Hardness of Staircase Guarding

Therese Biedl [1] Saeed Mehrabi [2]

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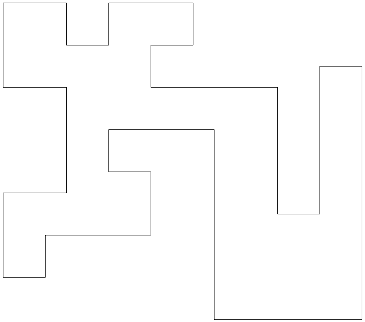
Staircase Guarding (s-guarding)



- Given: orthogonal polygon P

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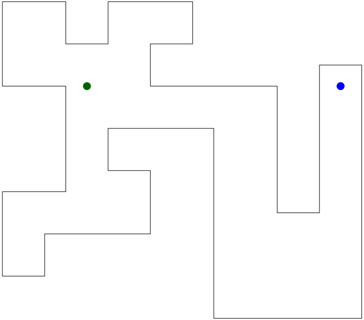

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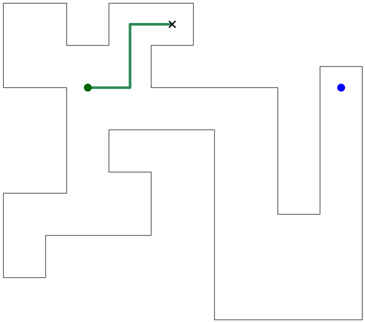

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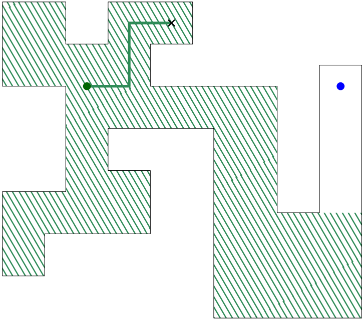

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  - staircase: xy-monotone

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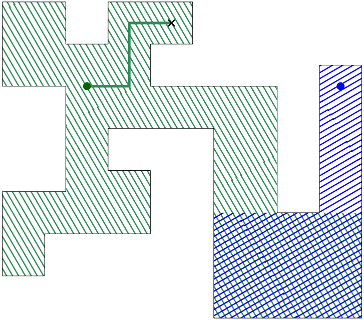

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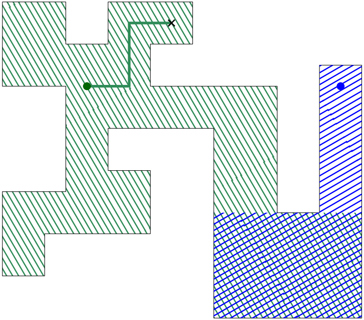

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  - "Given polygon P, guard with few guards"
  - Introduced by Klee and Chvátal in 1973
  - Many variations of polygons and guards studied, usually they are NP-hard.

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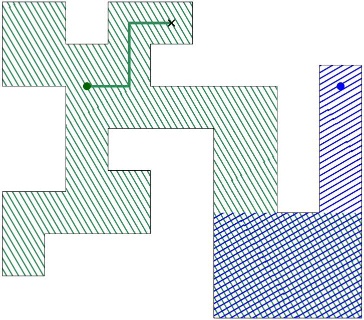

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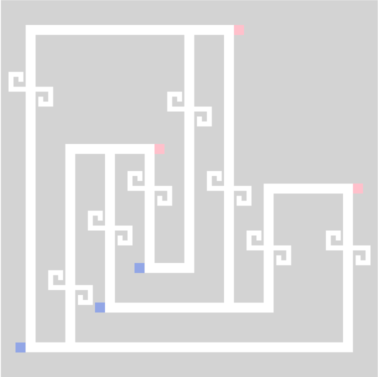

