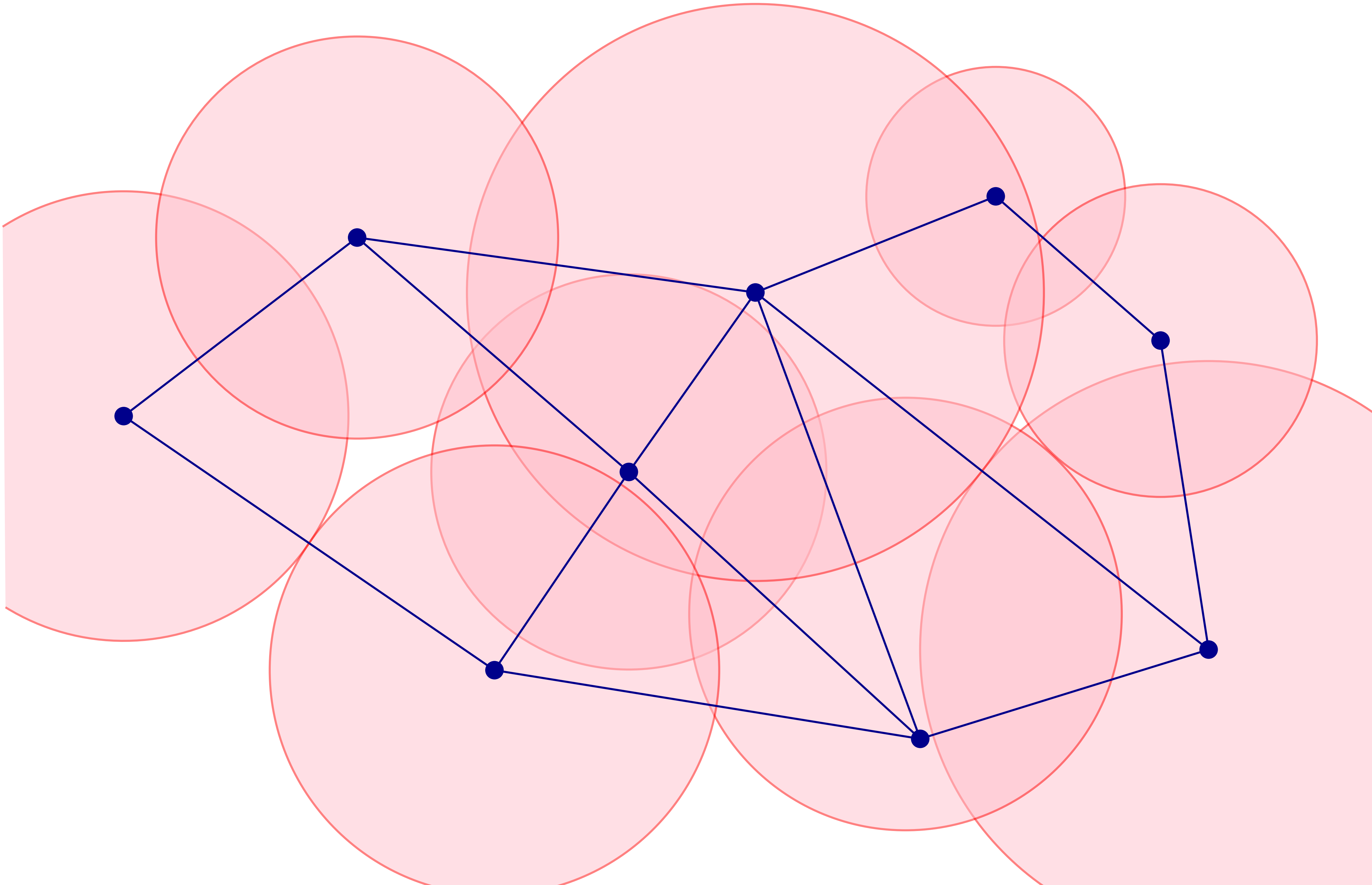


For each vertex, add a disk centered at it whose radius is half of the longest incident edge

Ply of a straight-line drawing



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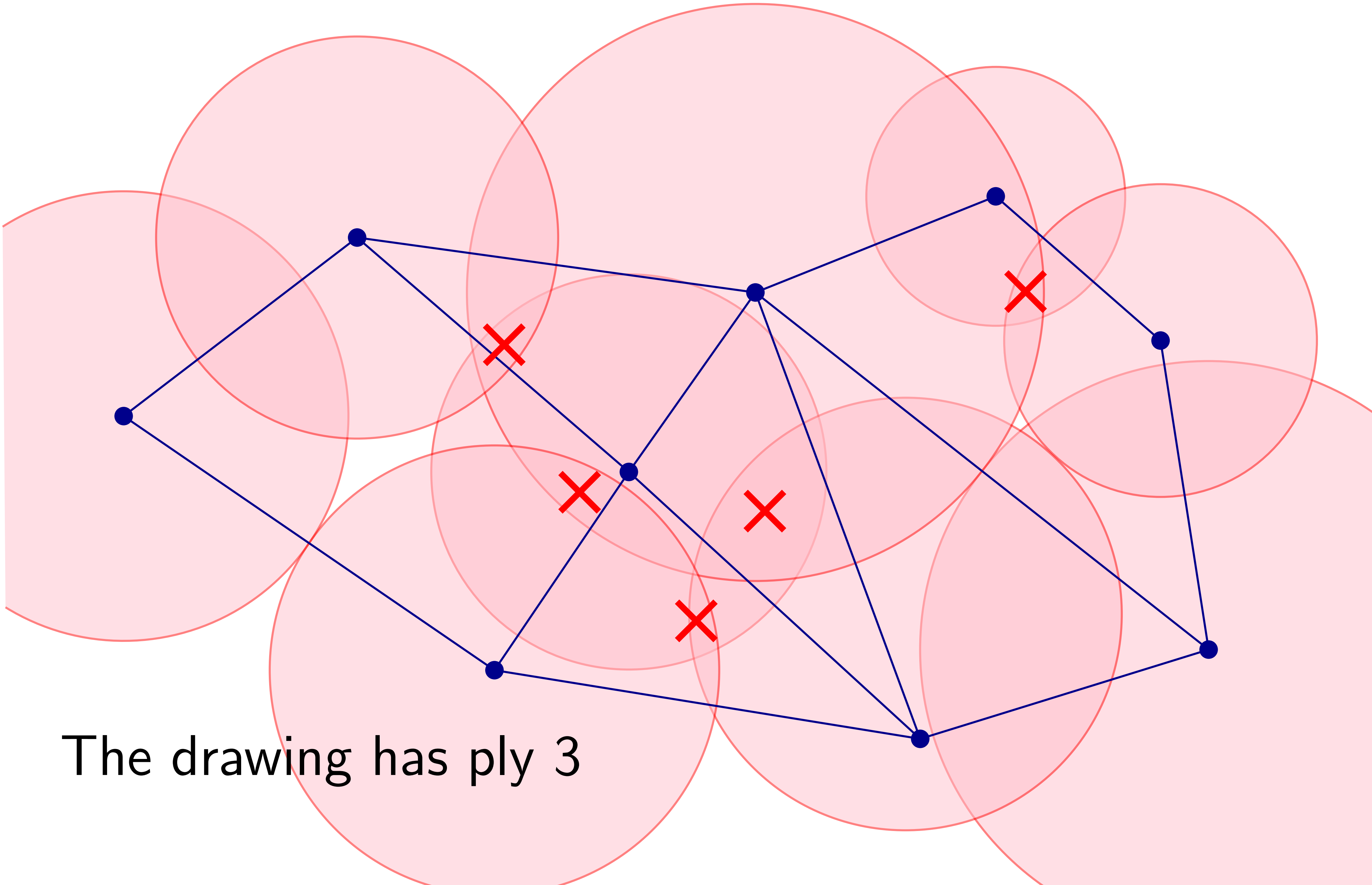

The drawing has ply 3

State of the art

- **Complexity of the testing problem**
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  Exponential area is required for stars
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- **Experimental results**
  [*De Luca et al. - WALCOM 2017*]
  [*Heinsohn and Kaufmann - next talk*]

Motivation

- **Empirical observation that road networks have low ply, when interpreted as subgraphs of disk intersection graphs**
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- **Intuition that ply is related to proximity drawings**
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  Studied in this paper

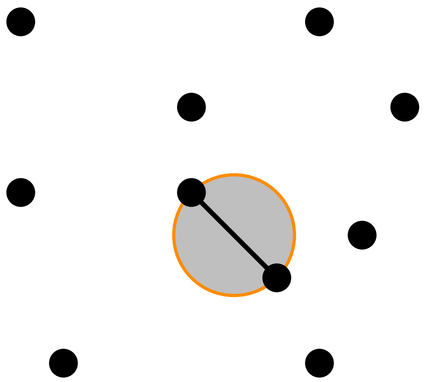
Proximity drawings

A straight-line drawing in which, for every two vertices u, v, their proximity region is empty if (and only if) edge (u, v) exists.

The proximity region of two vertices $u, v \in G$ is a region of the plane determined by their position.

Different proximity regions determine different proximity drawings:

- Gabriel graphs



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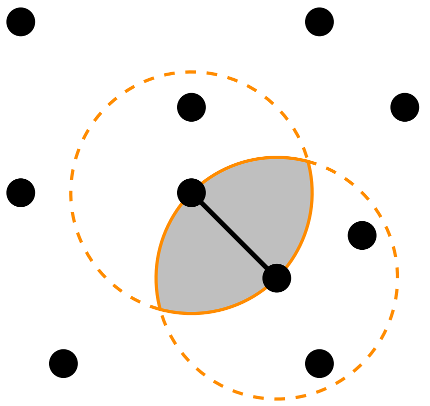

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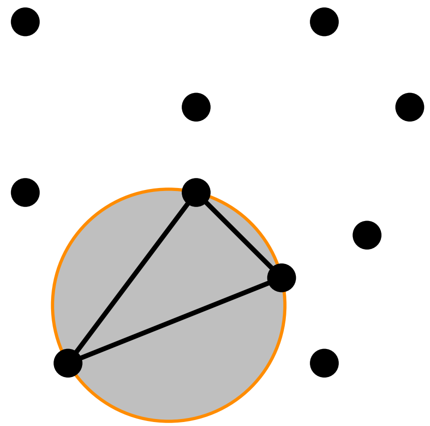
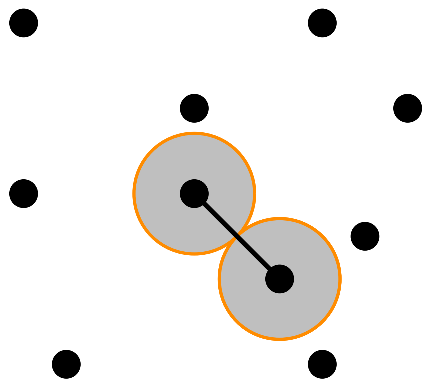
- Delaunay triangulations



Ply vs. proximity drawings

Proximity region for ply drawings



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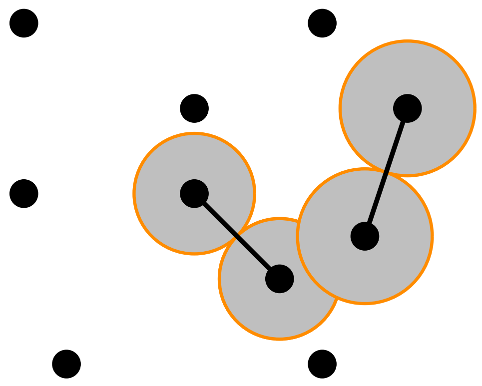

Ply depends on disk overlapping and not on vertex-disk containment

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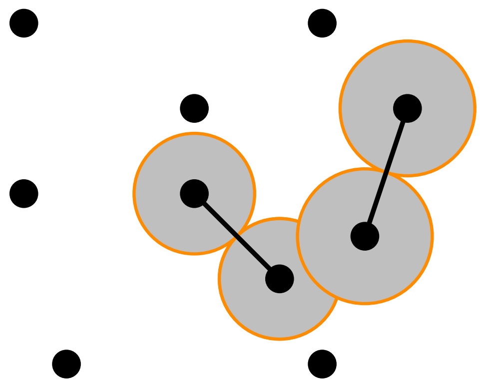

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We define the vertex-ply of a drawing as the ply computed only on the points where vertices are placed

