3D Visibility Representations of 1-planar Graphs

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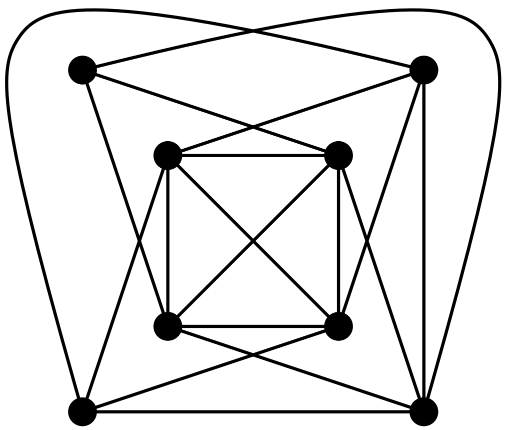
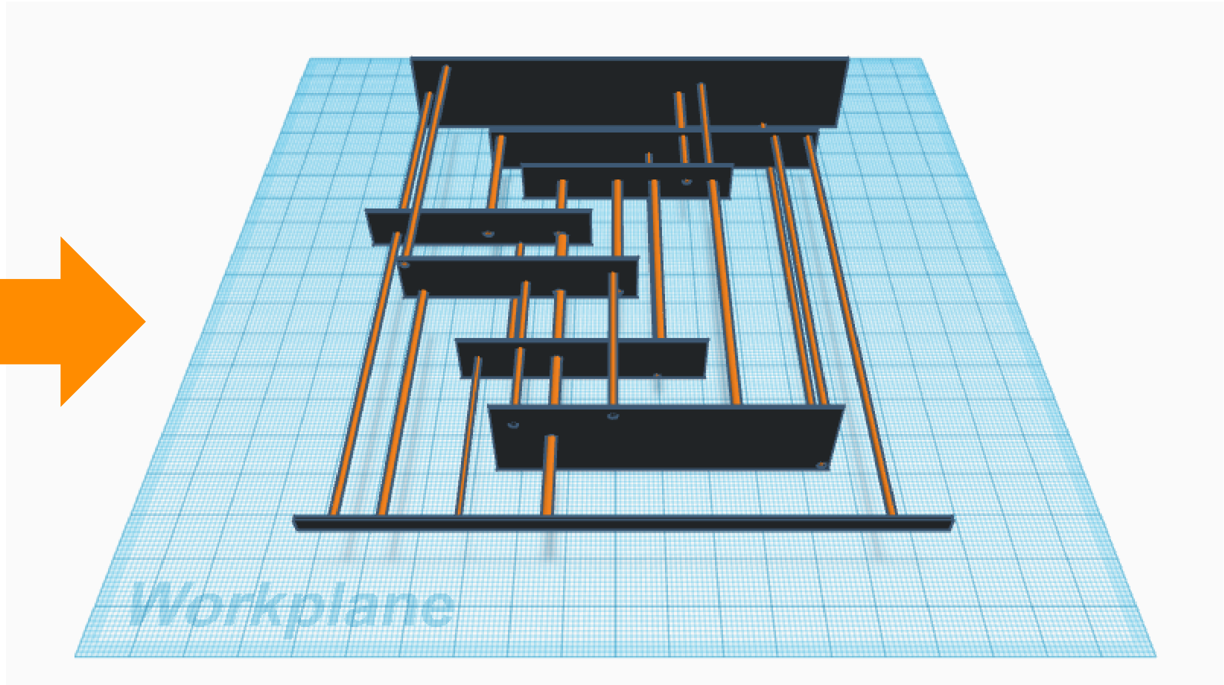
In this presentation...

We will show that every 1-planar graph can be realized as a visibility representation of parallel rectangles in 3D





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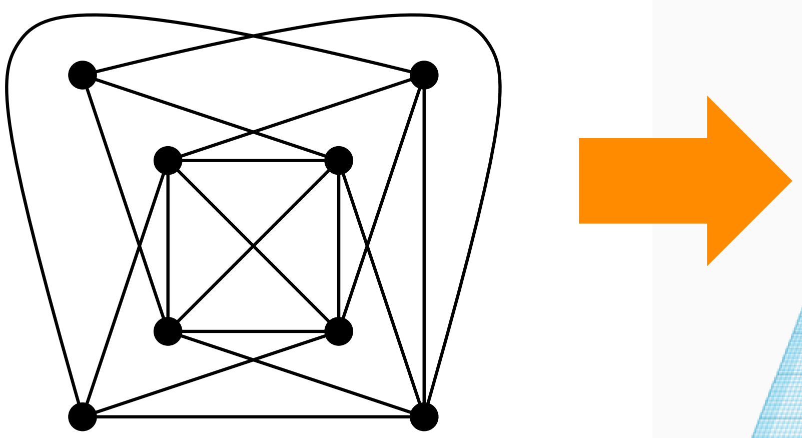
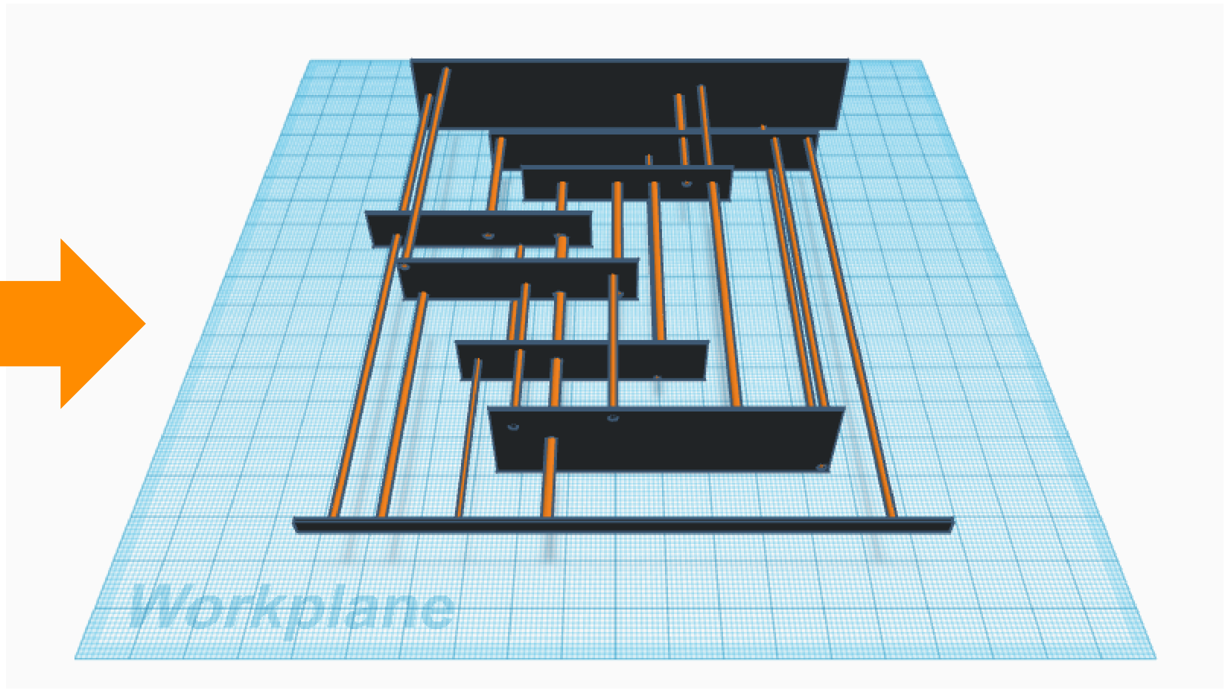