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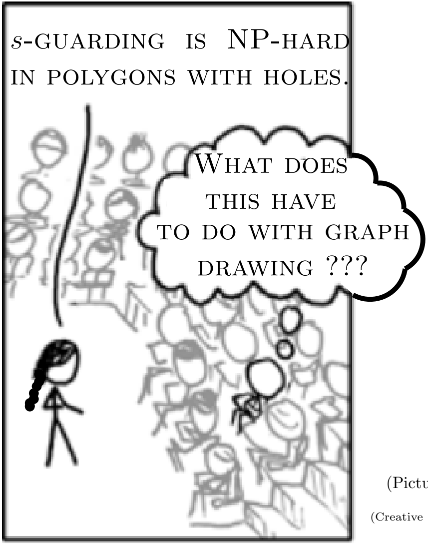
  - Gwali & Naftos 1992: NP-hard in 3D.
- **This talk:** It's NP-hard in 2D (in polygons with holes).



(Picture shamelessly stolen from xkcd.com.)

Grid-obstacle representations of graphs

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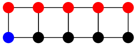
Obstacle representations (Alpert et al. 2010)



- Given: graph $G = (V, E)$

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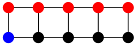

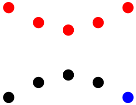

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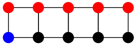

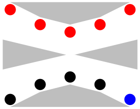

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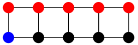

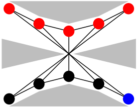

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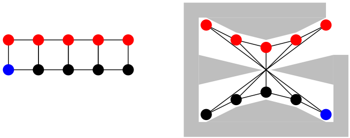

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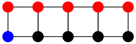

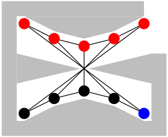

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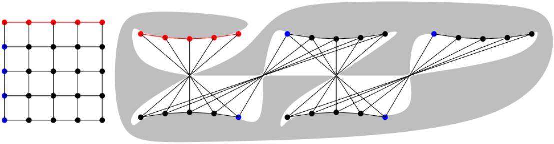
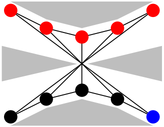


(done by Fabrizio Frati, as mentioned by Dujmovic and Morin.)

- Can always do it with $O(n^2)$ obstacles.
- Various results on how many obstacles required/enough.



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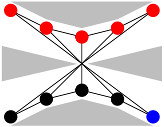

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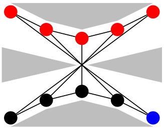

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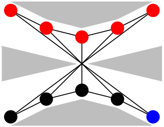
Grid-obstacle representations (Bishnu et al., 2015)



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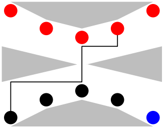

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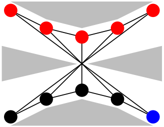

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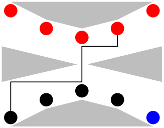

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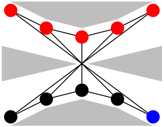

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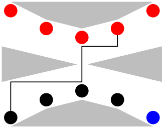

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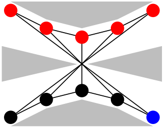

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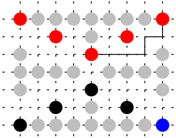

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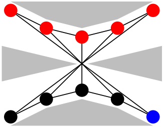

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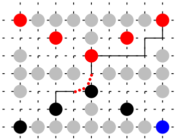

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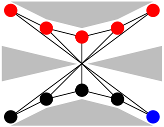

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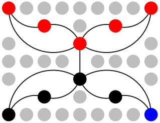

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Completely different model, results don't transfer.