Vertex-ply of a straight-line drawing

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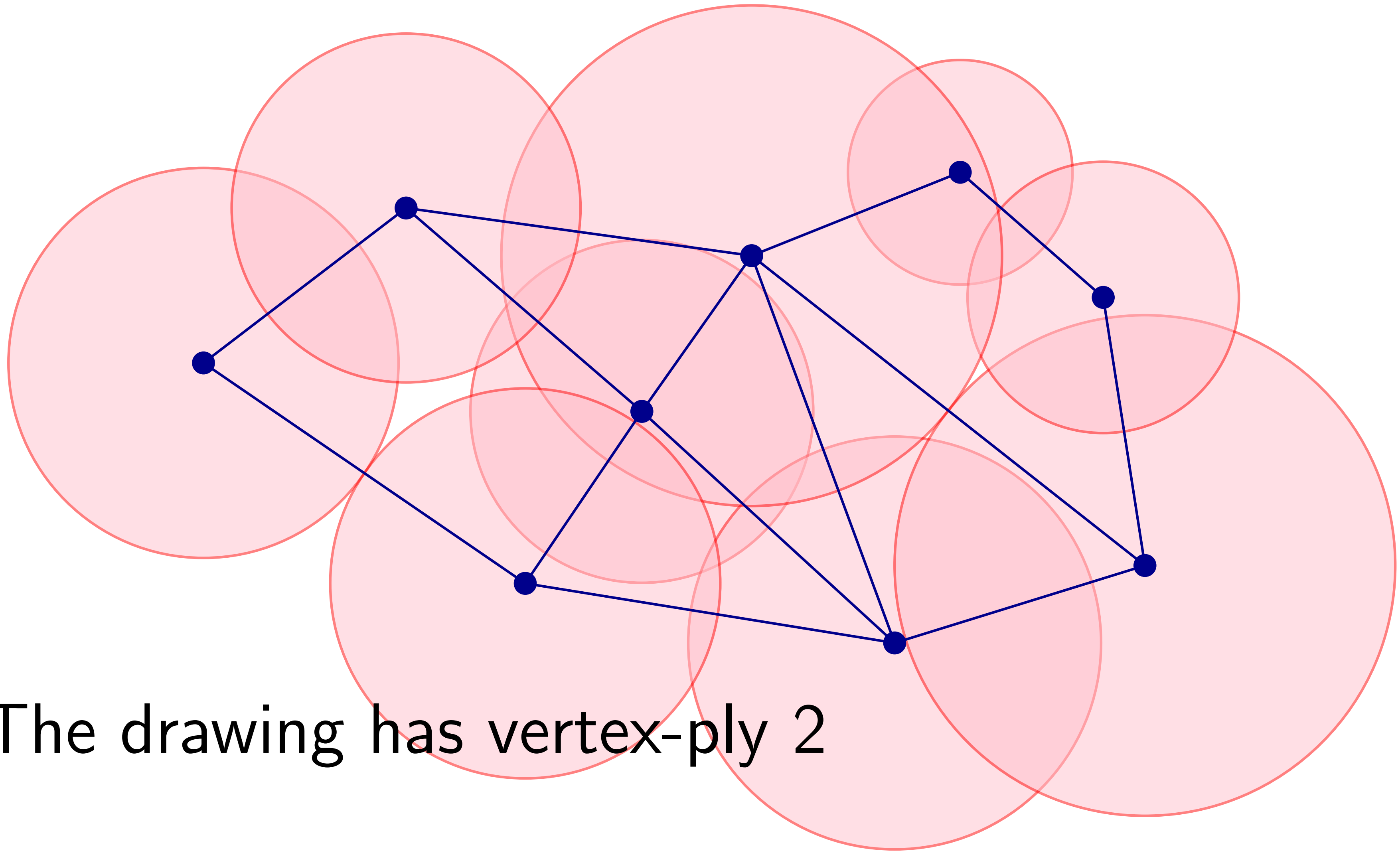
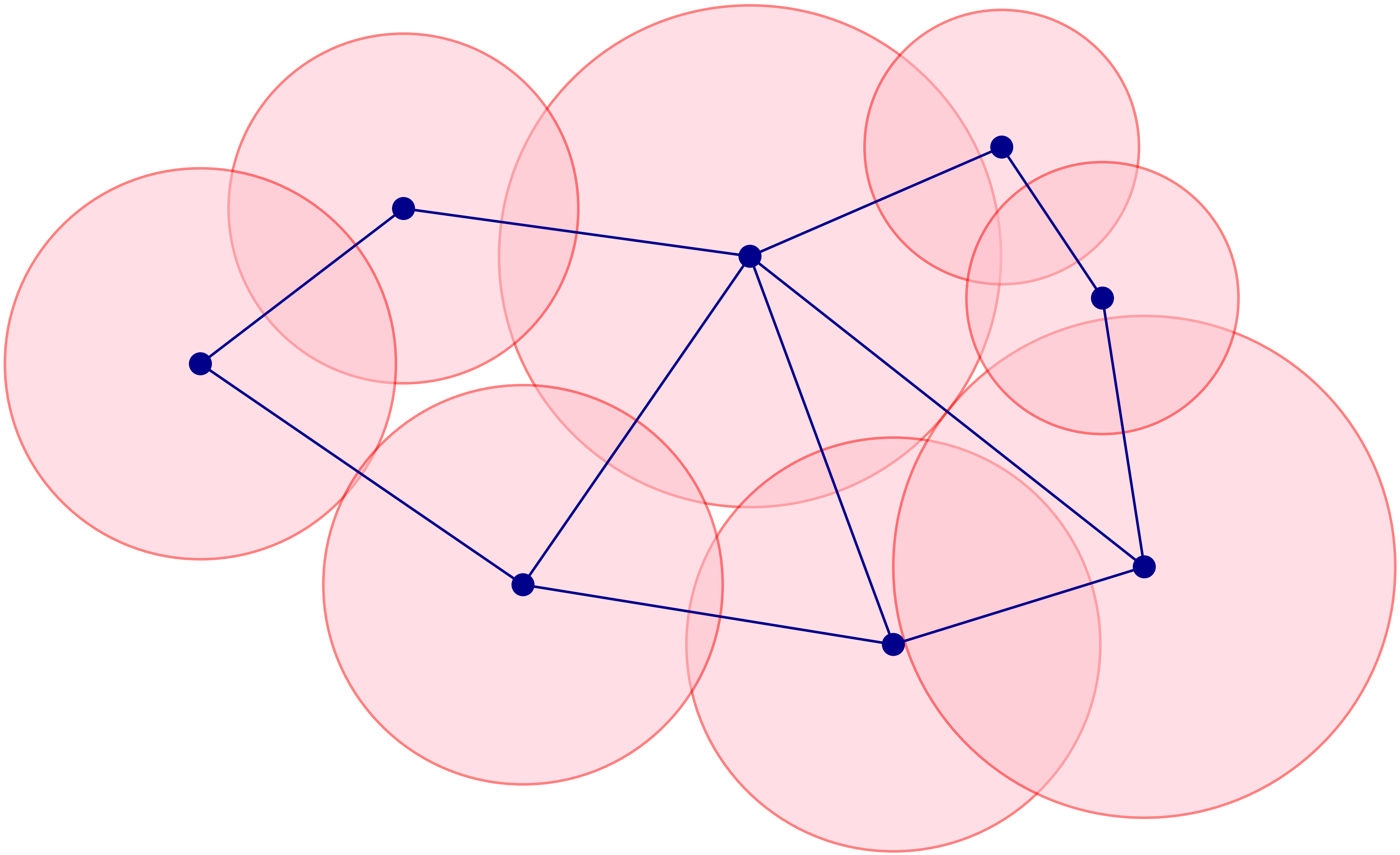


The drawing has vertex-ply 2

Empty-ply drawings

When vertex-ply $= 1$, we say that the drawing is empty-ply



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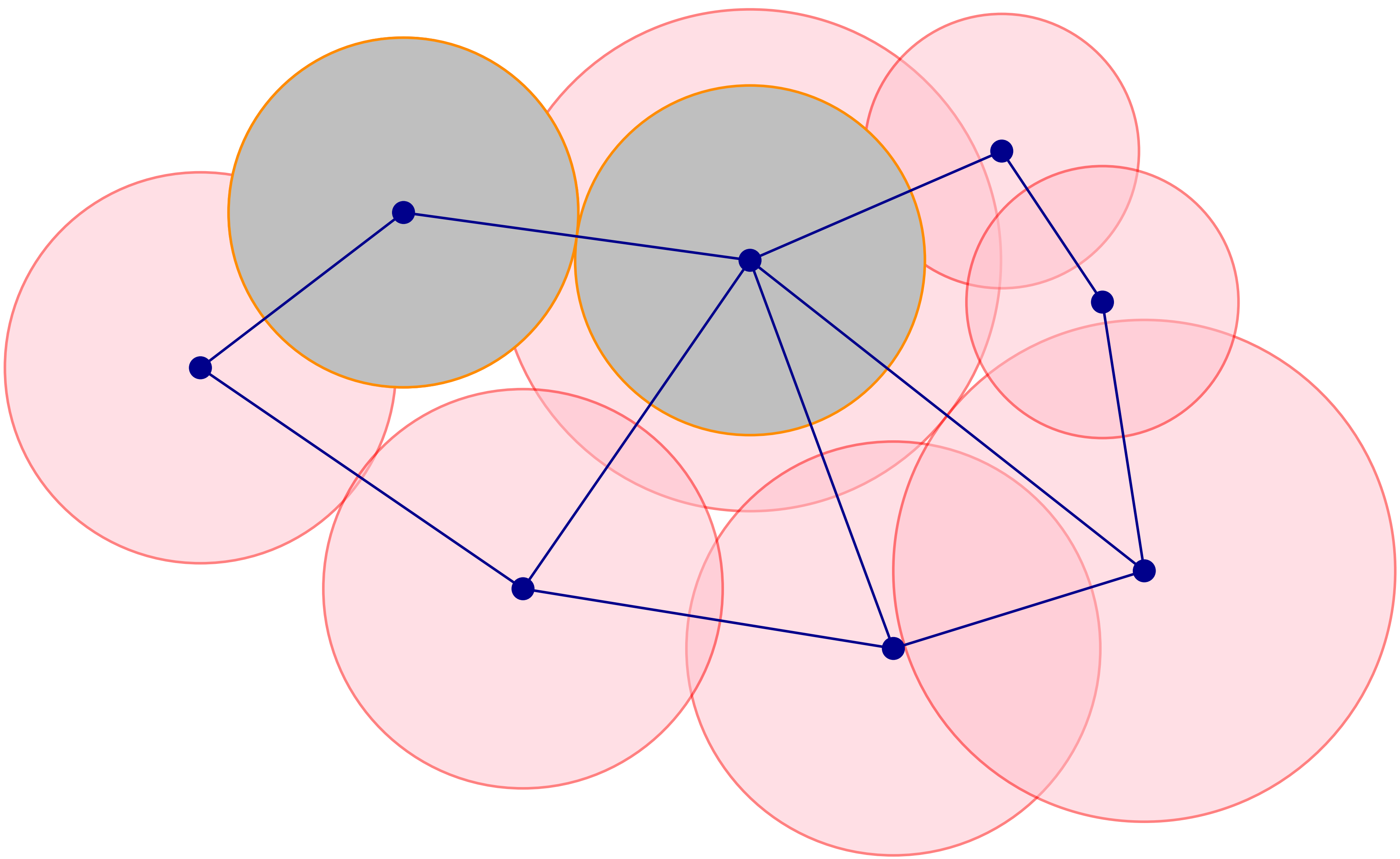

Relationship between ply and vertex-ply

Question [Di Giacomo et al. IISA 2015]

What is the ply of an empty-ply drawing?

Relationship between ply and vertex-ply

Theorem

An empty-ply drawing has ply at most 5

In general, a drawing with vertex-ply h has ply at most $5h$

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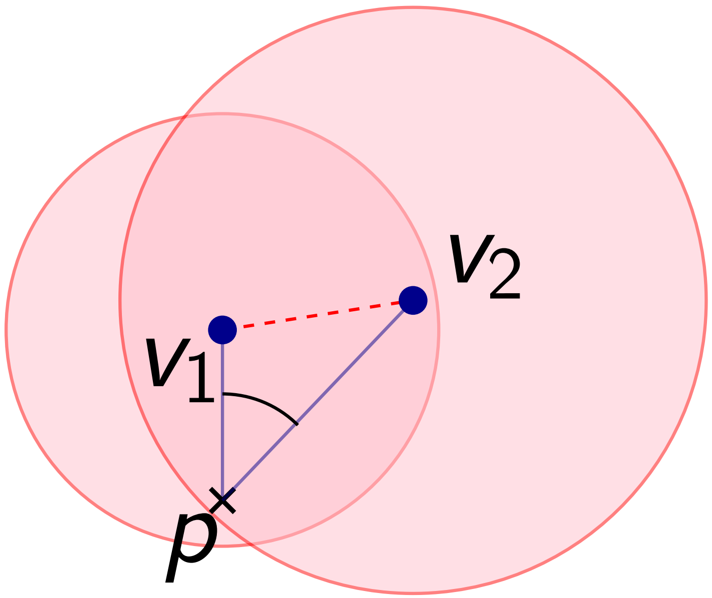
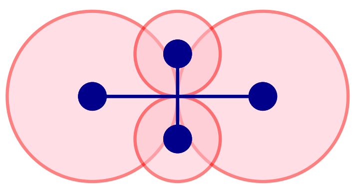
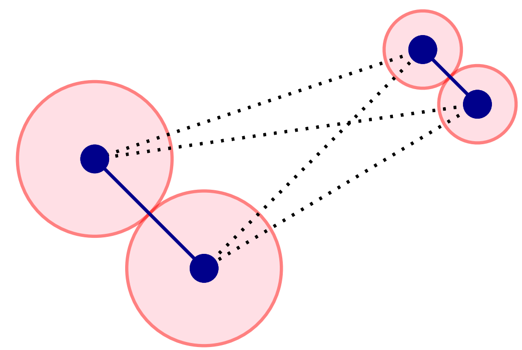


Empty-ply drawings: properties

Property

Empty-ply drawings may be non-planar and non-connected (not true for other proximity drawings)





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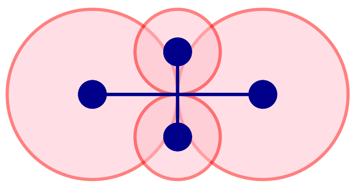
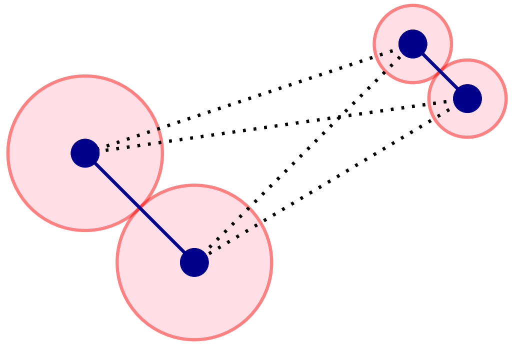
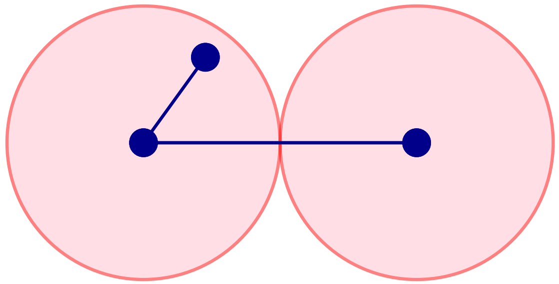