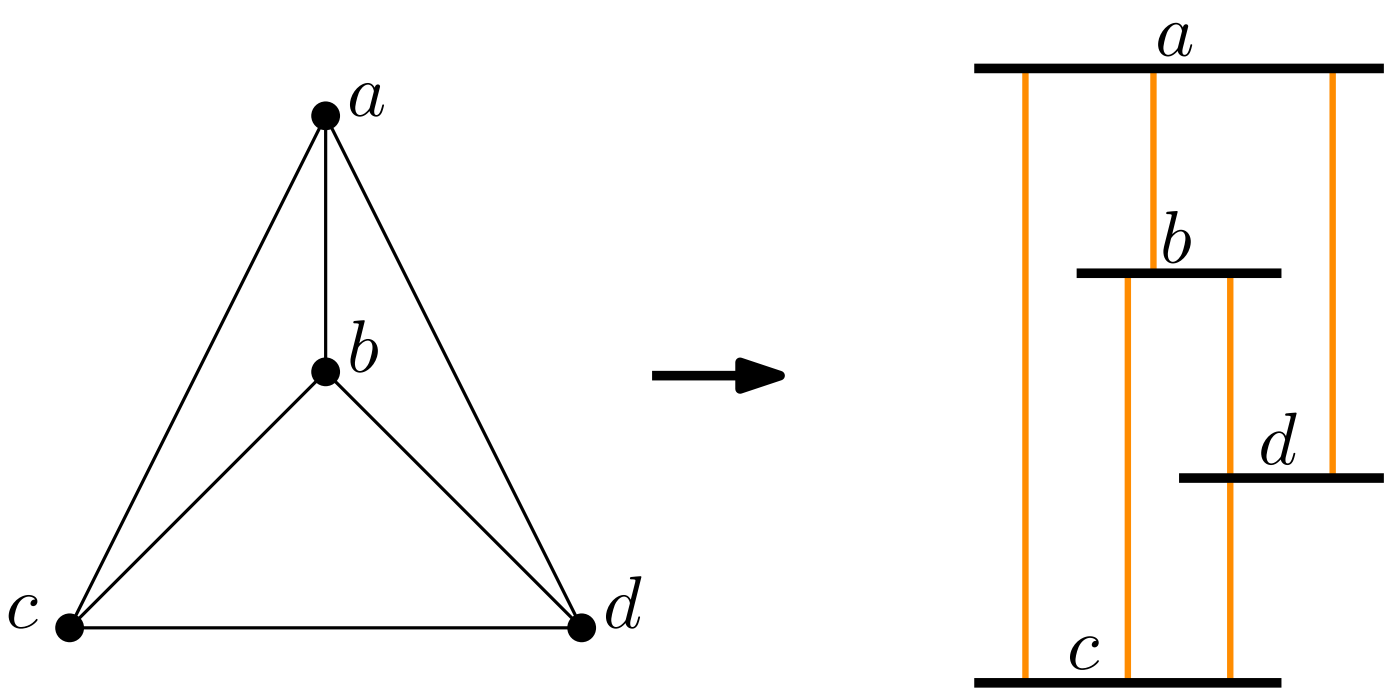


2D Planar Visibility Representations

Bar Visibility Representation (BVR) of a planar graph G:

Vertices $\rightarrow$ Horizontal bars

Edges $\rightarrow$ Vertical unobstructed visibilities

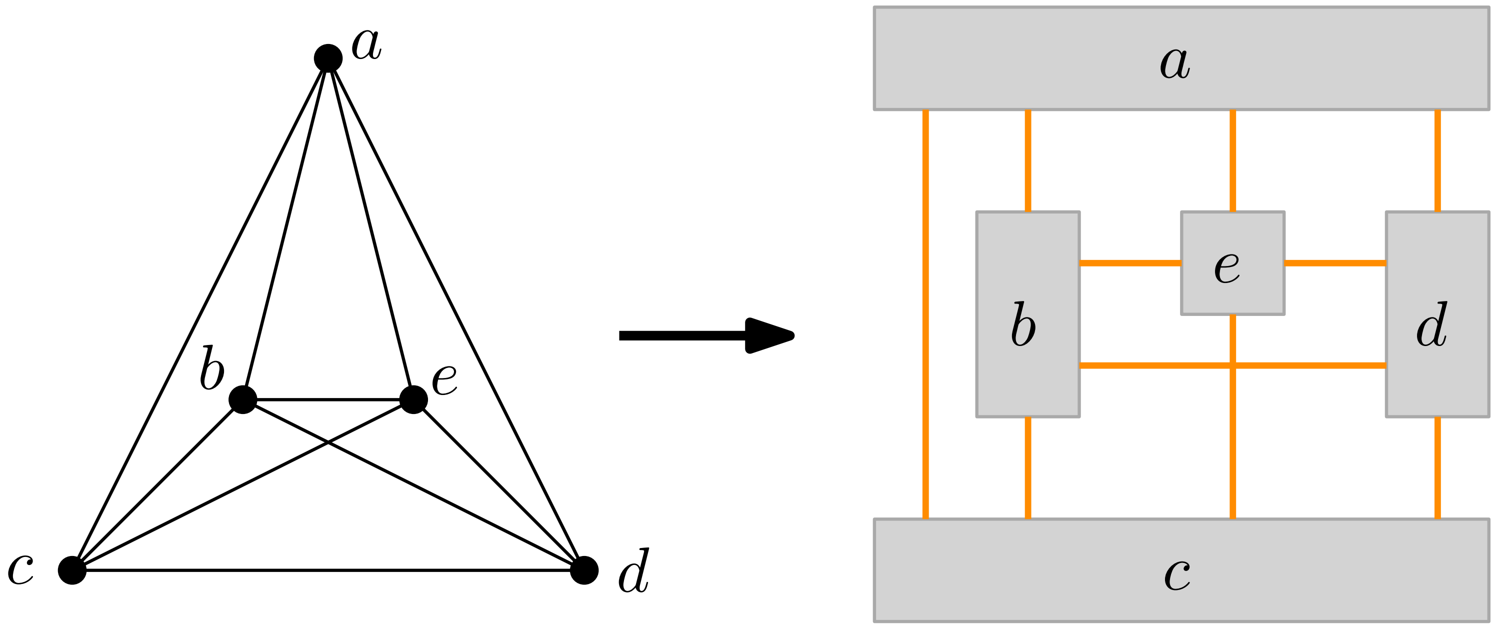


Every planar graph admits a (weak) BVR

[Duchet et al. 1983, Thomassen 1984, Wismath 1985, Rosenthiel & Tarjan 1986, Tamassia & Tollis 1986]