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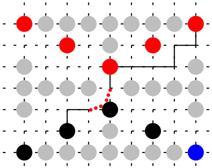

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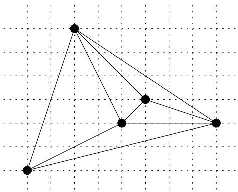

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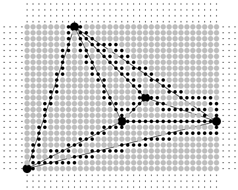

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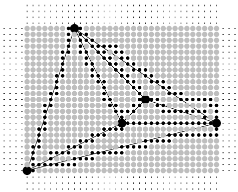

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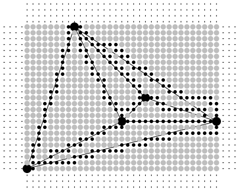

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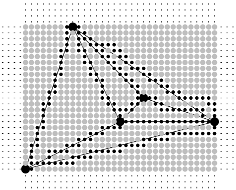

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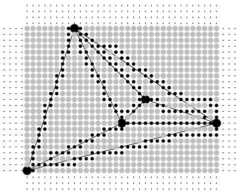

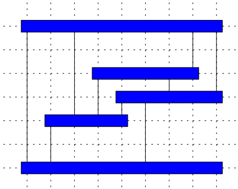

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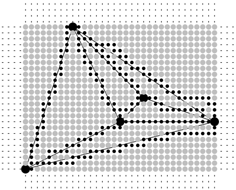

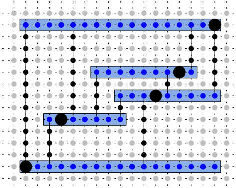

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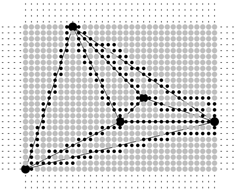
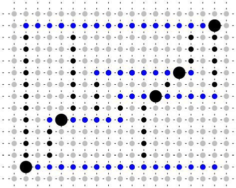
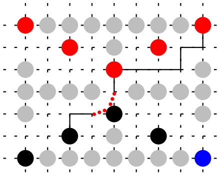
  - All graphs in 3D: $O(n^3)$ volume





- Bishnu et al. (see also poster):
  - Exists for all planar graphs in 2D.
  - Does not exist for all graphs in 2D.
  - Exists for all graphs in 3D.
- New objective: how big is the grid?
- Bishnu et al.:
  - Planar graphs in 2D: $O(n^8)$ area
  - All graphs in 3D: $O(n^{22})$ volume
- **This paper:**
  - Planar graphs in 2D: $O(n^2)$ area
  - All graphs in 3D: $O(n^3)$ volume

Non-blocking grid-obstacle representations



- Recall special rule 2: Vertex-points block grid-paths.

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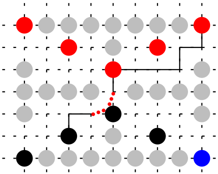

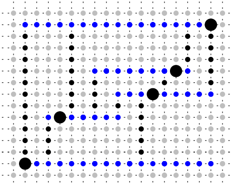

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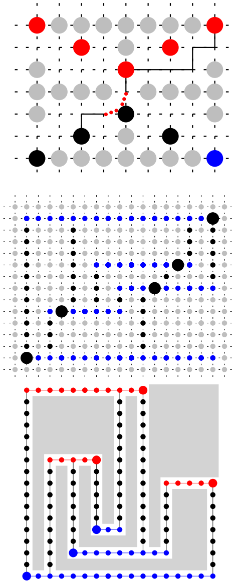

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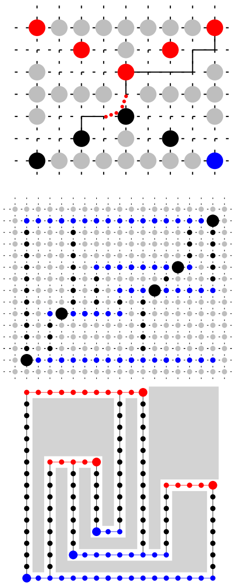
Non-blocking grid-obstacle representations



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Open Problem

Does every planar graph have a non-blocking grid-obstacle representation?

Non-blocking grid-obstacle representations

Theorem

Every planar bipartite graph $G = (A \cup B, E)$ has a non-blocking grid-obstacle representation.

