On the Edge-length Ratio of Outerplanar graphs

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Outline

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Drawing Aesthetics

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Efforts to produce drawings that reflect these two aesthetics have proven hard to come by

Prescribed Edge Lengths

Some NP-Hardness Results

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Bhatt and Cosmadakis (1987)

Does a tree T with $\Delta(T) \leq 4$ admit a unit-length PSLD with vertices on integer grid points?

Relative Edge Lengths

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Hoffmann, van Kreveld, Kusters, and Rote (2014)
: Construct a family of embedded planar graphs having unbounded edge-length ratio

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The edge-length-ratio $\rho(\Gamma)$ of a drawing Γ of a graph G is the ratio of the lengths of the longest to the shortest edge in Γ

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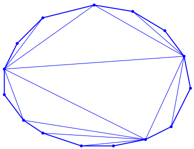

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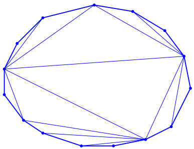

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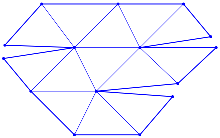

$\rho(\Gamma_2) < 2$

Edge-Length Ratio of a Planar Embedding

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The edge-length-ratio $\rho(\mathcal{G})$ of a planar embedding $\mathcal{G}$ of a graph G is the infimum of $\rho(\Gamma)$ over all embedding-preserving planar straight-line drawings Γ of $\mathcal{G}$

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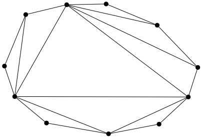

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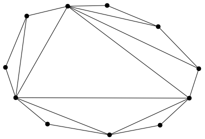

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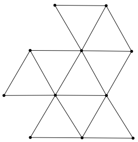

A drawing of $\mathcal{G}$ with $\rho(\mathcal{G}) = 1$

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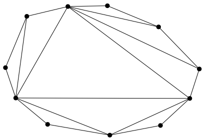

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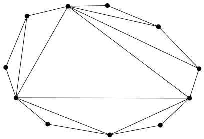

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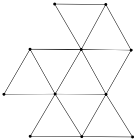

A drawing of G with $\rho(\mathcal{G}) = 1$

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For any $k > 0$, there exists an embedded outerplanar graph Γ with $\rho(\Gamma) > k$

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Constructing an Outerplanar Graph with $\rho > 2 - \epsilon$

Warning: High School Trig Ahead...

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The edge length ratio of a triangle with smallest angle θ *is at least* $2\cos(\theta)$

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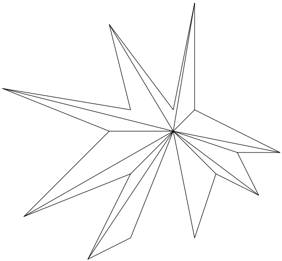

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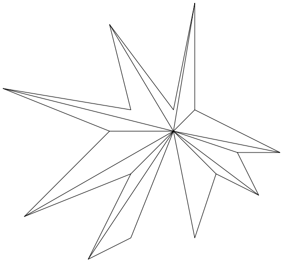

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Corollary

Let G_k be the fan of degree $k+1$. Then $\rho(G) \geq 2\cos(\frac{2\pi}{k})$

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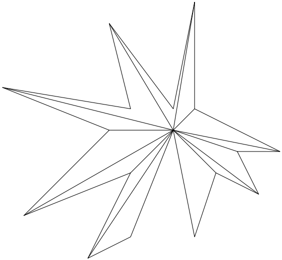

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Let G_k be the fan of degree $k+1$. Then $\rho(G) \geq 2\cos(\frac{2\pi}{k})$ So

$$\lim_{k\to\infty} \rho(G_k) = 2$$

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- Note that the set of strips naturally forms a tree
- Recursively draw each strip starting at root of tree