2D Nonplanar Visibility Representations

Rectangle Visibility Representation (RVR) of a graph G:
Vertices $\rightarrow$ Axis-aligned rectangles
Edges $\rightarrow$ Horizontal/Vertical unobstructed visibilities



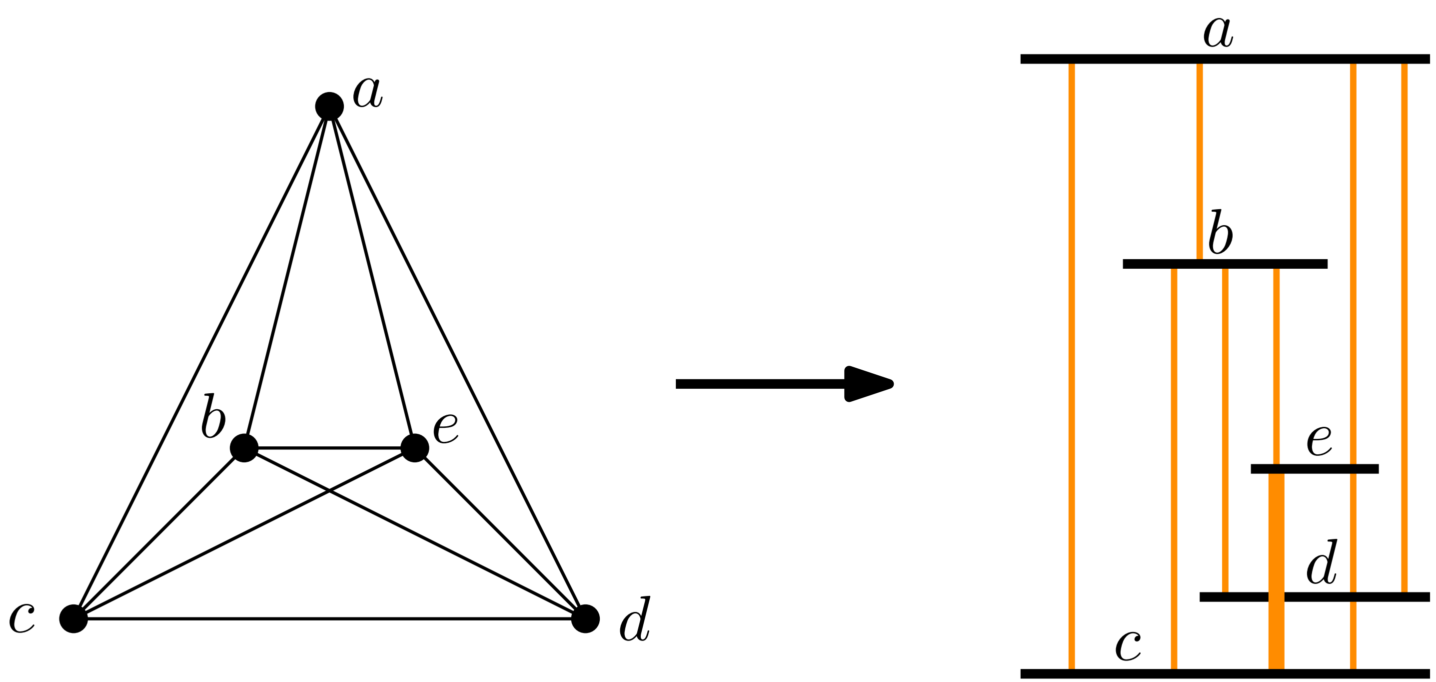
An n-vertex graph that admits an RVR has thickness at most two and at most $6n - 20$ edges [Hutchinson et al. 1999]

2D Nonplanar Visibility Representations

Bar k-Visibility Representation (BkVR) of a graph G:

Vertices $\rightarrow$ Horizontal bars

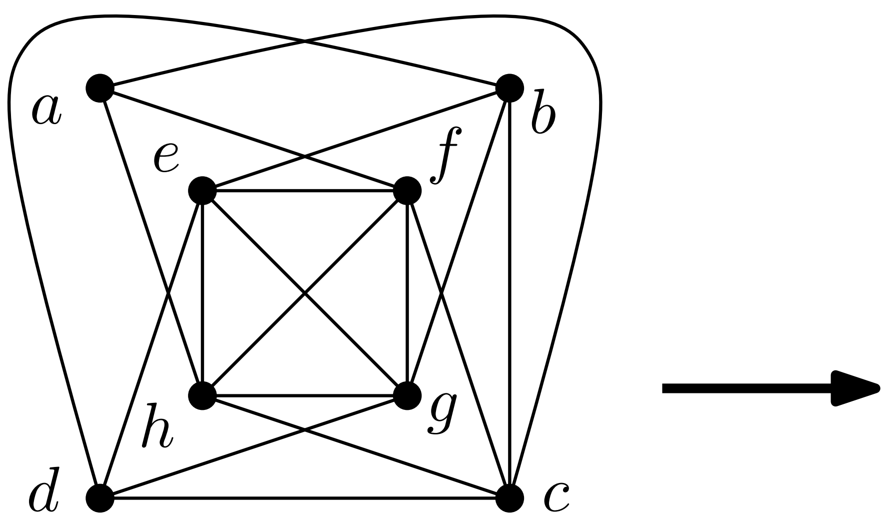
Edges $\rightarrow$ Vertical visibilities that can traverse at most k bars

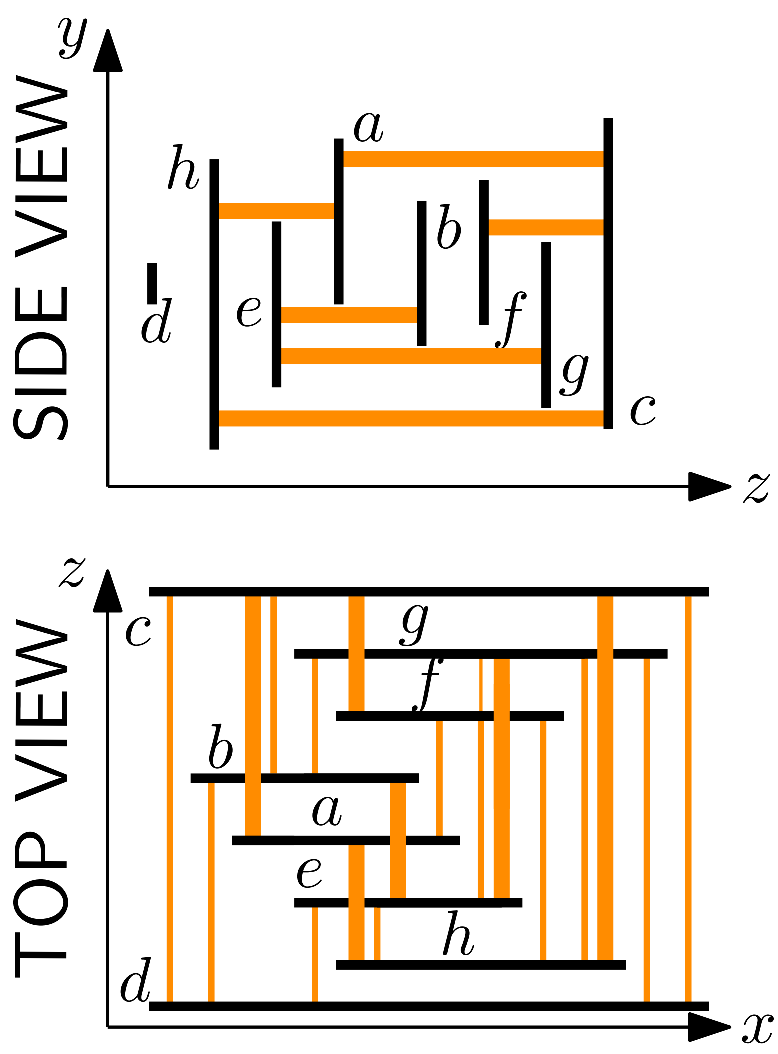


An n-vertex graph that admits a BkVR has thickness $O(k^2)$ [Dean et al. 2007] and $O(kn)$ edges [Dean et al. 2007, Hartke et al. 2007].