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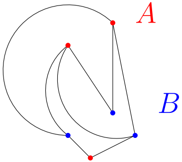

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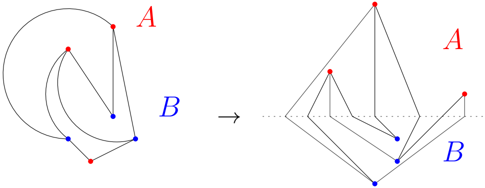

- Create *HH-drawing* [B., Kaufmann, Mutzel, 1998]
  - Vertices of A above x-axis
  - Vertices of B below x-axis
  - Edges have one bend on x-axis

Non-blocking grid-obstacle representations

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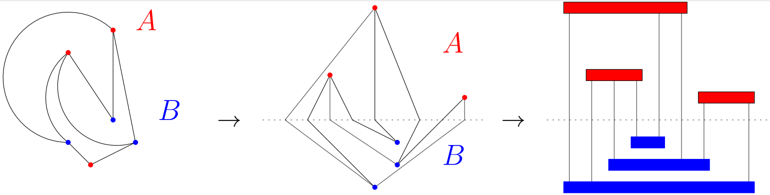

- Create *HH-drawing* [B., Kaufmann, Mutzel, 1998]
- Turn into visibility representation [B., GD'14]
  - Feasible since edges y-monotone
  - All x-coordinates unchanged

Non-blocking grid-obstacle representations

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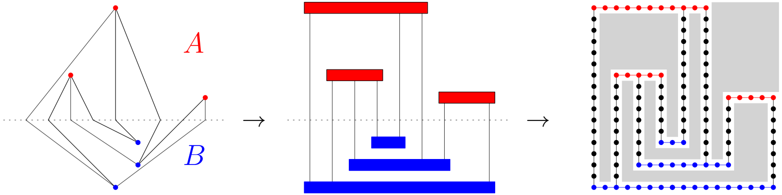

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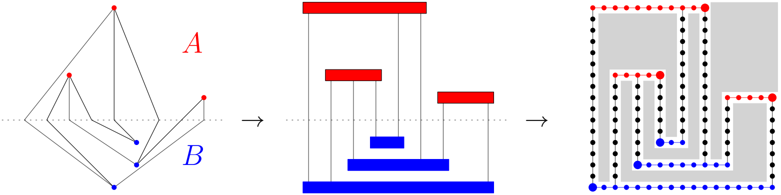

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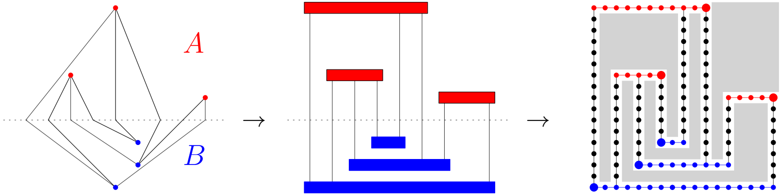
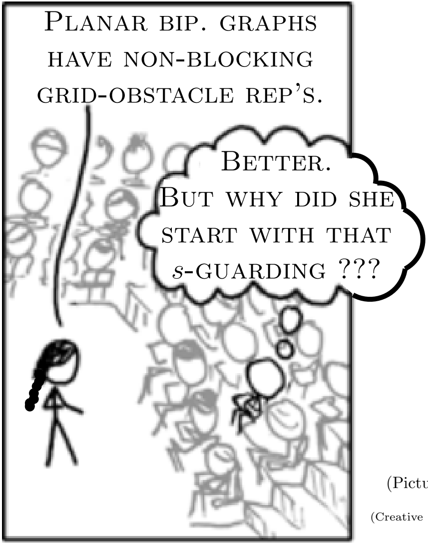


- Create *HH-drawing* [B., Kaufmann, Mutzel, 1998]
- Turn into visibility representation [B., GD'14]
- Interpret as grid, fill complement with obstacles.
- Point for v: rightmost / leftmost point in box of v
- Argue: $(v, w) \in E \Leftrightarrow xy$-monotone grid-path



(Picture shamelessly stolen from xkcd.com.)