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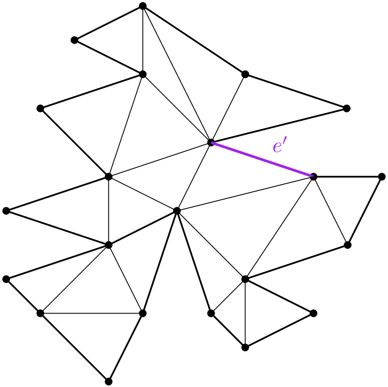
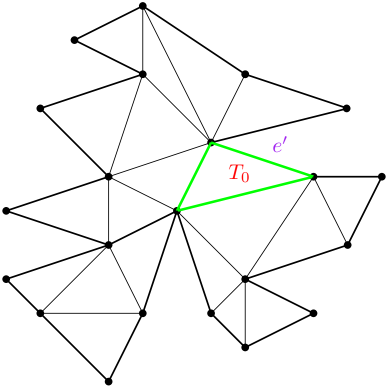


Choose an external edge of G

Drawing Outerplanar Graphs So That $\rho < 2$

Partition triangles of G into strips



It determines a unique triangle T_0...

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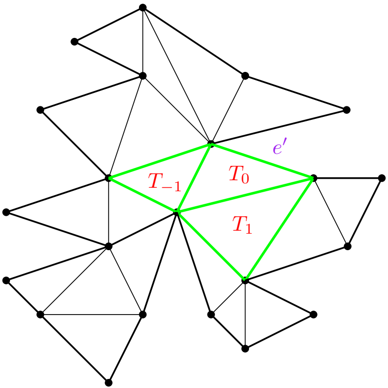

...which determines unique triangles T_1 and T_{-1}

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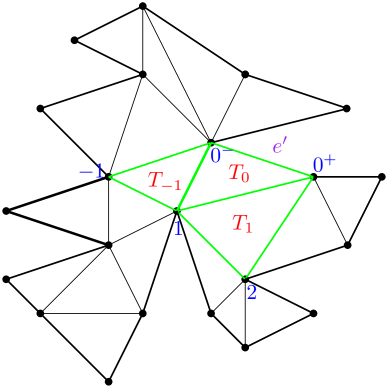

Now cleverly label the vertices...

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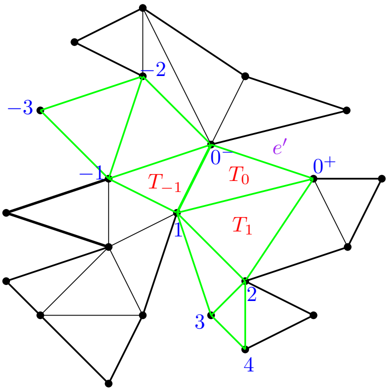

...to uniquely determine a chain of triangles with a single bend

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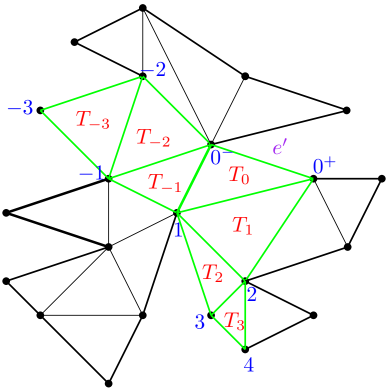

A U-chain of triangles in *G*

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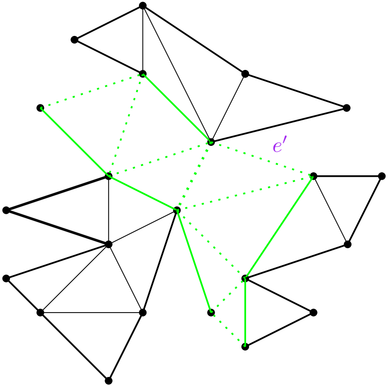

Removing short edges decomposes *G* into 2-connected components

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- Recursively compute the U-chain decomposition for each G_i from e_i