3D Visibility Representations

z-parallel Visibility Representation (ZPR) of a graph G:
Vertices $\rightarrow$ Rectangles with sides parallel to x- and y-axis
Edges $\rightarrow$ Unobstructed visibilities parallel to z-axis





K_{22} admits a ZPR [Bose et al. 1998]
while K_{51} does not admit any
ZPR [Štola 2009]

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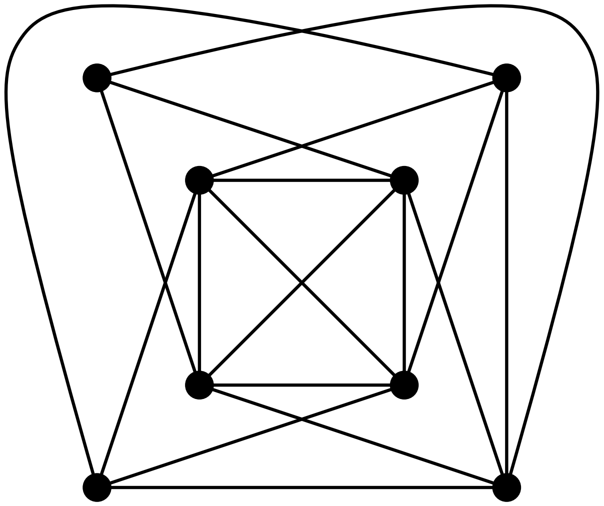
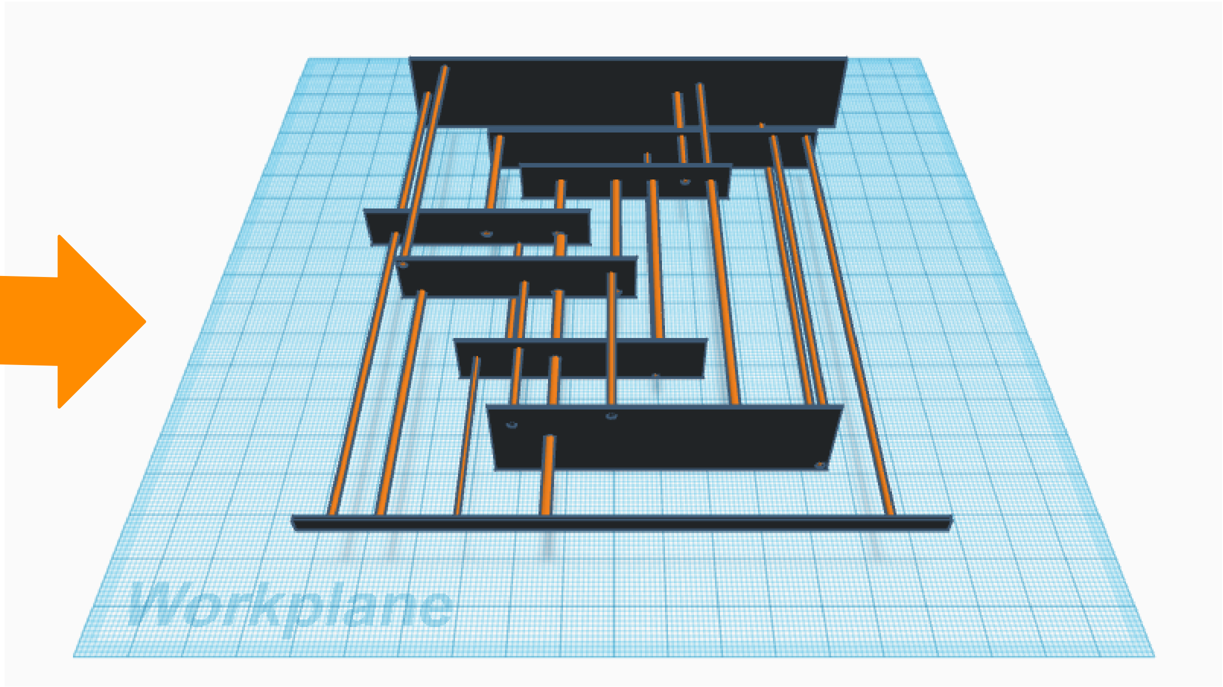
We will show that every 1-planar graph can be realized as a z-**parallel visibility representation**