Property

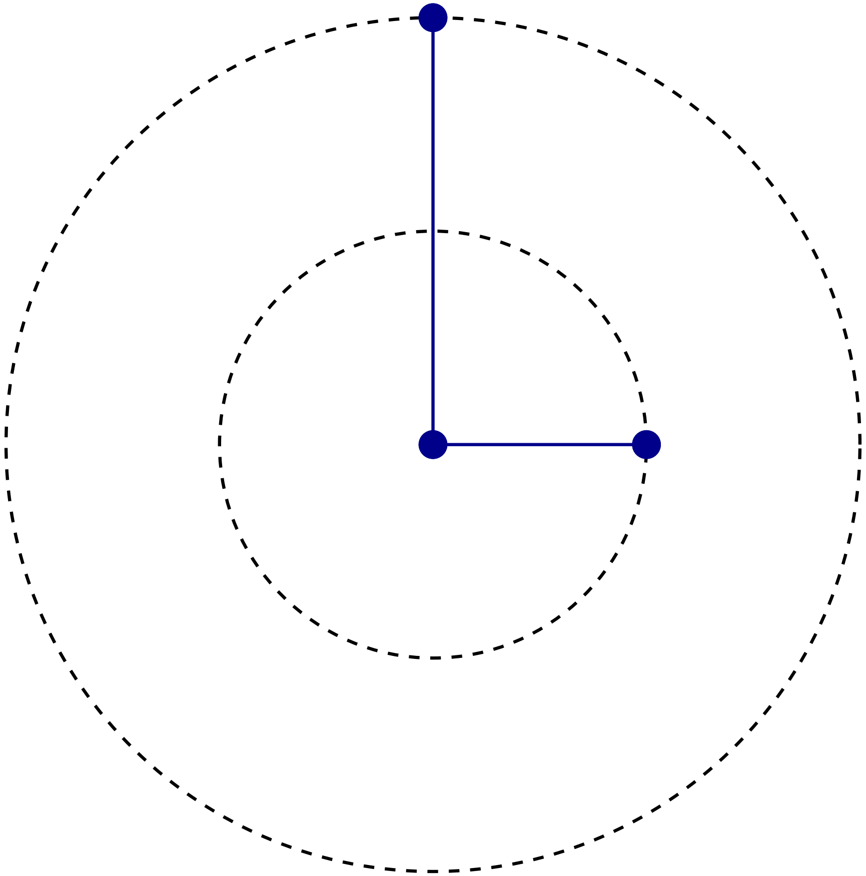
In an empty-ply drawing, the length of adjacent edges differs at most by a factor of 2



Empty-ply drawings: properties

Property

In an empty-ply drawing, no vertex has degree larger than 24



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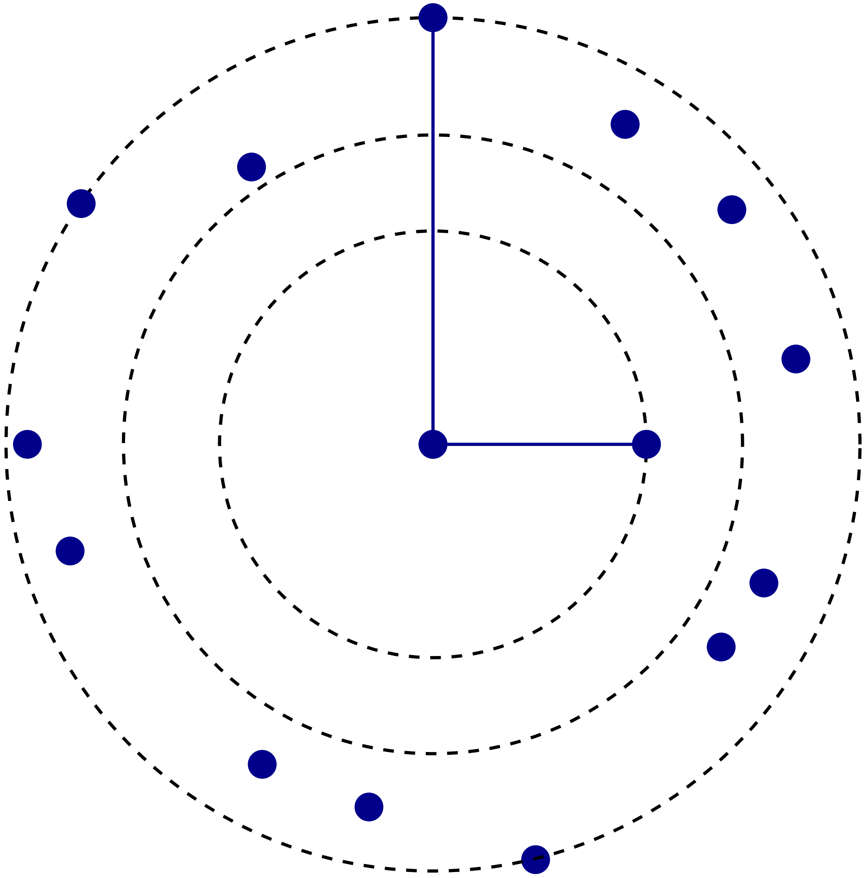

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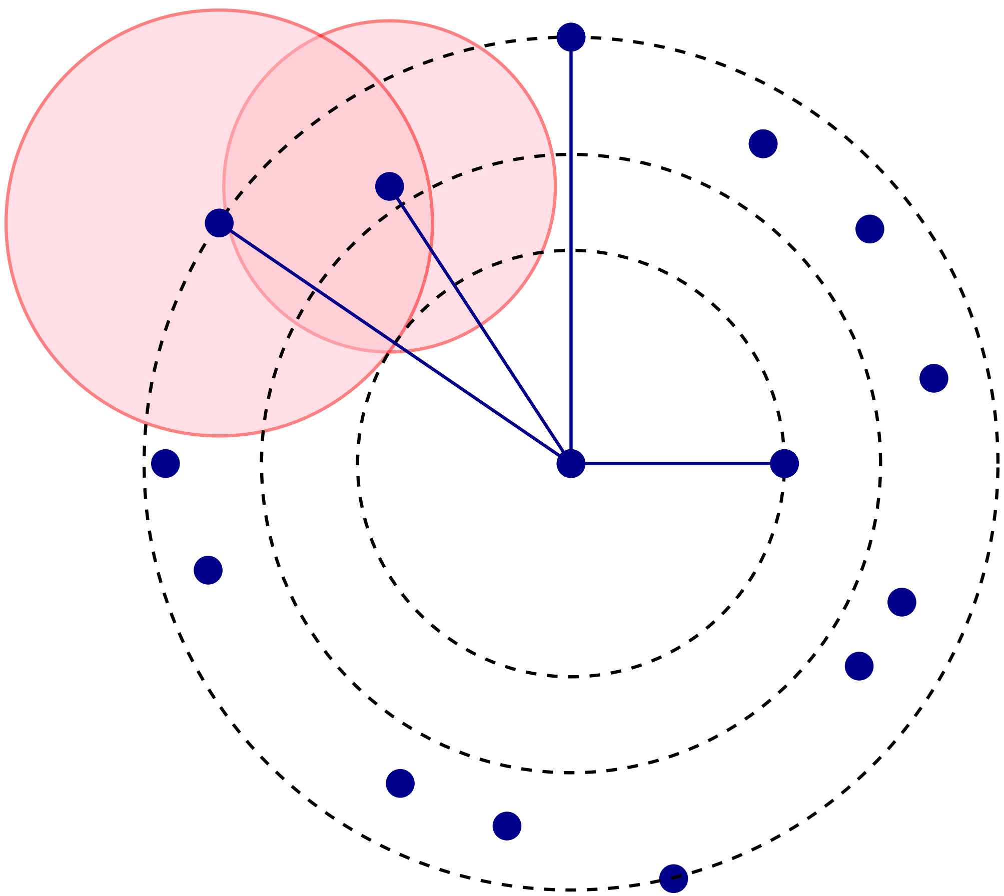
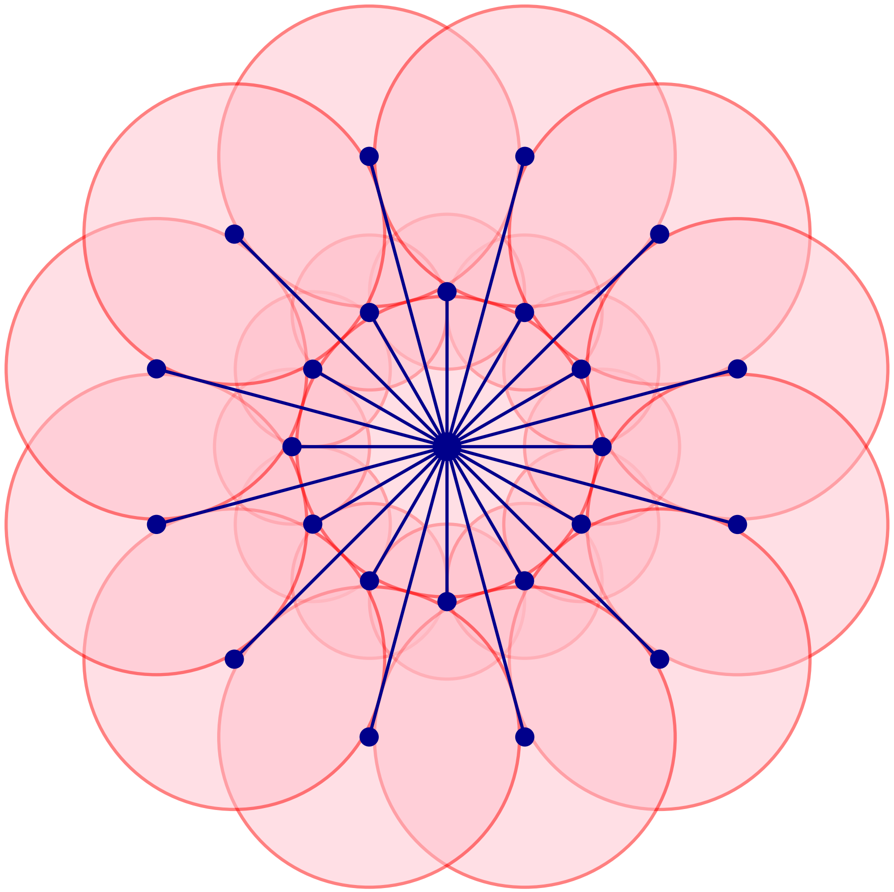


Graphs admitting empty-ply drawings

Theorem

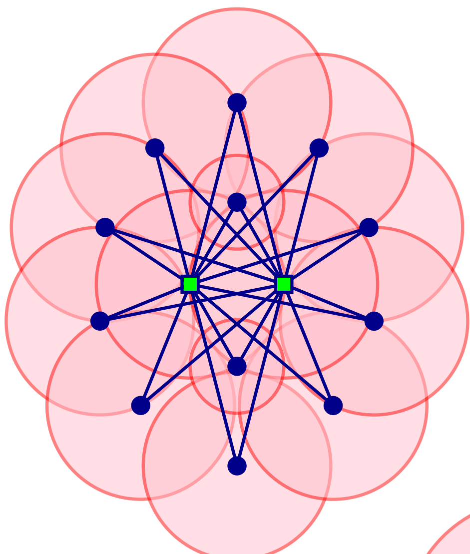
The graph $K_{1,24}$ (the star graph with 24 leaves) admits an empty-ply drawing

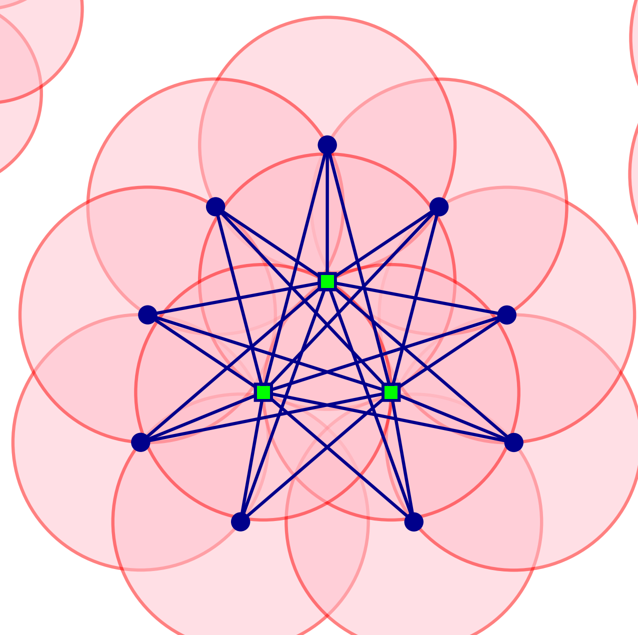


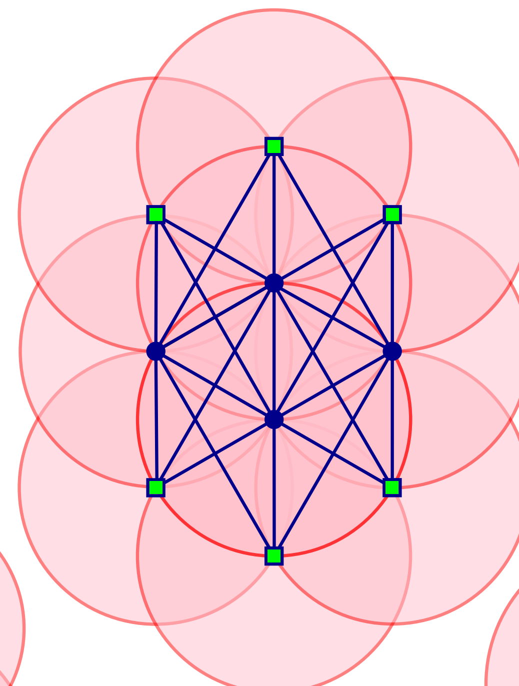
Empty-ply drawings of complete graphs

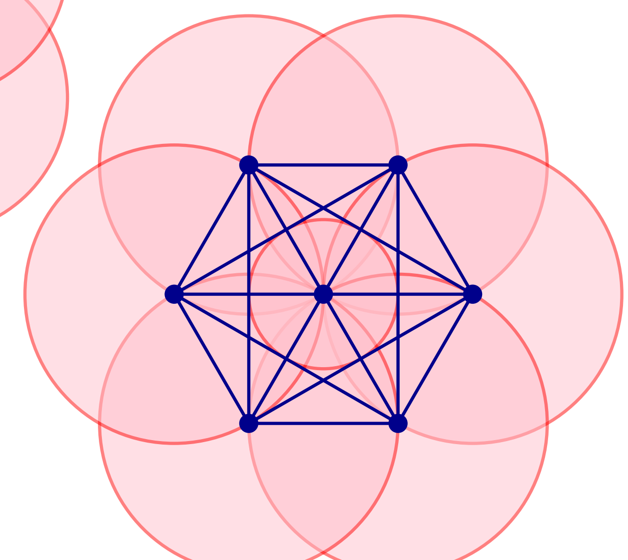
Theorem

The graphs $K_{2,12}$, $K_{3,9}$, $K_{4,6}$, K_7 admit empty-ply drawings









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The graphs $K_{2,12}$, $K_{3,9}$, $K_{4,6}$, K_7 admit empty-ply drawings