Grid-obstacle representations with connections to staircase guarding

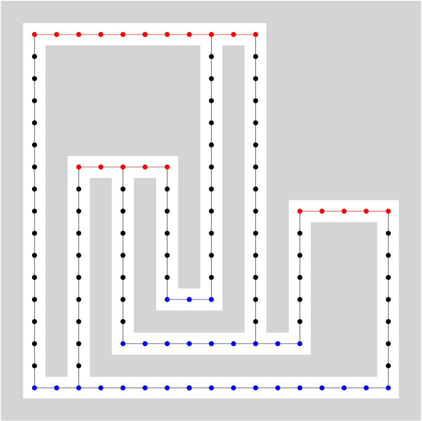
Therese Biedl [1]   Saeed Mehrabi [2]

[1]University of Waterloo, *biedl@uwaterloo.ca*

[2]was at UW, now at Carleton University, *mehrabi235@gmail.com*

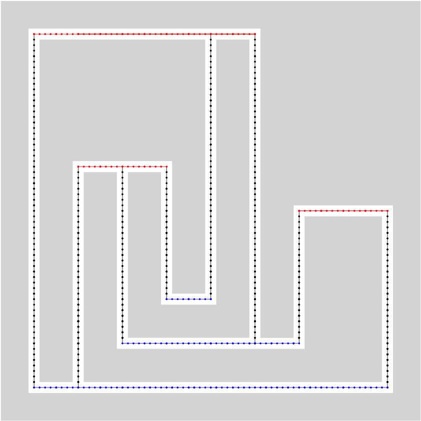
September 25, 2017

From obstacle-representation to hardness



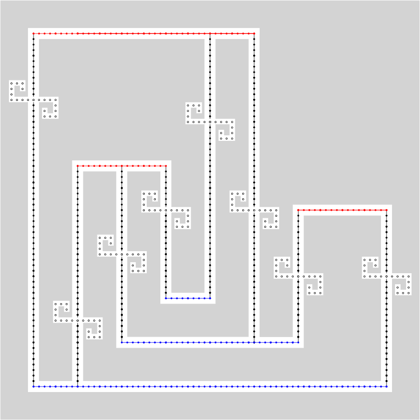
- Use non-blocking grid-obstacle representation of planar bipartite graph.

From obstacle-representation to hardness



- Use non-blocking grid-obstacle representation of planar bipartite graph.
- Refine grid.

From obstacle-representation to hardness



- Use non-blocking grid-obstacle representation of planar bipartite graph.
- Refine grid.
- Add *swirl* at every edge.

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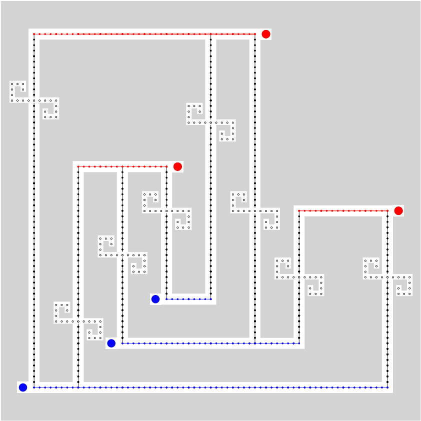

- Use non-blocking grid-obstacle representation of planar bipartite graph.
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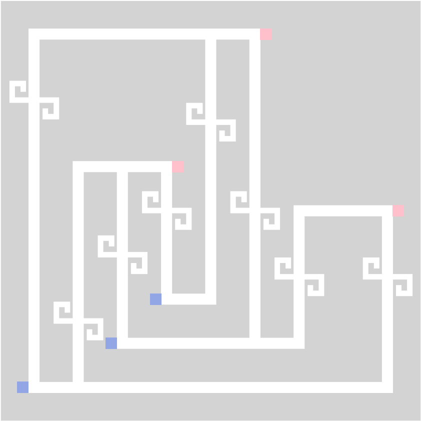
- Add *swirl* at every edge.
- Add *vertex-squares*.