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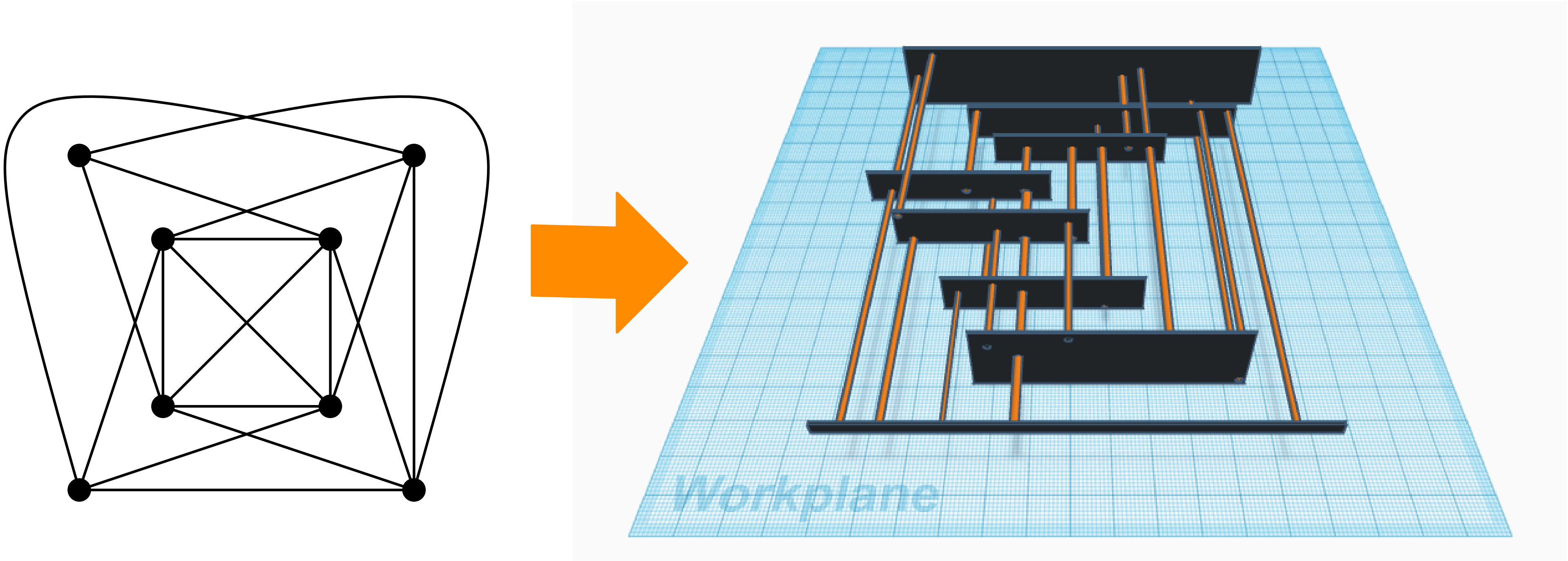
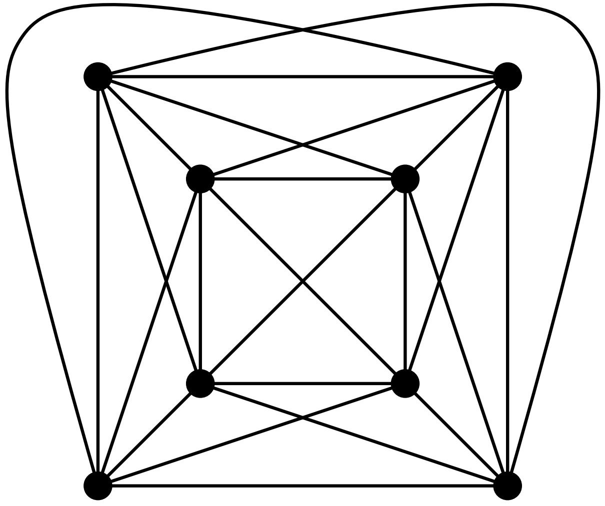
We will show that every **1-planar graph** can be realized as a z-parallel visibility representation



1-planar Graphs

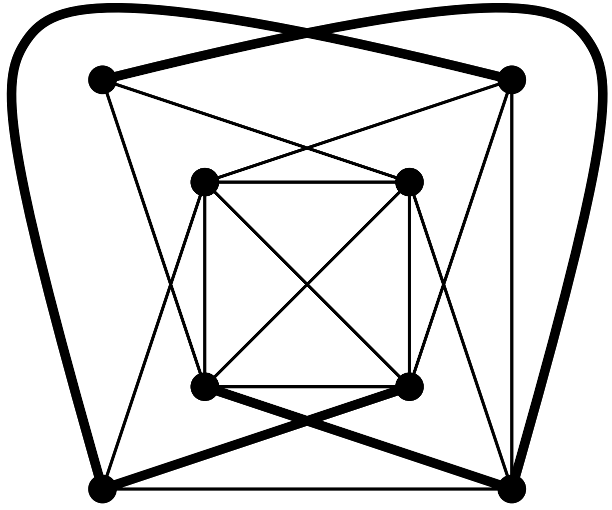
A graph is 1-planar if it can be drawn with at most one crossing per edge

A 1-planar graph has at most $4n-8$ edges (tight) [Bodendiek et al. 1983; Pach and Tóth 1997]



1-planar Graphs and Visibility Representations

There are 1-planar graphs that do not admit any rectangle visibility representation [Biedl, Liotta, M. 2016]



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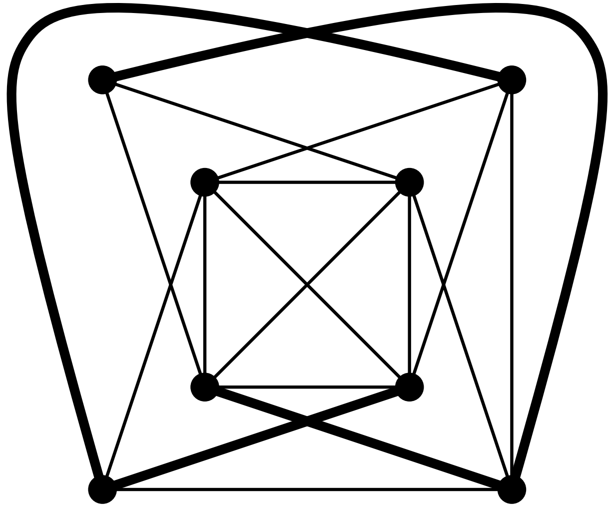
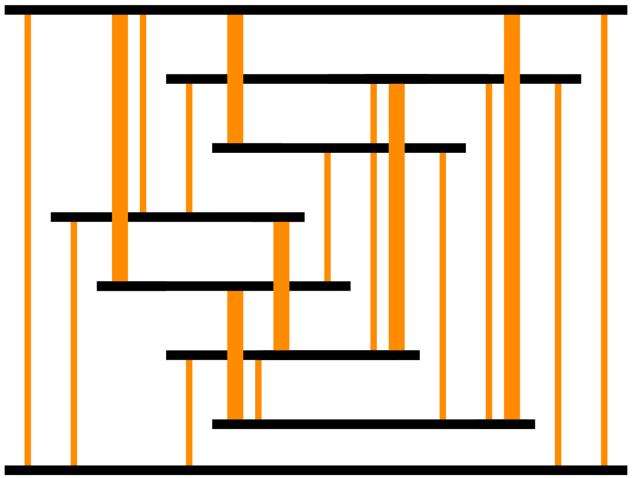
Every 1-planar graph admits a bar 1-visibility representation [Brandenburg 2014 & Evans et al. 2014]





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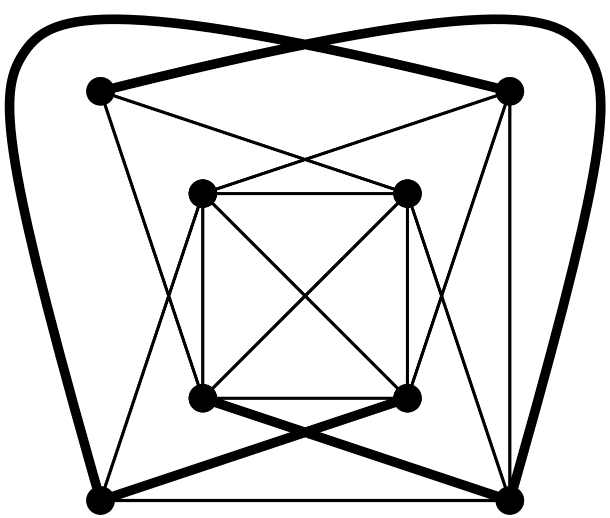
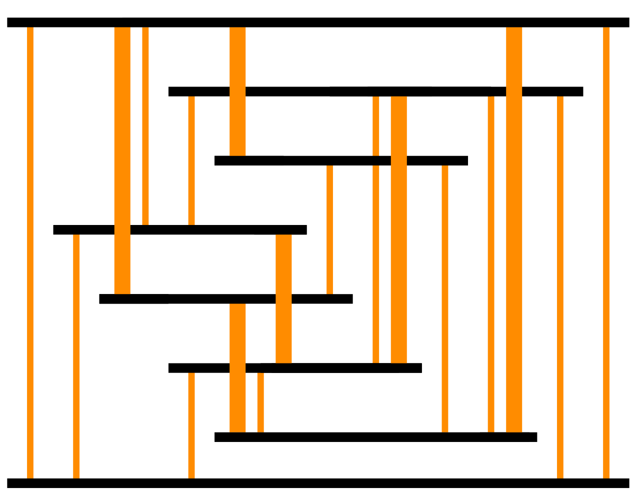
There are 1-planar graphs that do not admit any rectangle visibility representation [Biedl, Liotta, M. 2016]