The graphs $K_{2,15}$ and K_8 do not admit empty-ply drawings

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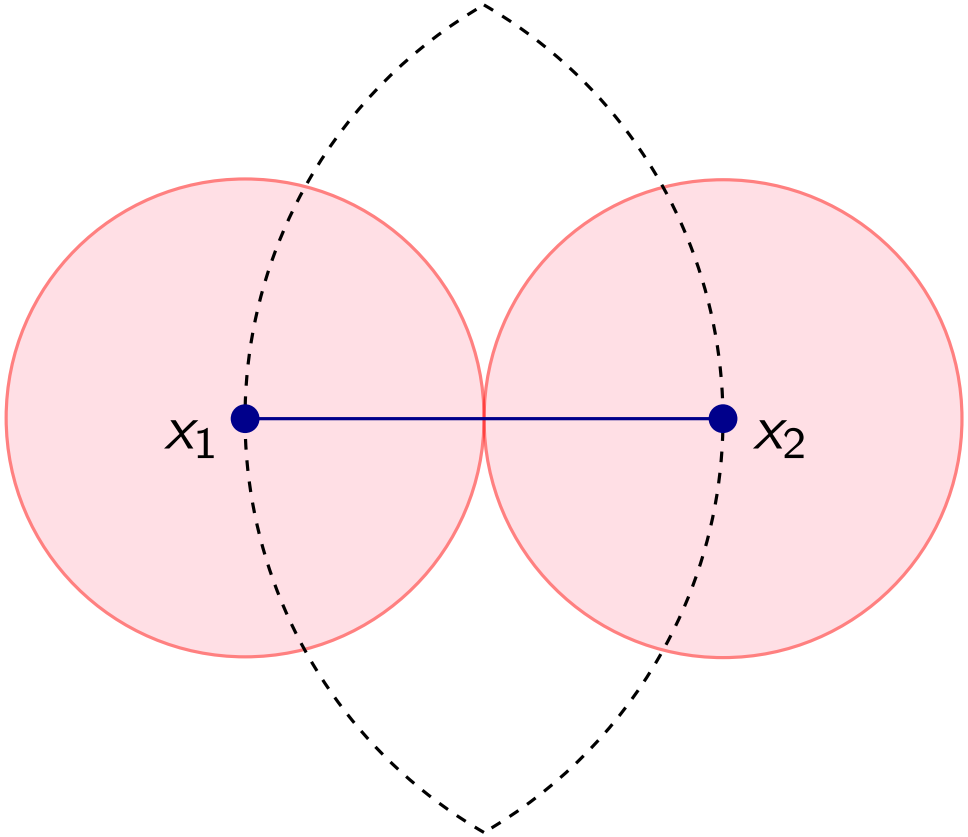

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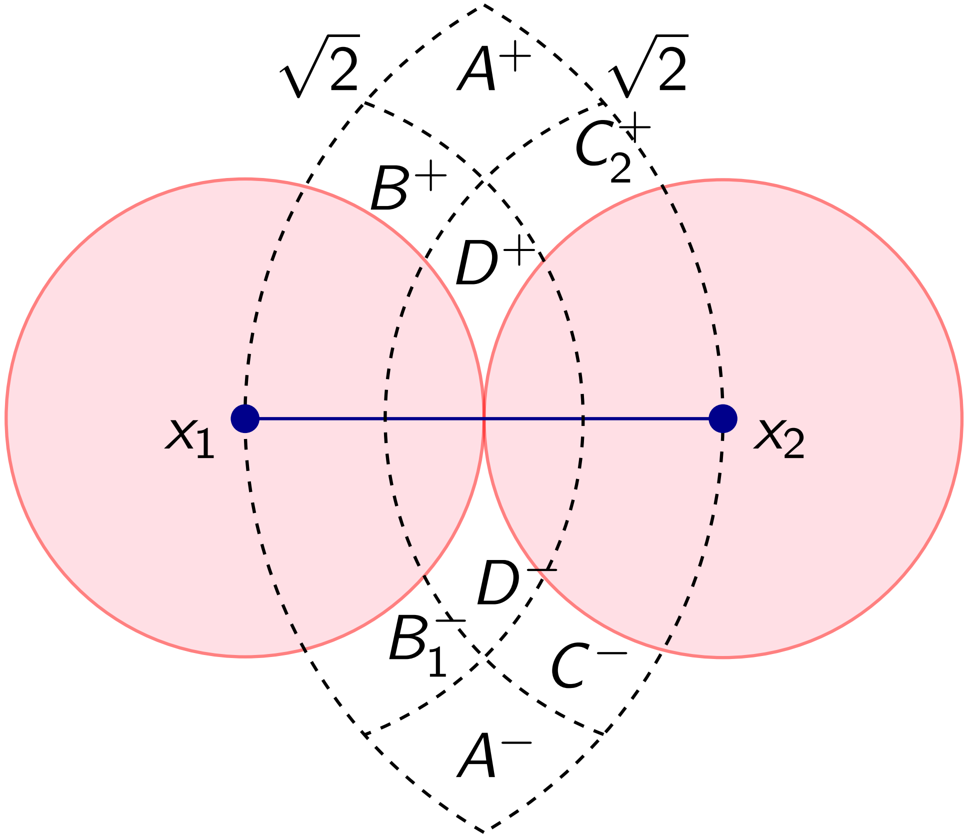

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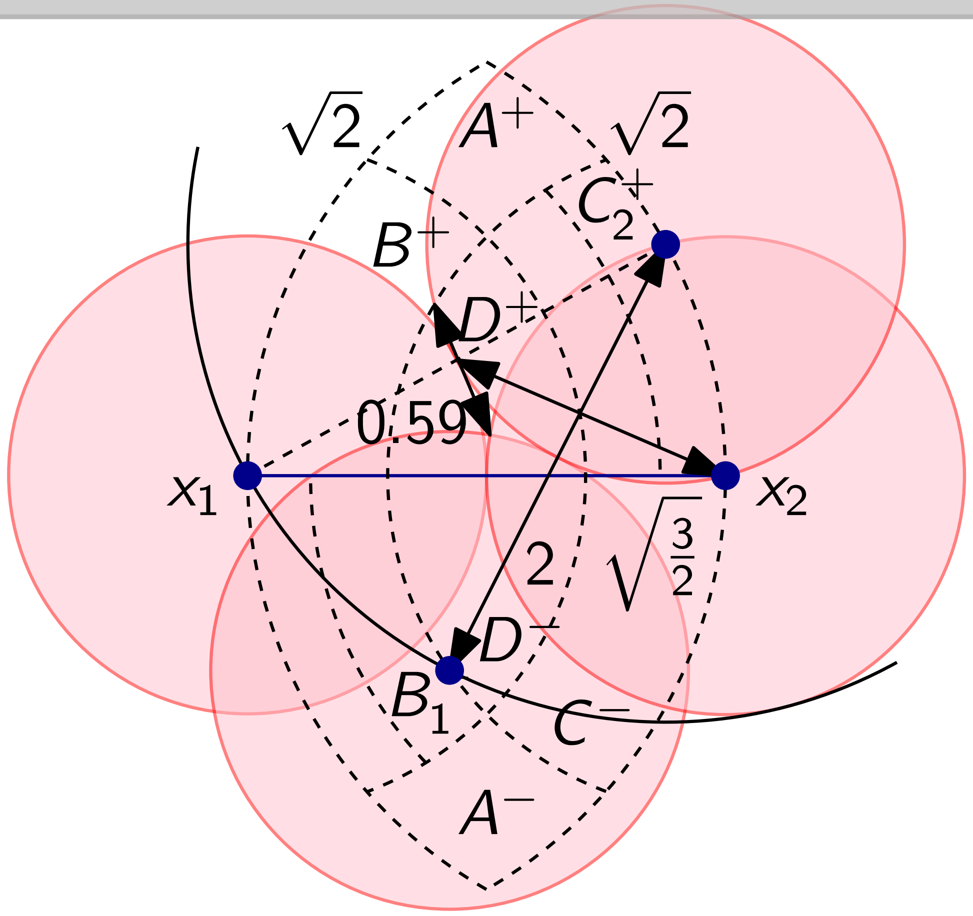

Empty-ply drawings of trees

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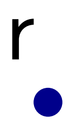

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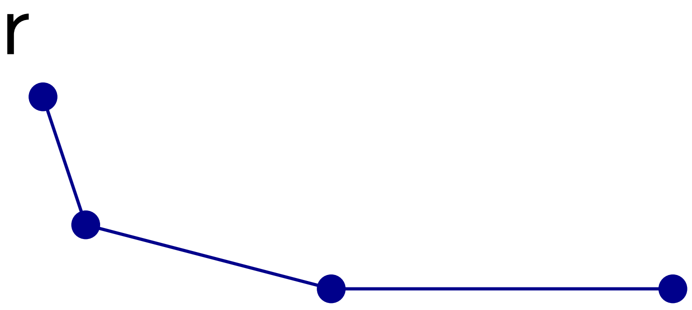

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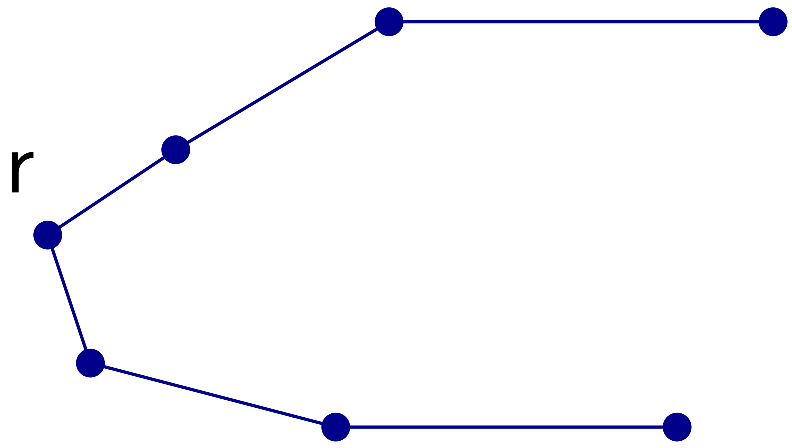

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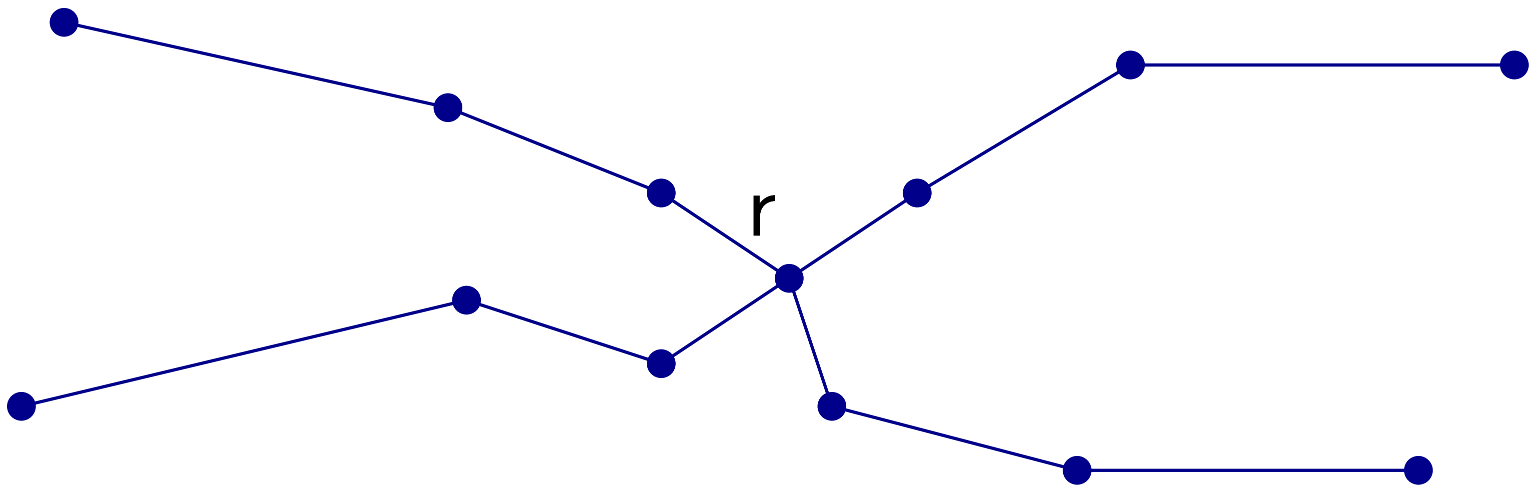