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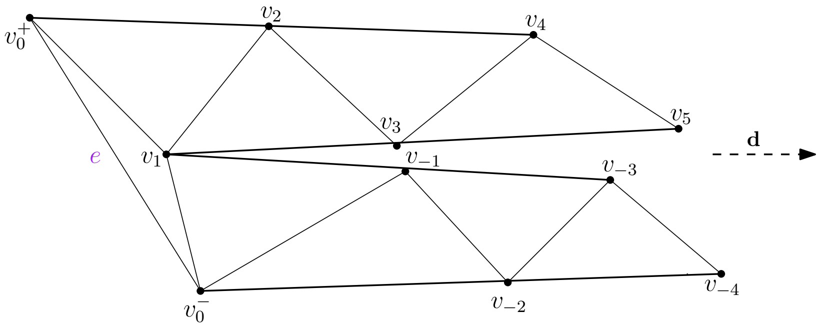

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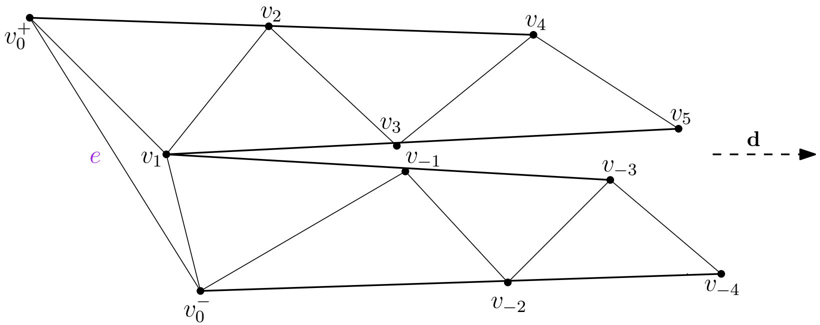

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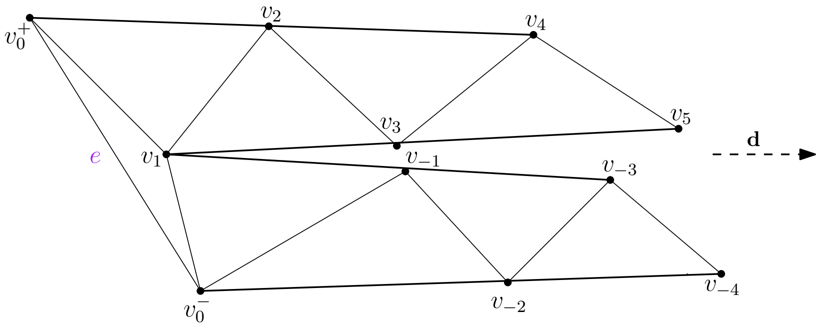

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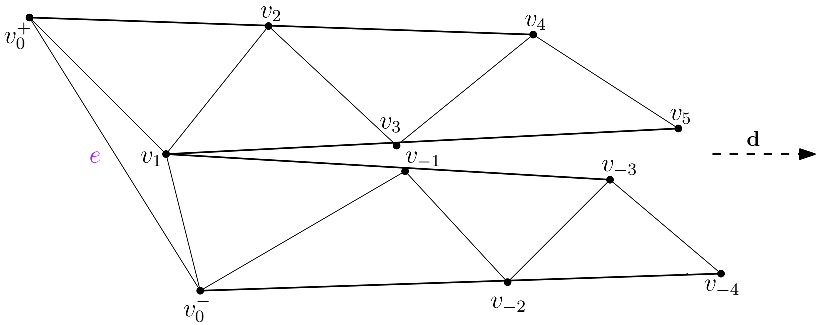

- *e* is drawn unit length with non-zero slope (w.r.t. **d**)
- The *long* edges are drawn unit-length with non-zero slope
- The *short* edges each have length in the range $(\frac{1}{2}, 1)$