Every 1-planar graph admits a bar 1-visibility representation [Brandenburg 2014 & Evans et al. 2014]

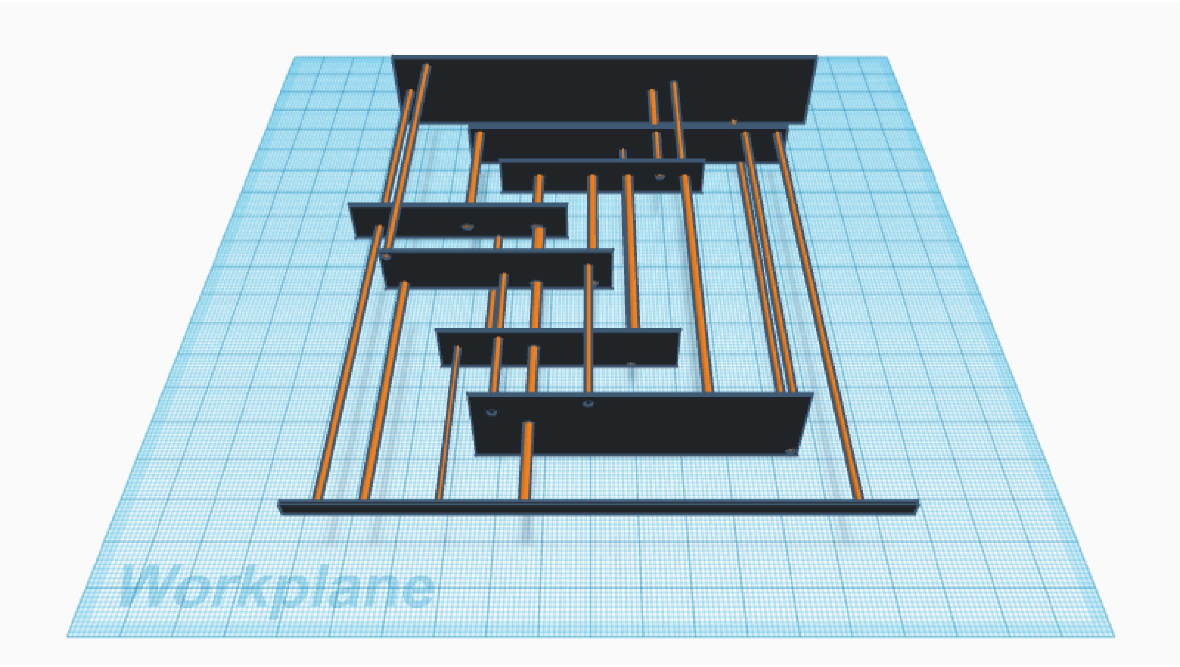
Question: Can we realize every 1-planar graph as a visibility representation of rectangles with unobstructed visibilities by exploiting the 3rd dimension?





In this presentation...

Theorem 1 *Every 1-planar graph G with n vertices admits a ZPR γ in $O(n^3)$ volume. Also, if a 1-planar embedding of G is given as part of the input, then γ can be computed in $O(n)$ time.*



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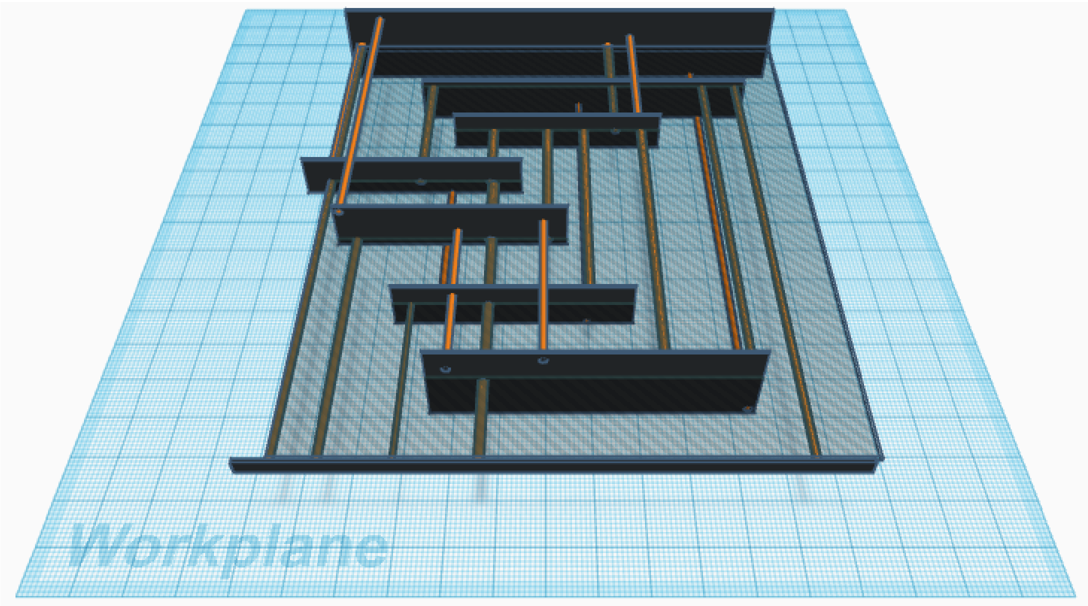
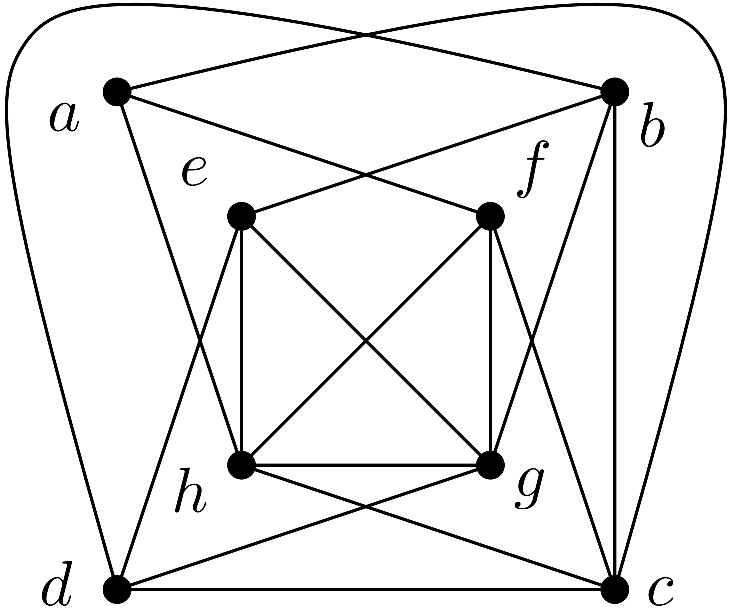
The ZPR γ is 1-visible:
$\exists$ a plane orthogonal to the rectangles of γ and whose intersection with γ is a B1VR