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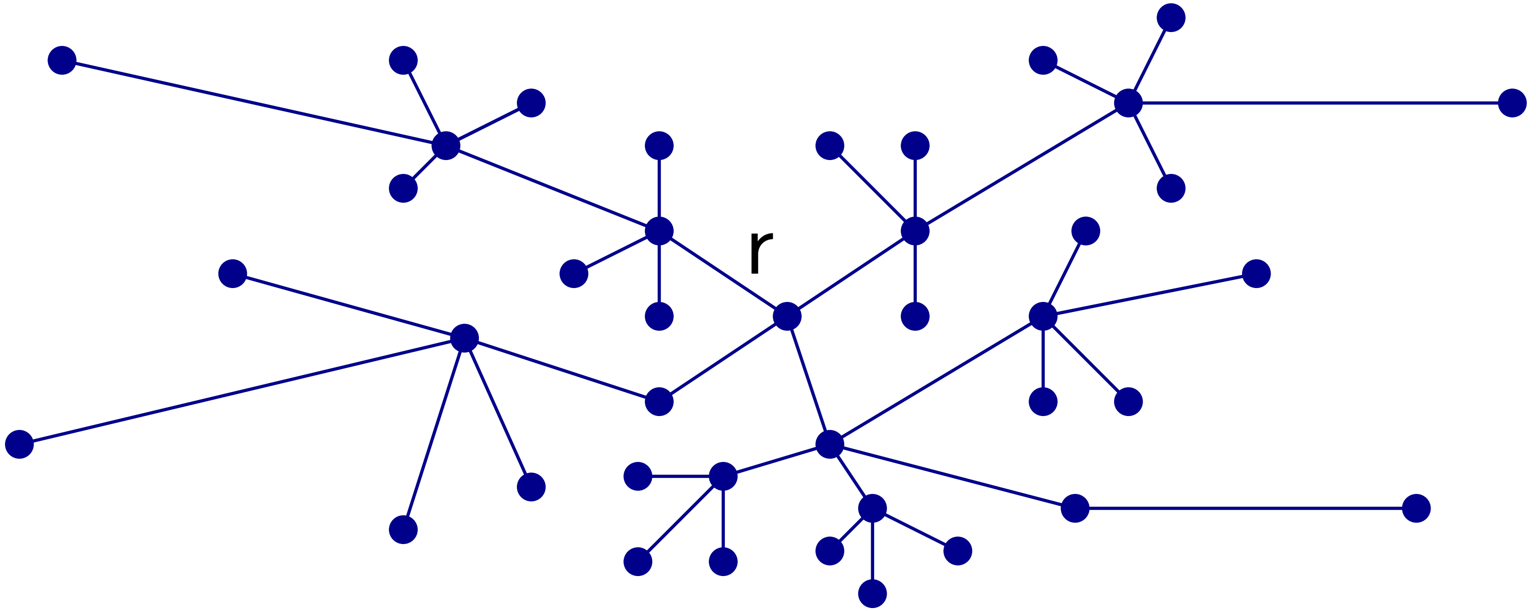

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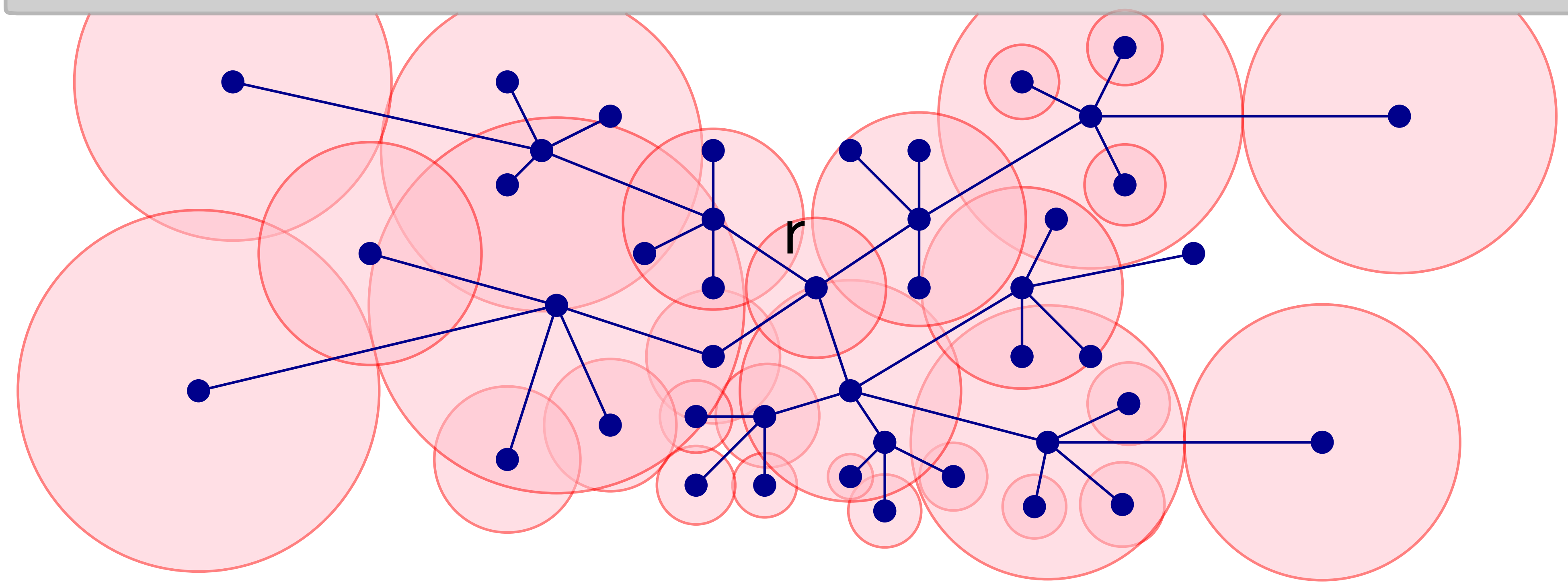

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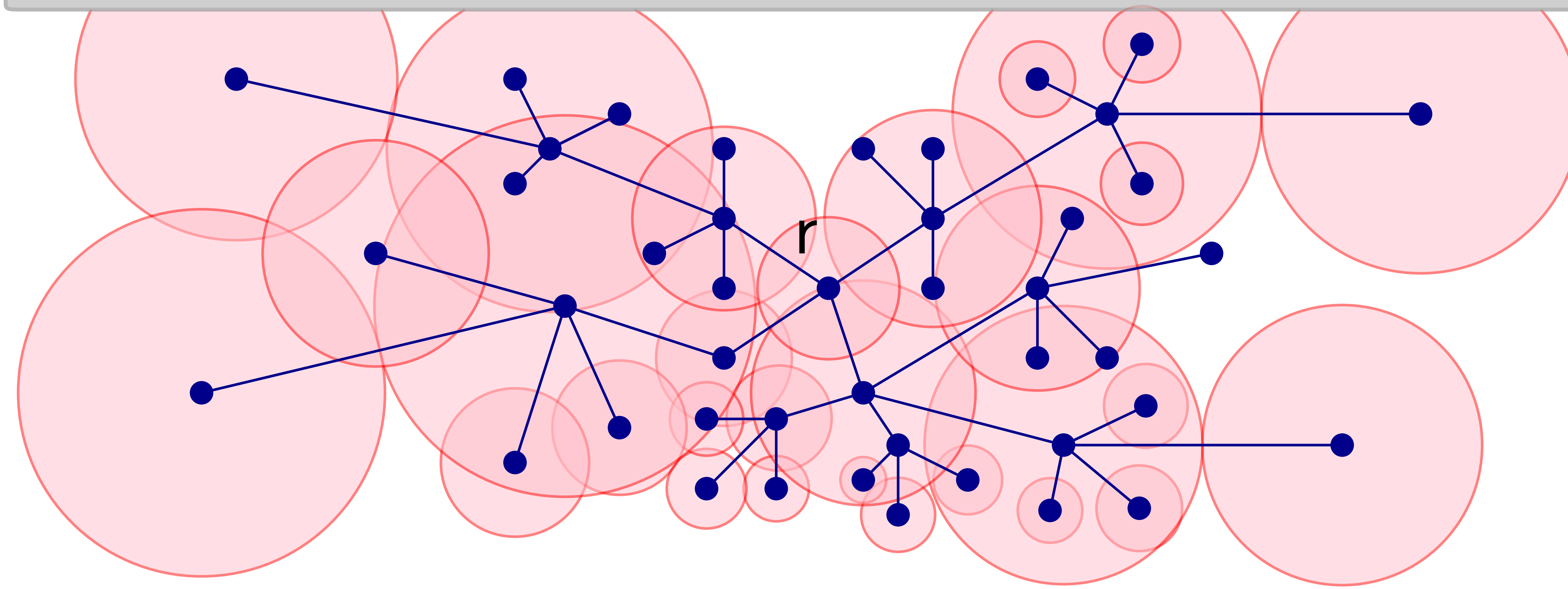


Theorem [Di Giacomo et al. IISA 2015]

Binary trees (maximum degree 3) admit drawings with ply 2, which are also empty-ply

Ply and vertex-ply of planar drawings

Question [Di Giacomo et al. IISA 2015]

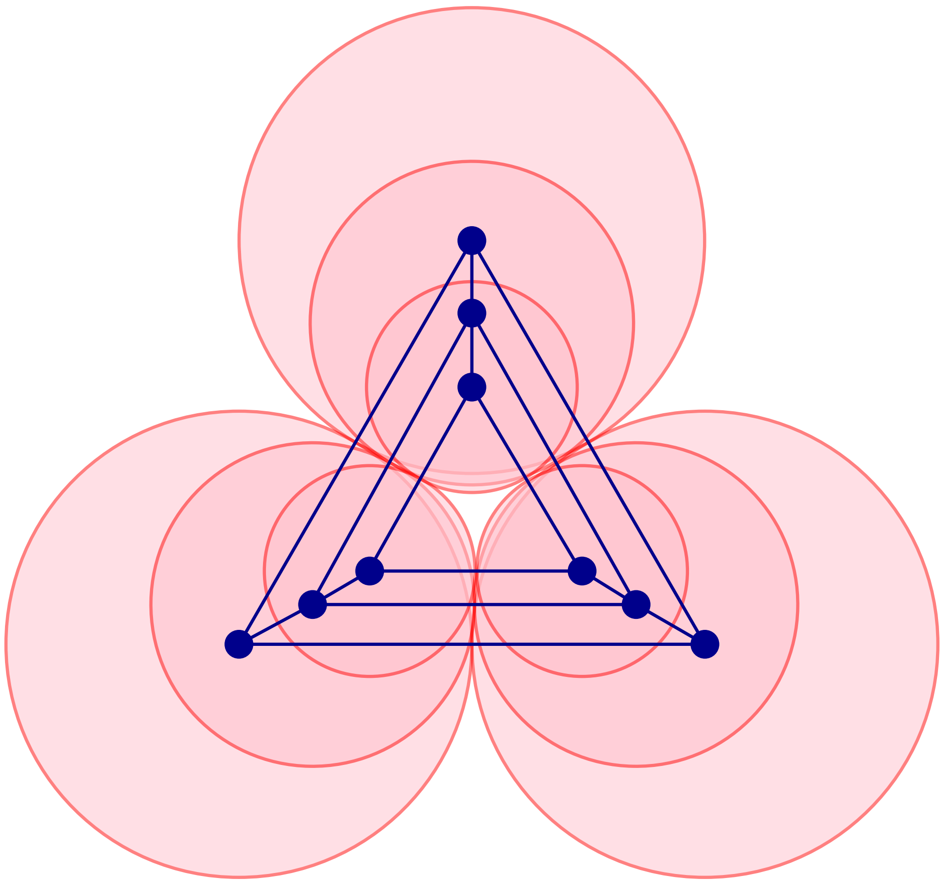
What happens if we restrict to planar drawings?

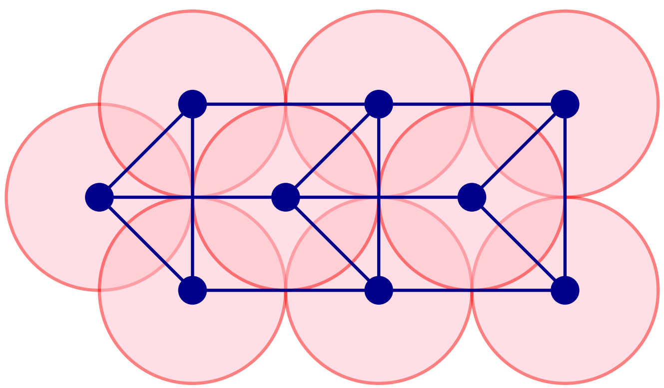
Is there a trade-off between ply and number of crossings?