From obstacle-representation to hardness



- Use non-blocking grid-obstacle representation of planar bipartite graph.
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- Add *swirl* at every edge.
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- Forget grid $\Rightarrow$ polygon.

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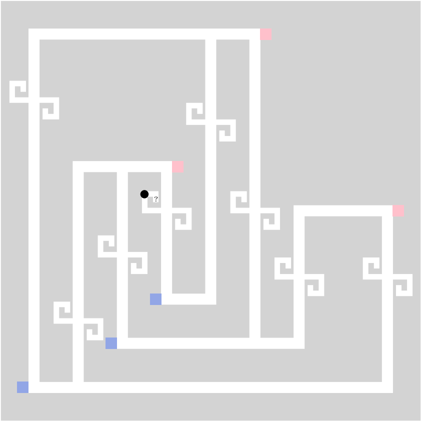

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    - Need guard on left swirl-part.

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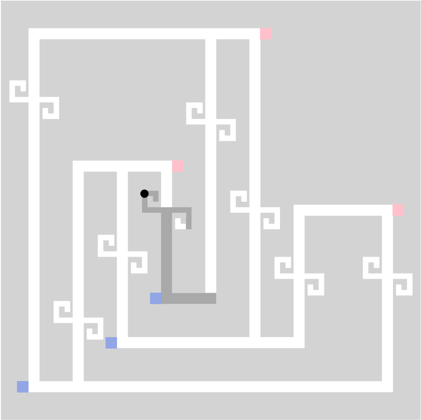

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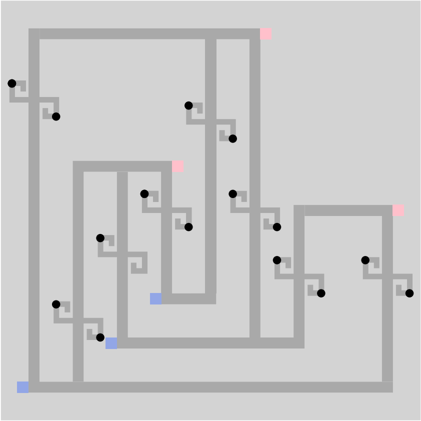

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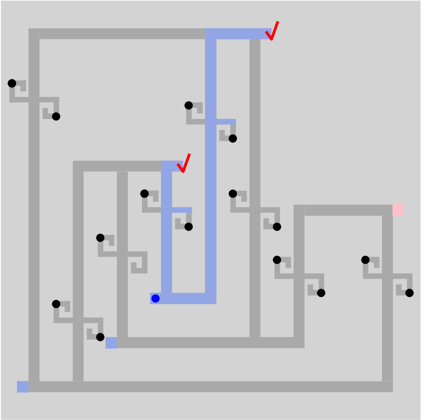

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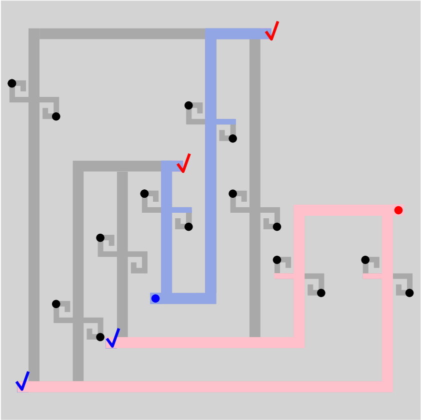

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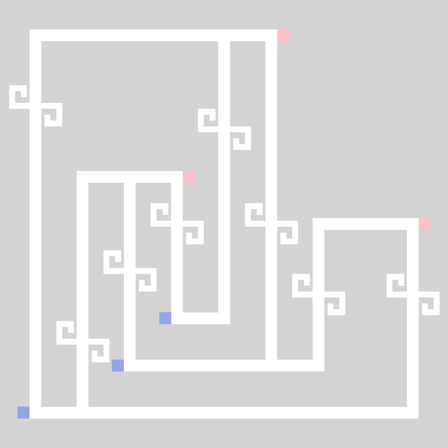
- Guard at v sees all squares of all neighbours of v.