Proof overview

Input: 1-plane graph G



Proof overview

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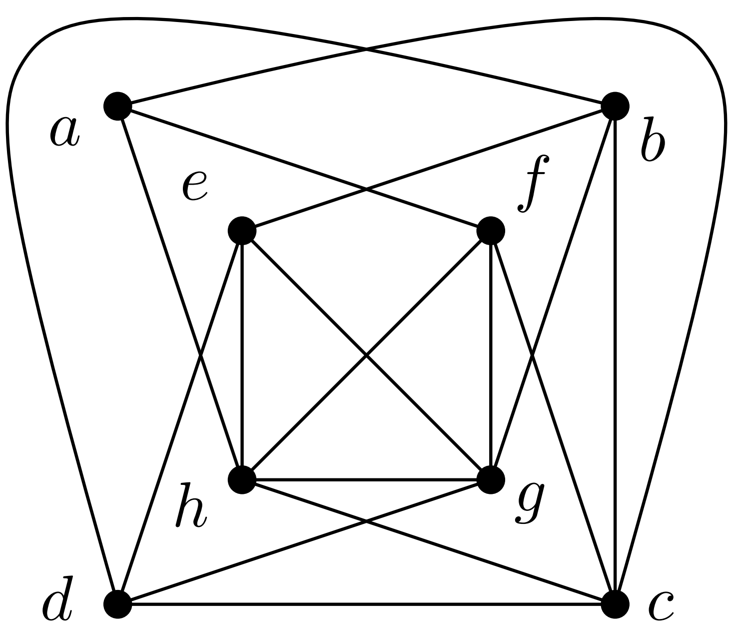
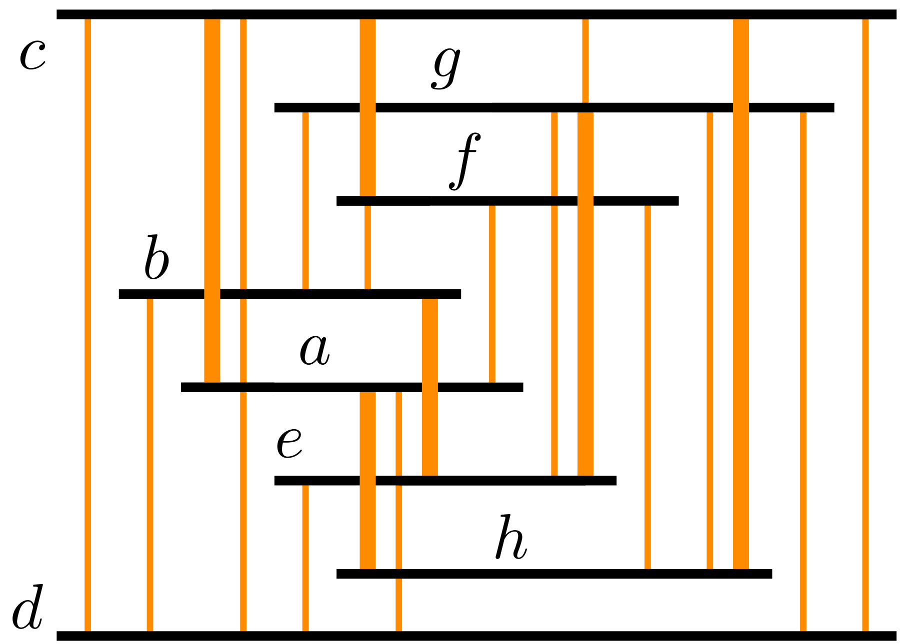
Step 1: Compute a B1VR γ_1



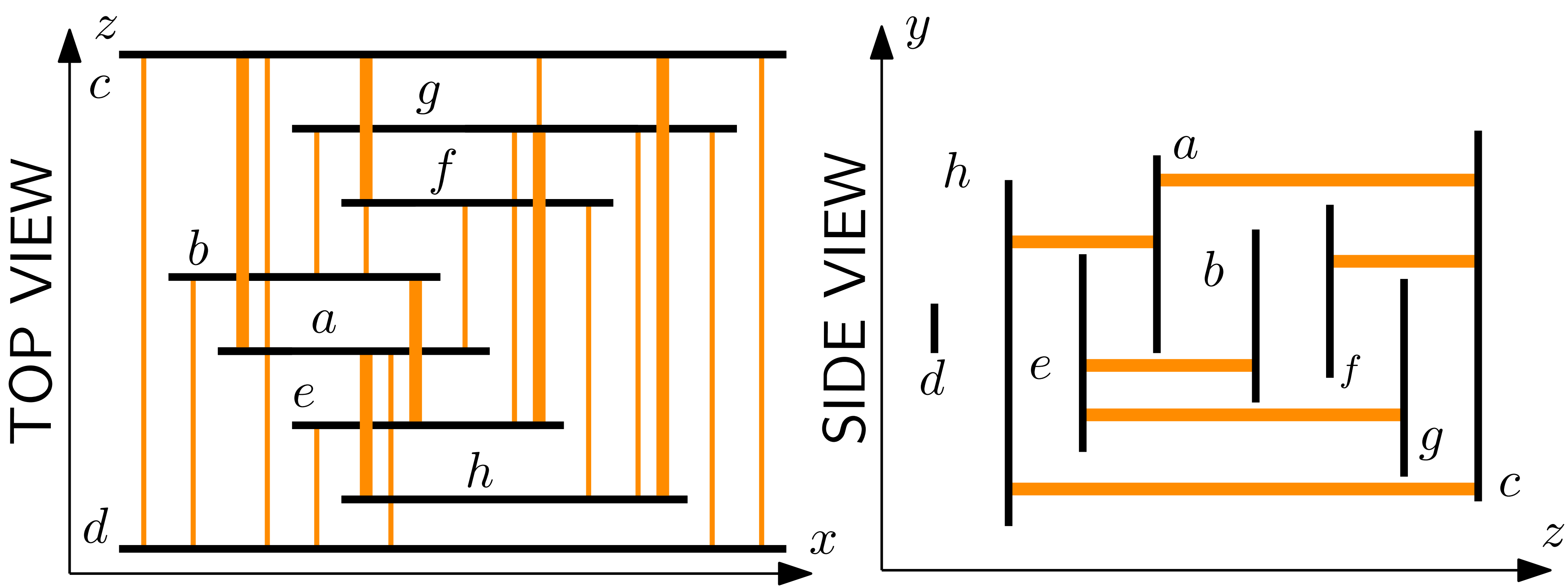


Proof overview

Input: 1-plane graph G

Step 1: Compute a B1VR γ_1

Step 2: Transform every bar into a rectangle by computing the y-coordinates of top and bottom sides, s.t. each visibility that traverses a bar in γ_1 can be moved upward or downard so to avoid the obstacle