Ply and vertex-ply of planar drawings

Observation [Di Giacomo et al. IISA 2015]

The natural drawing of a nested-triangle graph has linear (vertex-)ply, but they admit non-planar drawings with ply 2

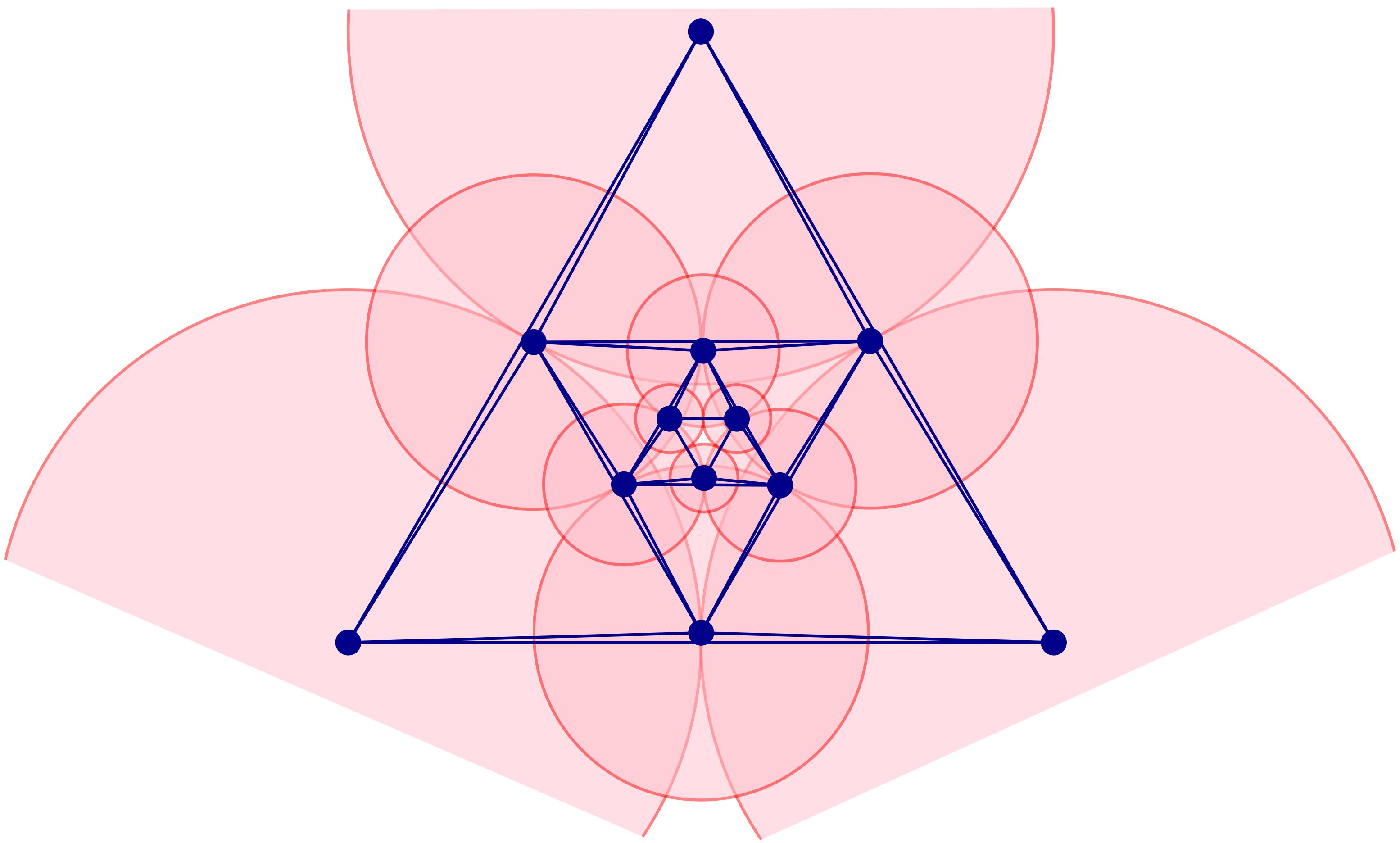




Ply and vertex-ply of planar drawings

Theorem

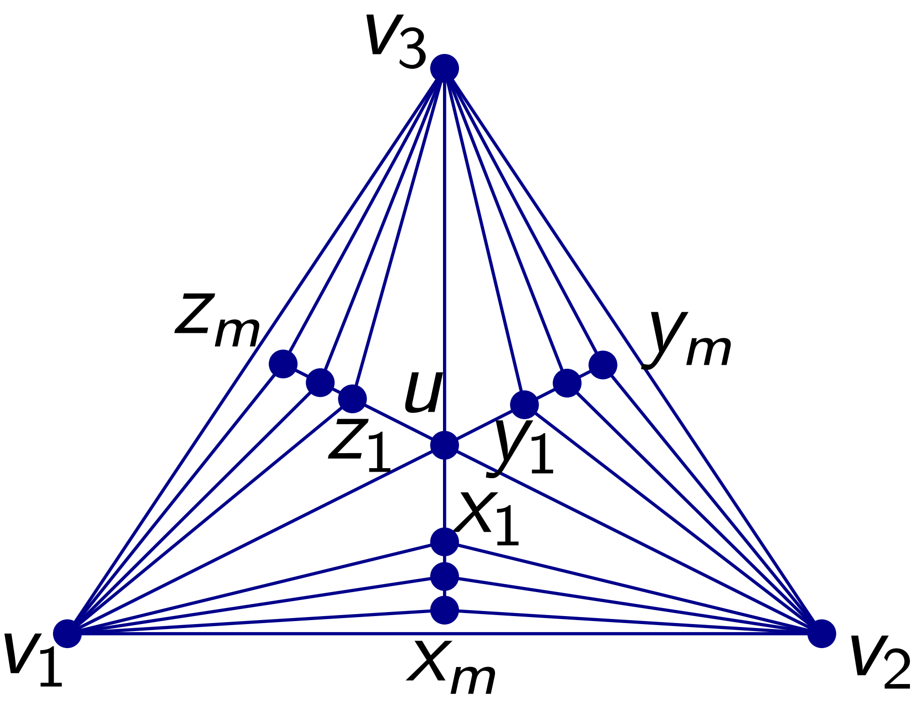
Nested-triangle graphs admit planar drawings with ply 4



Ply and vertex-ply of planar drawings

Theorem

There exist graphs that require $\Omega(n)$ vertex-ply in any planar drawing, but that admit drawings with ply 5 and 3 crossings



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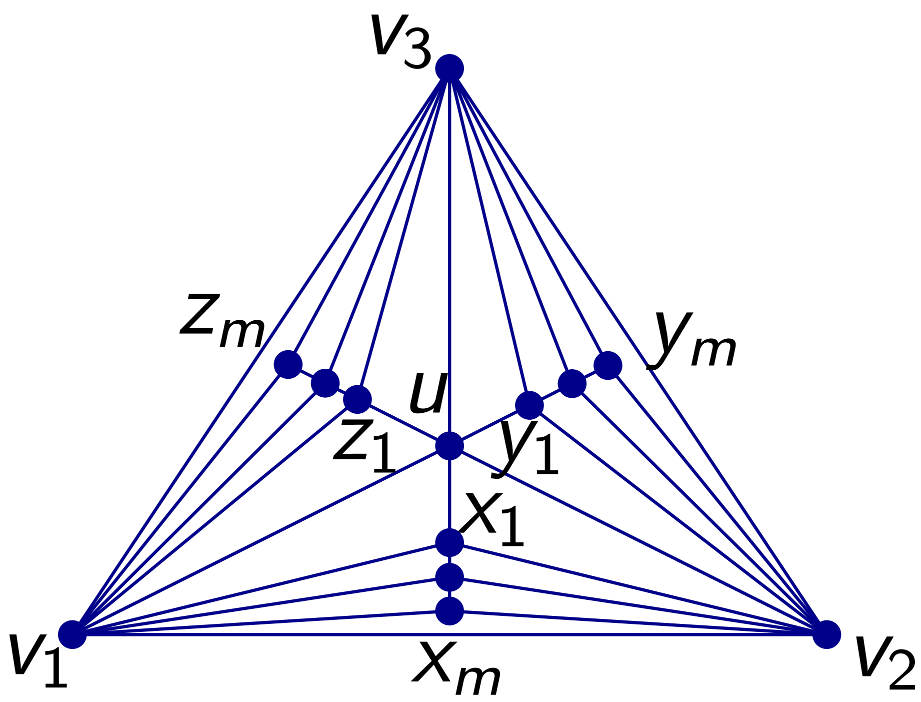

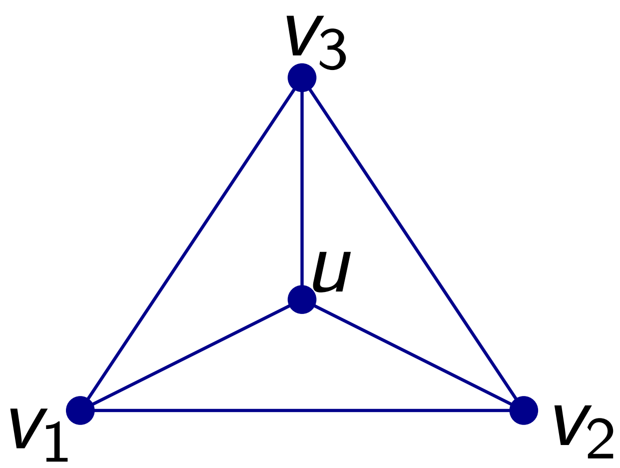

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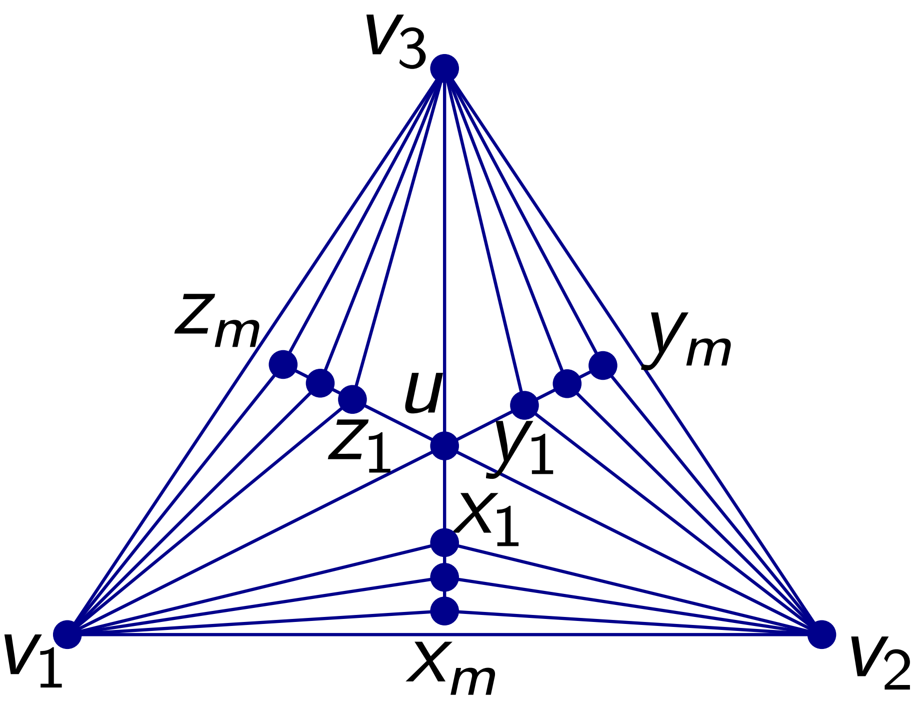

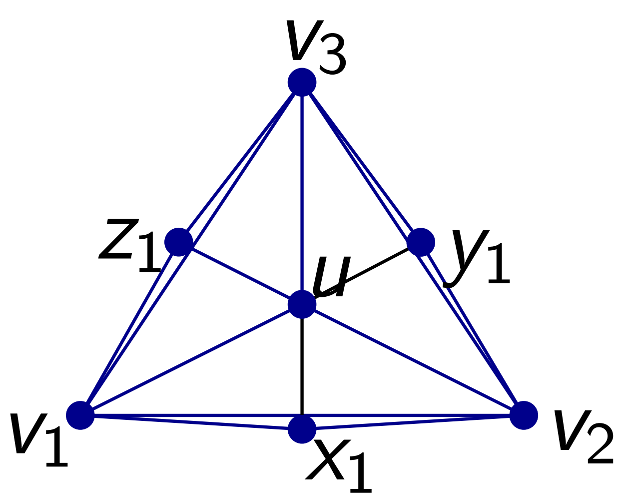

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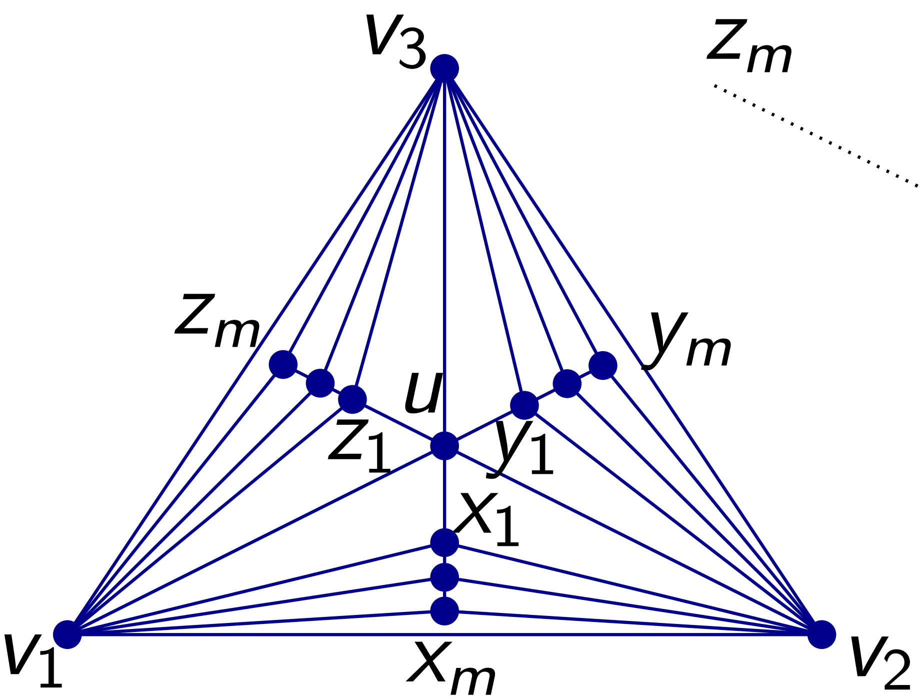