$\Rightarrow$ $2m + k$ guards suffice $\Leftrightarrow$ G has dominating set of size k.

From obstacle-representation to hardness

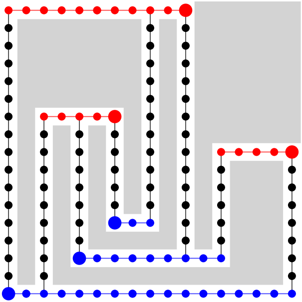
Dominating set is NP-hard in planar bipartite graphs $\Longrightarrow$



Theorem

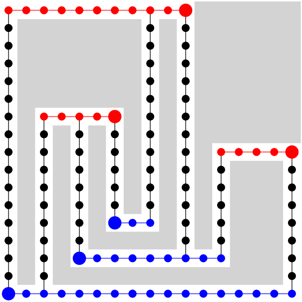
Staircase-guarding is NP-hard in orthogonal polygons with holes.

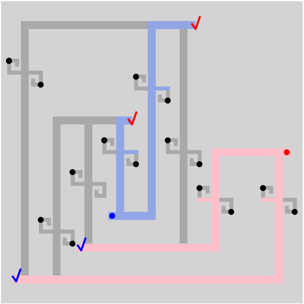
Conclusion



- Some improvements to grid-obstacle representations.

Conclusion





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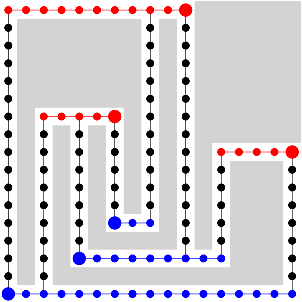

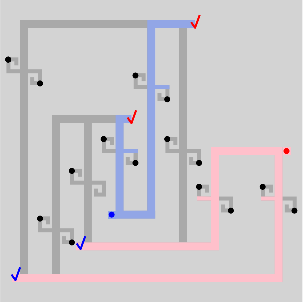

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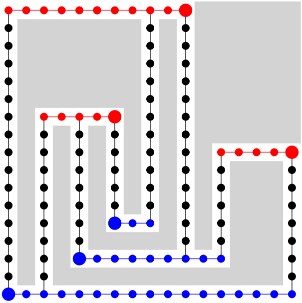

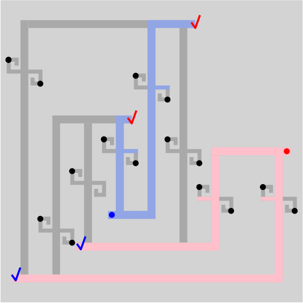

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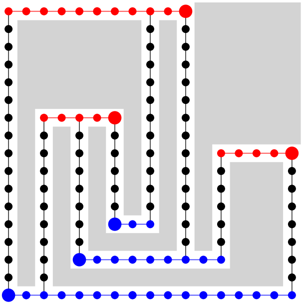
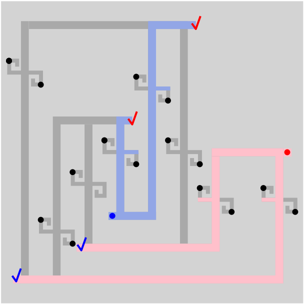
Conclusion





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  - Representation would have to "not look like planar drawing"
- Open: Could we do $o(n)$ obstacles in 2D?

Conclusion