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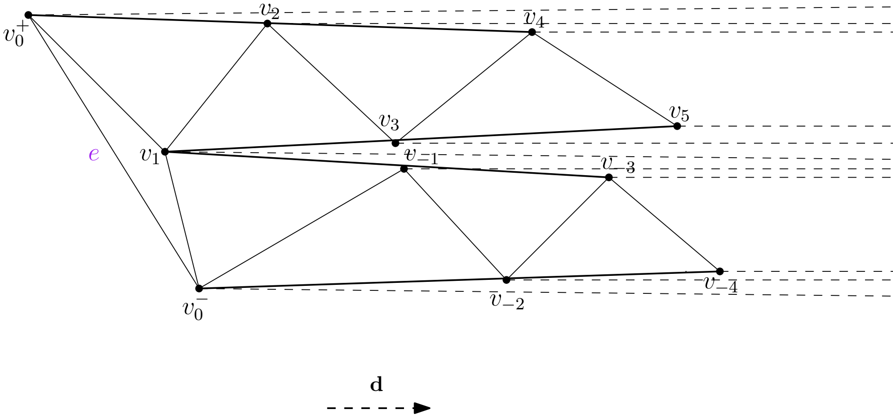

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- *Γ lies within the horizontal strip $S(s)$ with left boundary s*
- *Each long edge e' of C is drawn as unit-length s' with non-zero slope in the range $(-\theta, \theta)$*
- *The strips $S(s')$ intersect Γ only at s'*

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Moreover, such a drawing can be computed in $O(n)$-time in the real RAM model.

Proving the Lemma

Drawing the Chain

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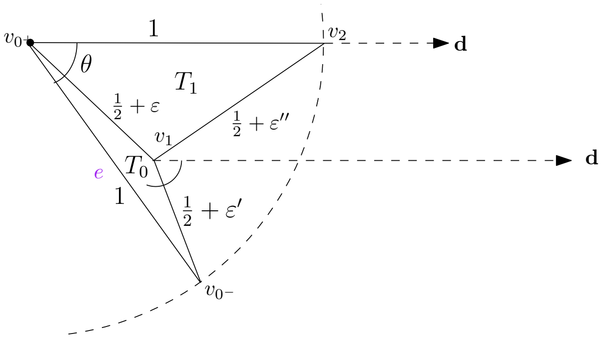
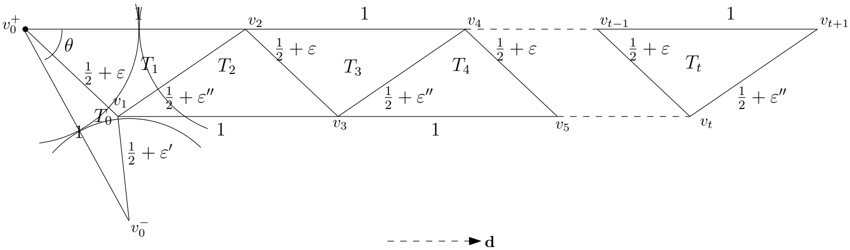
Drawing T_0 and T_1



Proving the Lemma

Drawing the Chain

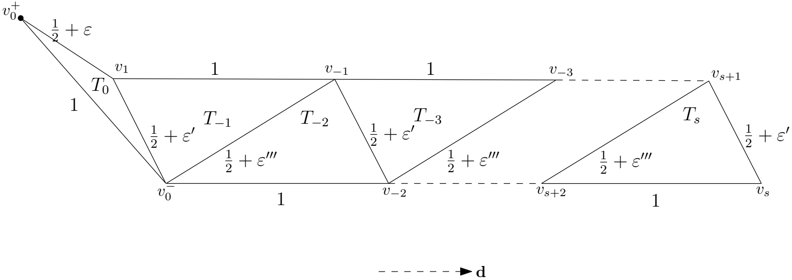
Drawing T_2, ...