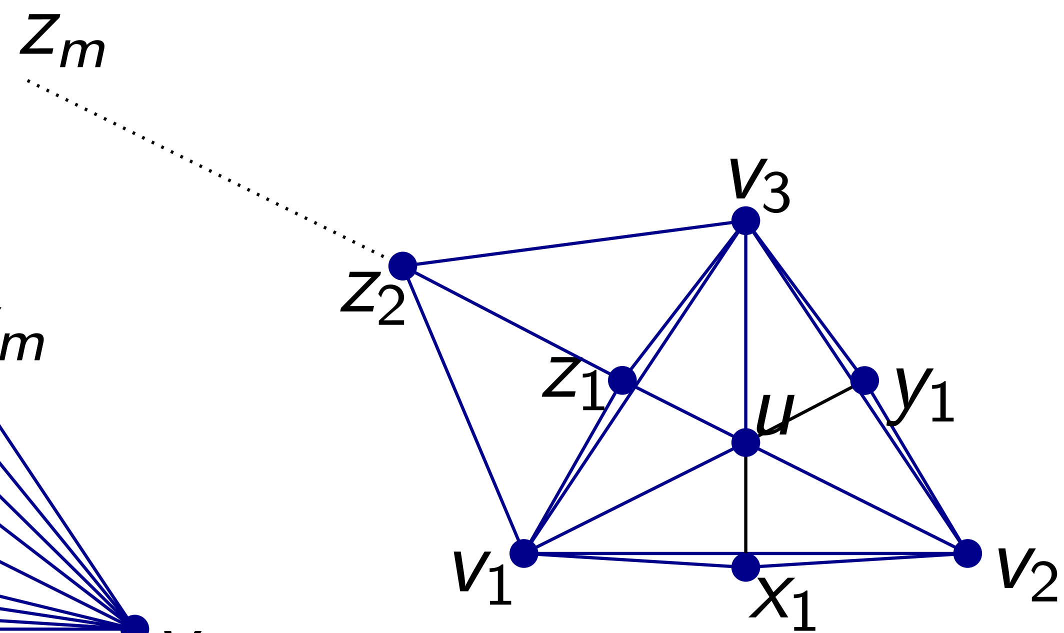

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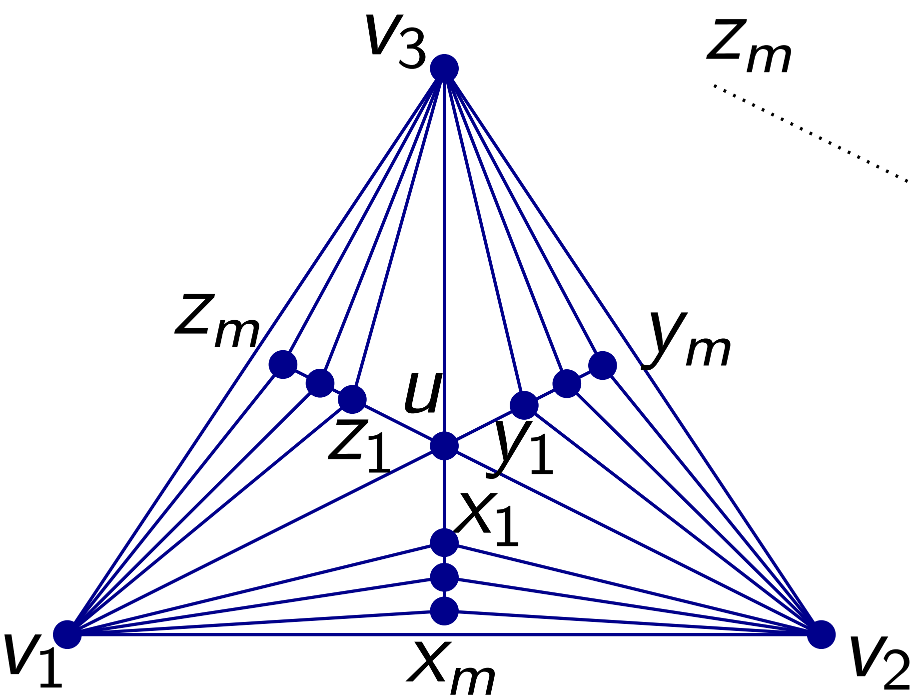

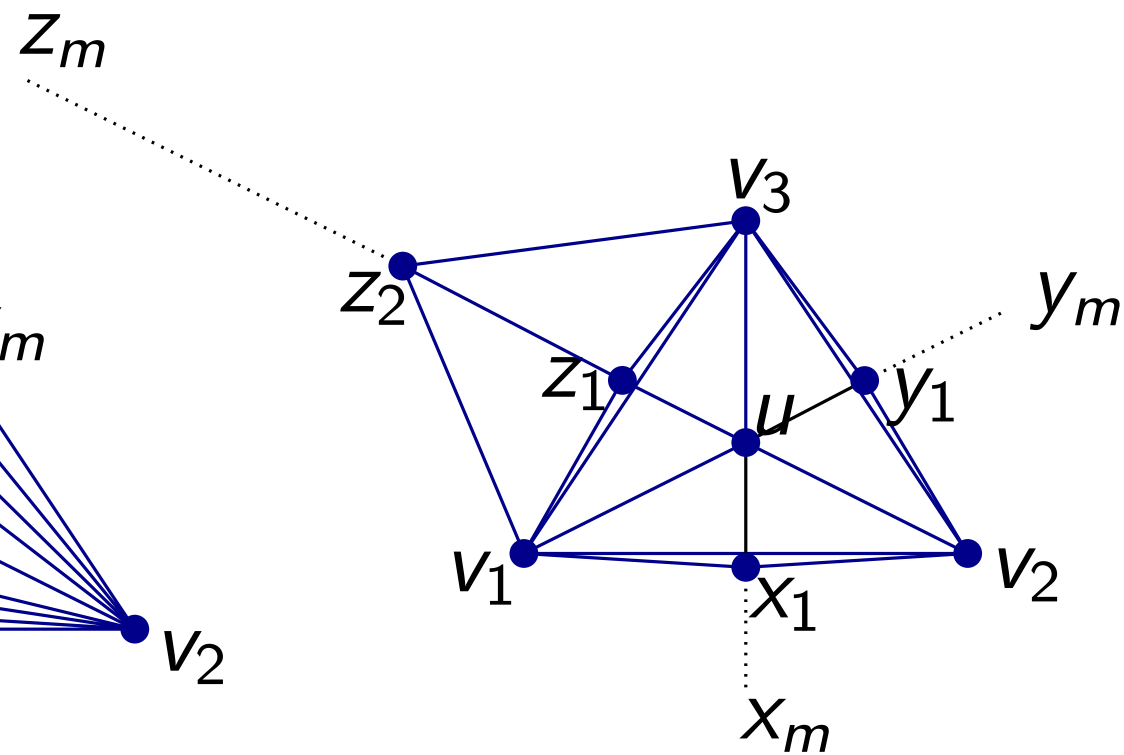

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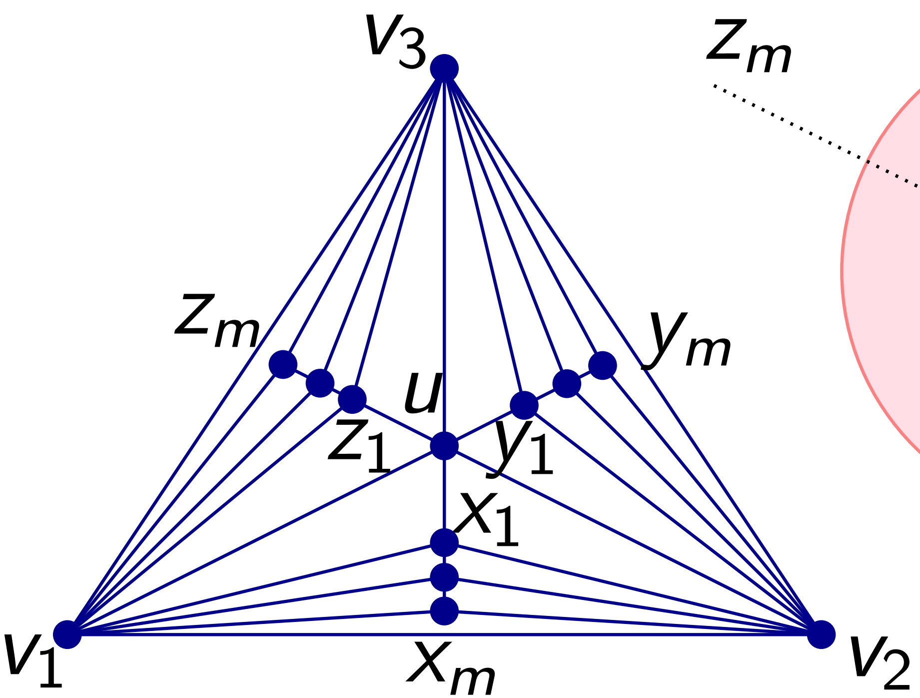