Proving the Lemma

Drawing the Chain

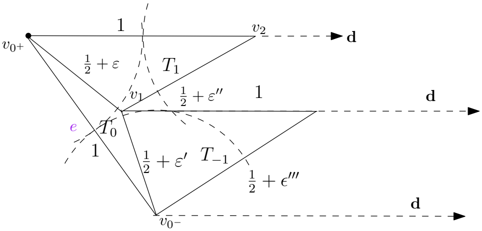
Drawing $T_{-1}, T_{-2}, \ldots$



Proving the Lemma

Drawing the Chain

Positioning v_1 appropriately



Unit-Length PSLDs for Bipartite Outerplanar Graphs

A similar construction

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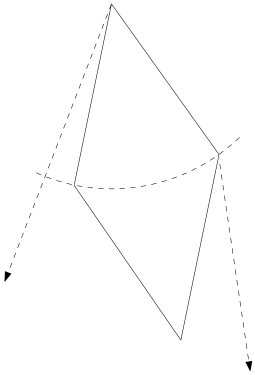

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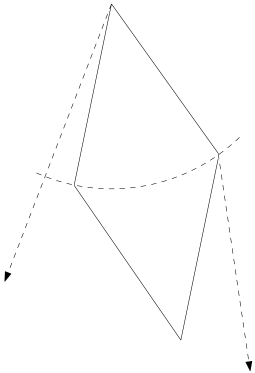
- Draw F within a wide wedge
- Removing e from G leaves up to 3 2-connected components

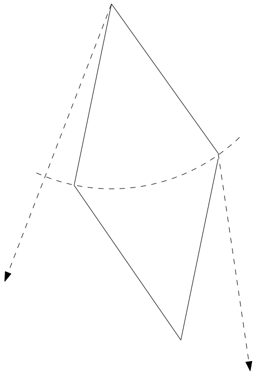


Unit-Length PSLDs for Bipartite Outerplanar Graphs

A similar construction

Assume that T is maximal: All faces are quadrilaterals

- Choose a face F with an external edge e
- Draw F within a wide wedge
- Removing e from G leaves up to 3 2-connected components
- Draw each component recursively in its own wide wedge



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Given a wedge of interior angle $\pi + \epsilon$

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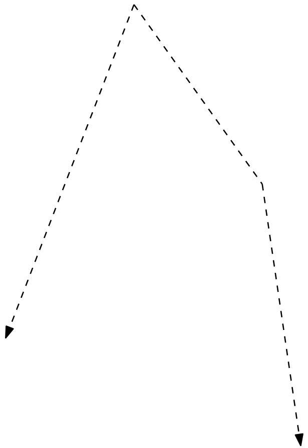
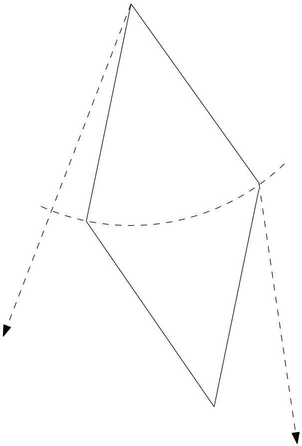