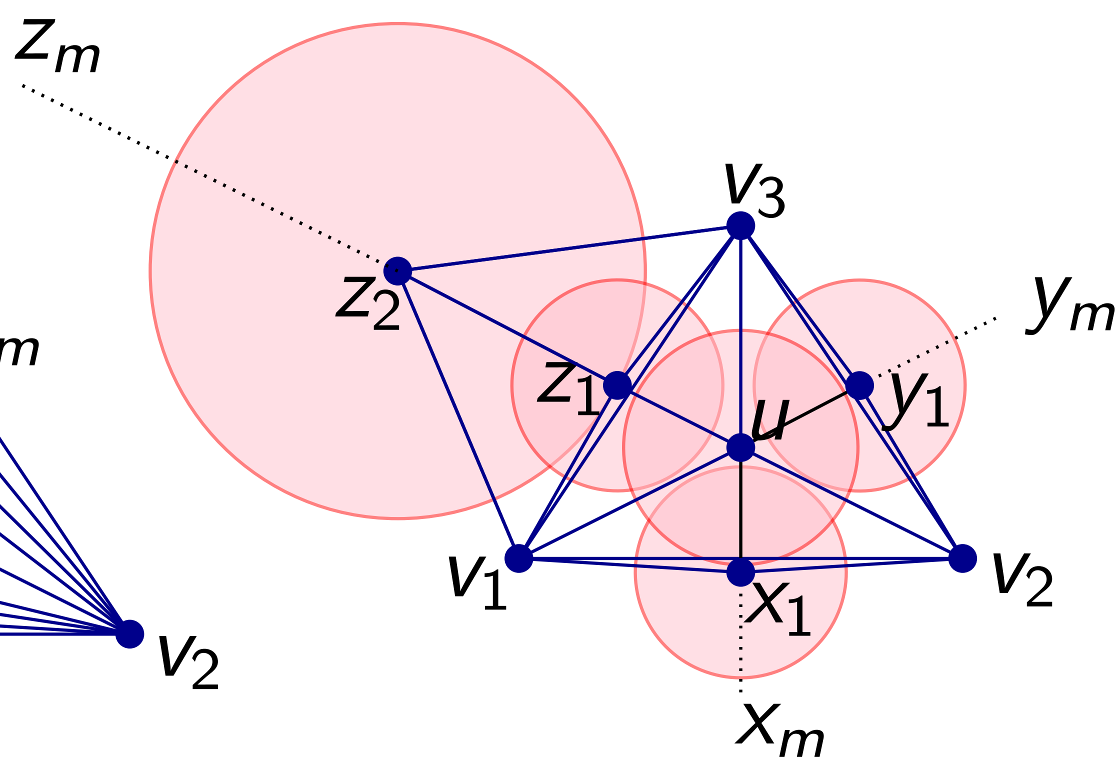

Ply and vertex-ply of planar drawings

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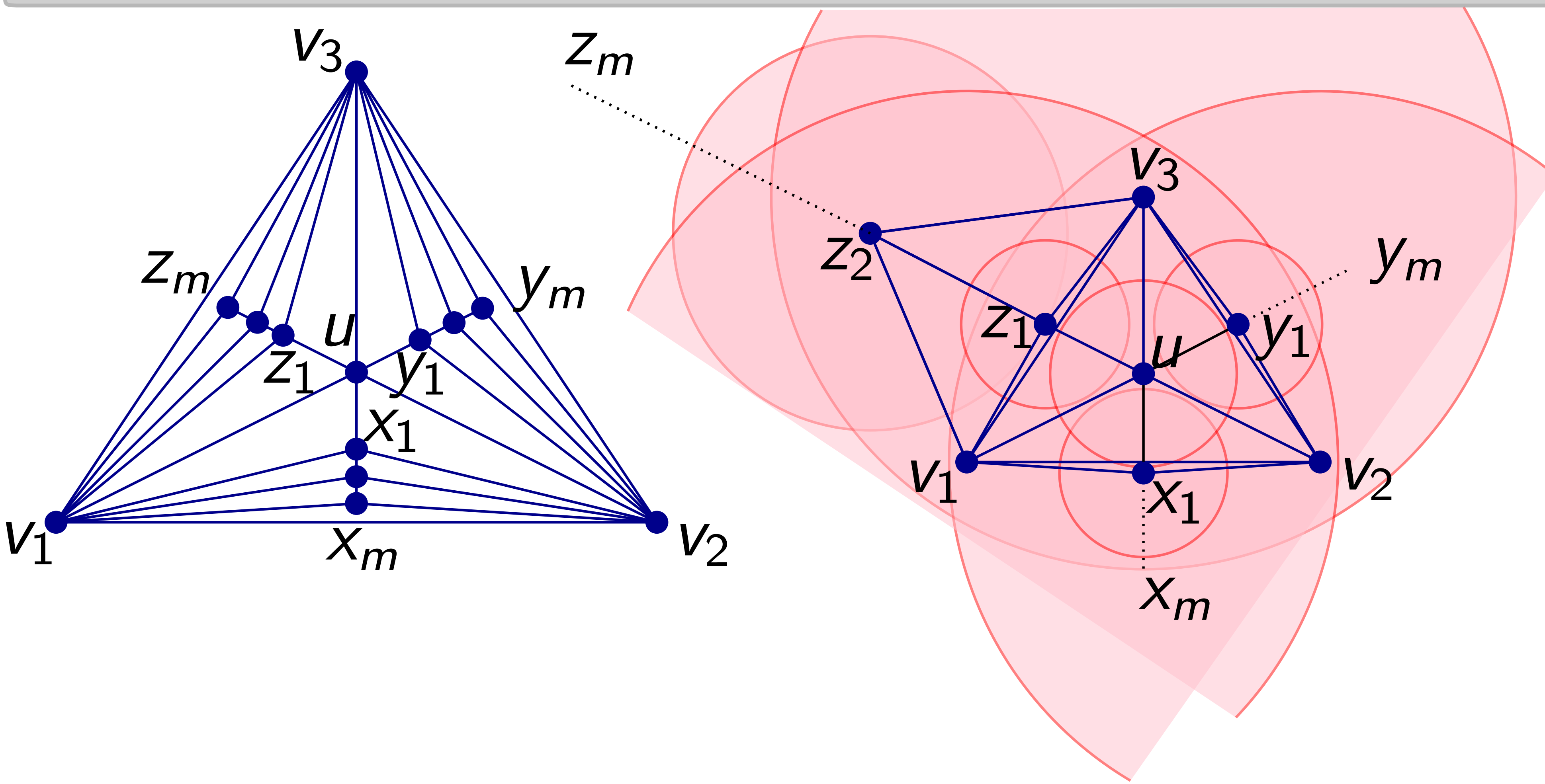

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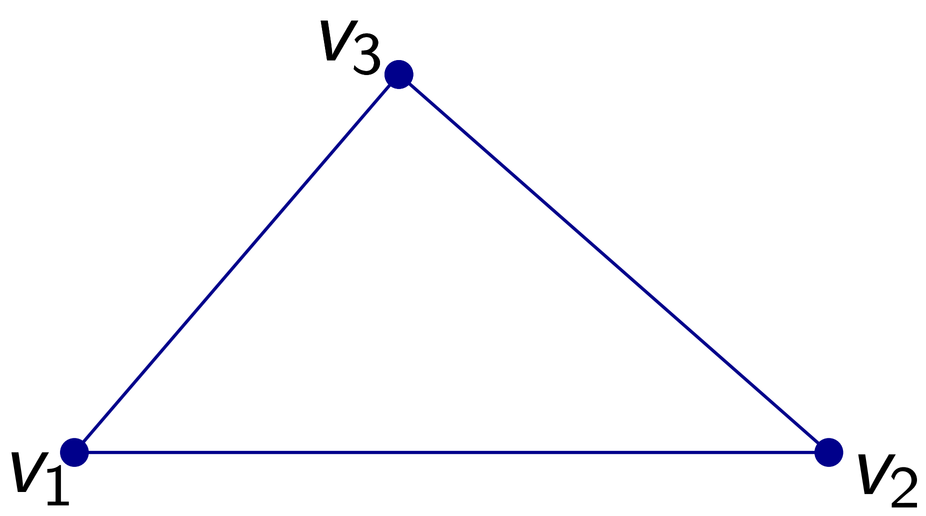
Suppose that (v_1, v_2) is the longest outer edge



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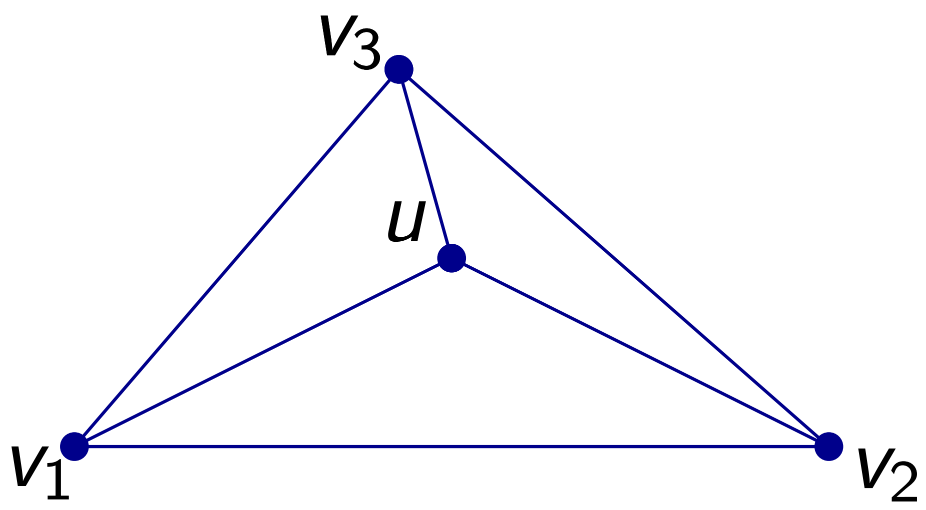
Edge (v_1, v_2) is longer than (u, v_1) and (u, v_2)



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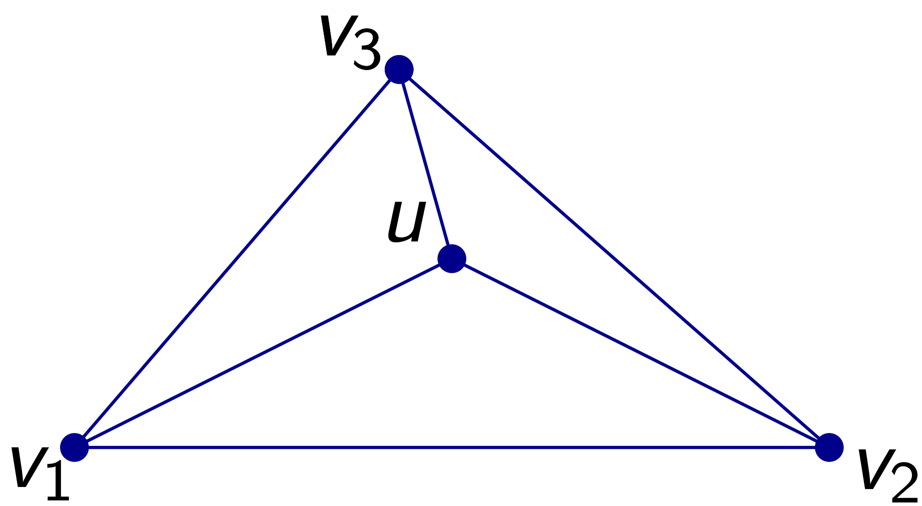
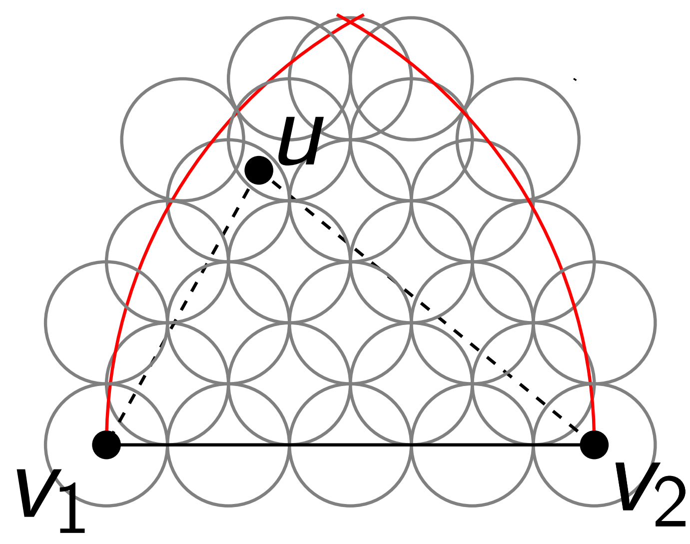
Triangle (u, v_1, v_2) can be covered with a constant number of circles whose diameter is $\frac{1}{4}$ of the length of (v_1, v_2)





Ply and vertex-ply of planar drawings

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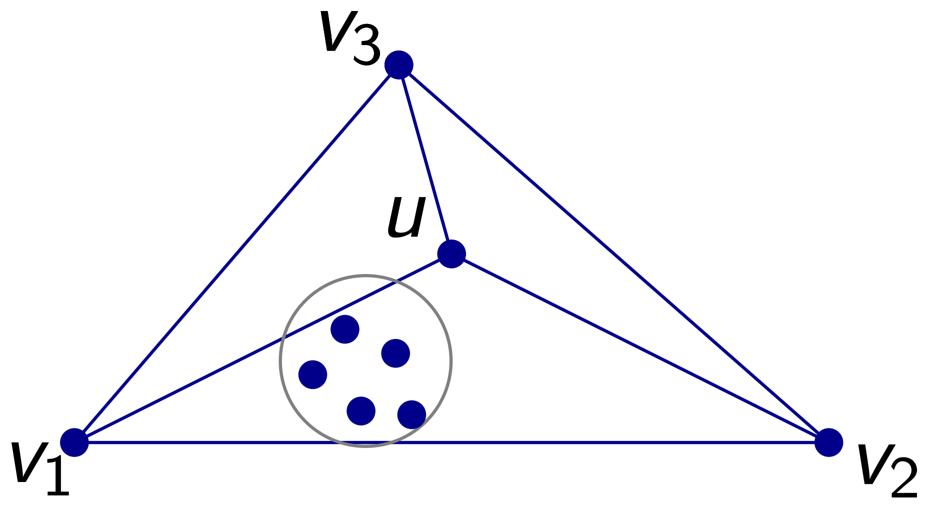
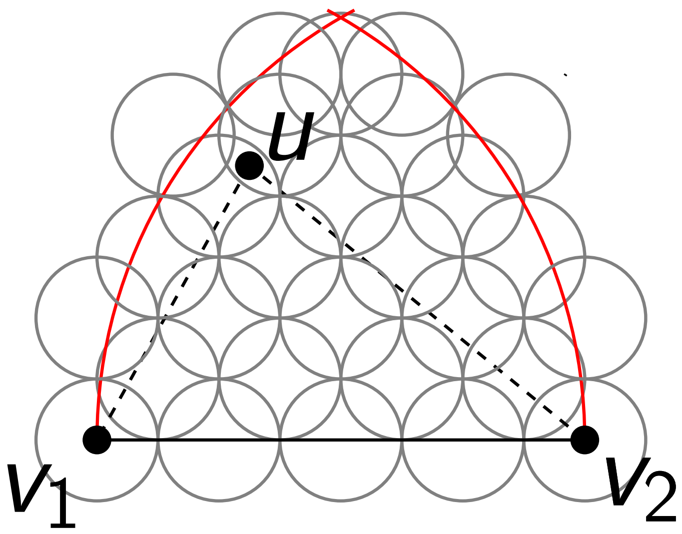
One of these circles contains a linear number of $x_1, \ldots, x_m$





Ply and vertex-ply of planar drawings

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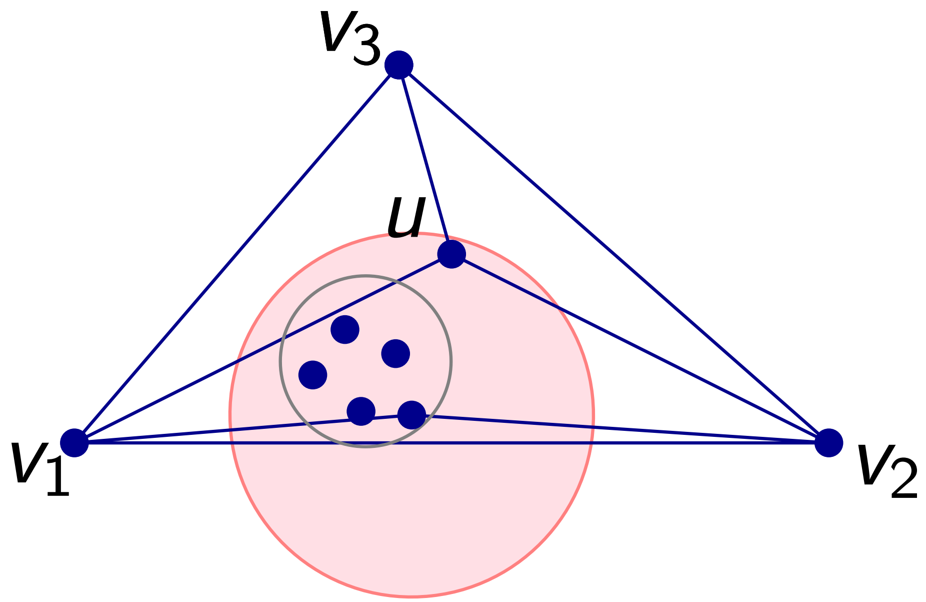
The disks of these vertices have radius at least $\frac{1}{4}$ of (v_1, v_2)
Thus, each of them contains all the vertices in the circle



Open problems

- How hard is to test the existence of an empty-ply drawing?
- Do 3-ary trees (max-degree-4) admit empty-ply drawings?
- Do max-degree-3 (planar) graphs admit empty-ply drawings?
- Fill the gaps on complete bipartite graphs.
  $K_{2,13}$, $K_{3,10}$, $K_{4,7}$, $K_{5,5}$?
- What if we allow at most k vertices to lie inside each disk?
  Typical generalization for proximity drawings
  k-ply drawings or k-empty-ply drawings

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Thanks for your attention!!!