Unit-Length PSLDs for Bipartite Outerplanar Graphs

A similar construction

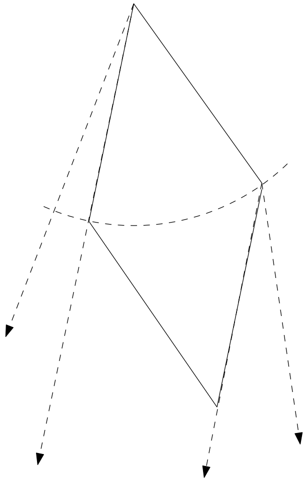
Draw a rhombus inside it



Unit-Length PSLDs for Bipartite Outerplanar Graphs

A similar construction

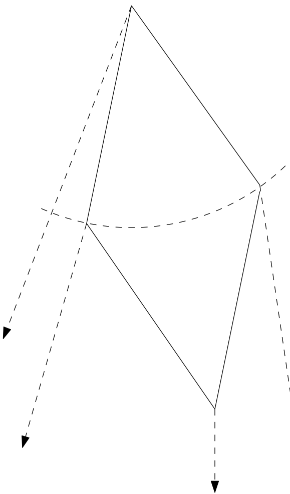
Consider the rays leaving the top side



Unit-Length PSLDs for Bipartite Outerplanar Graphs

A similar construction

Rotate them slightly to make three new wide wedges



Constructing an Embedding with $\rho > k$

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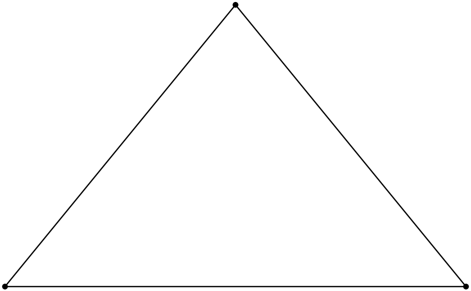
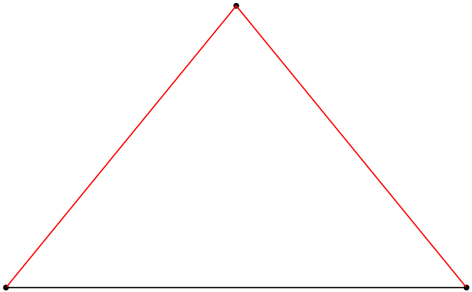


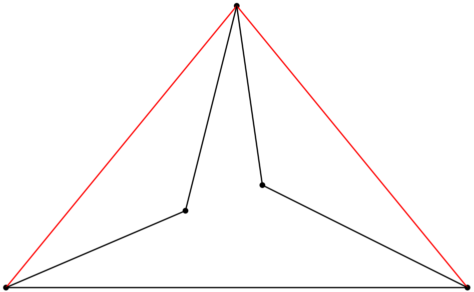
A triangle

Constructing an Embedding with $\rho > k$



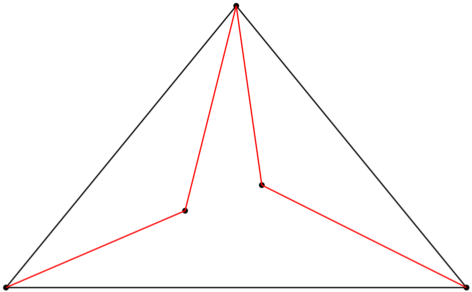
$\mathcal{G}_0$ with distinguished edges

Constructing an Embedding with $\rho > k$



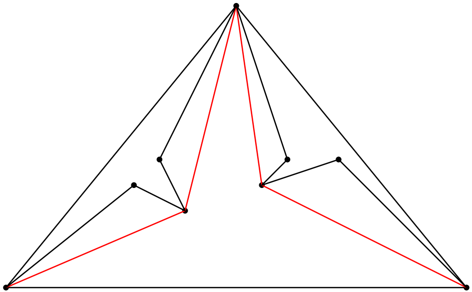
Add 2 triangles to each distinguished edge of $\mathcal{G}_0$

Constructing an Embedding with $\rho > k$



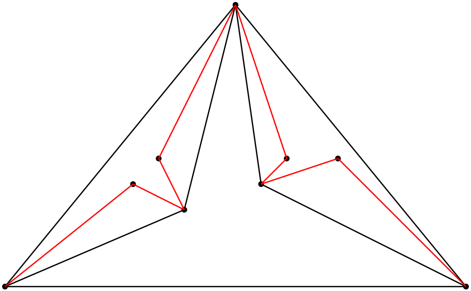
$\mathcal{G}_1$ with distinguished edges

Constructing an Embedding with $\rho > k$



Add 2 triangles to each distinguished edge of $\mathcal{G}_1$

Constructing an Embedding with $\rho > k$



$\mathcal{G}_2$ with distinguished edges

Outline

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- Given a constant $c < 2$, what is the complexity of deciding whether an outerplanar graph has edge-length ratio at most c?