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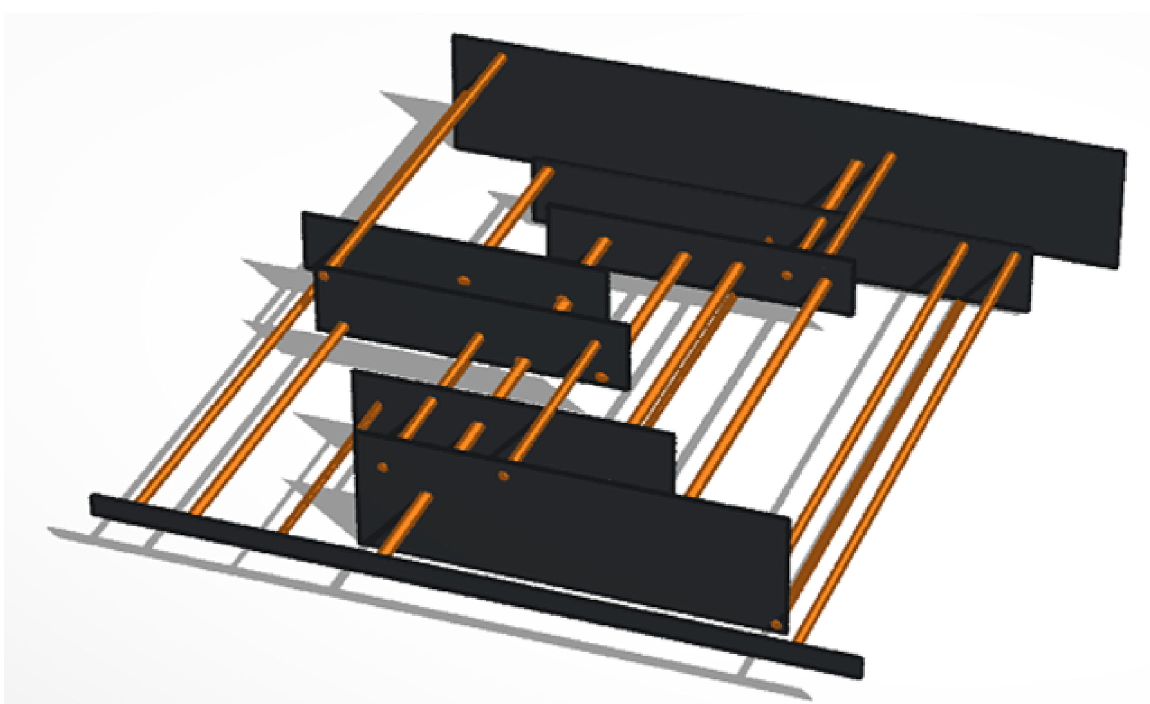
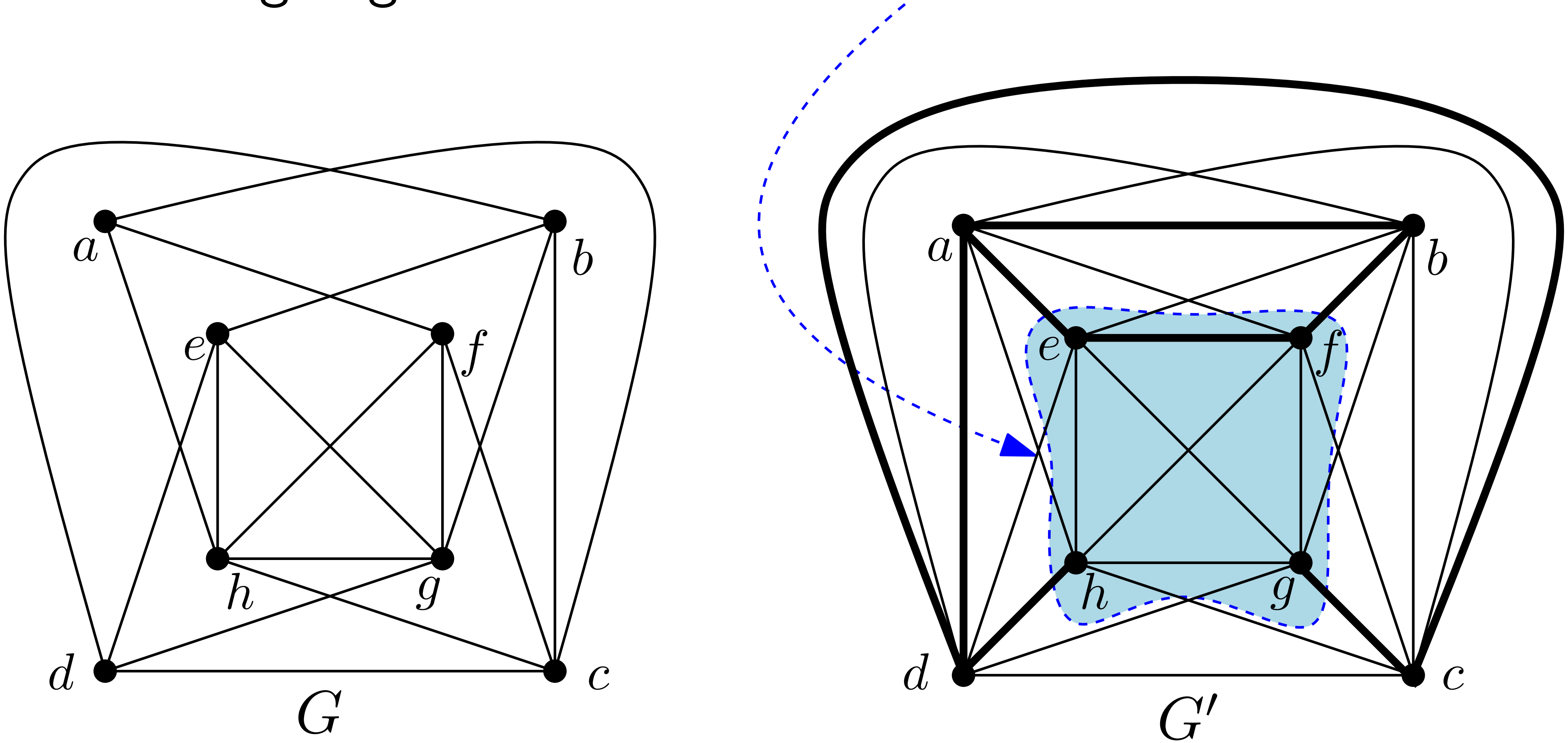
Output: 1-visibile ZPR γ of G



Step 1: B1VR

Step 1: We compute a B1VR γ_1 of G by applying Brandenburg's linear-time algorithm [Brandenburg 2014]

1.a A 1-plane multigraph $G' = (V, E' \supseteq E)$ is computed from G such that the four end-vertices of each pair of crossing edges of G' induce a kite



Step 1: B1VR

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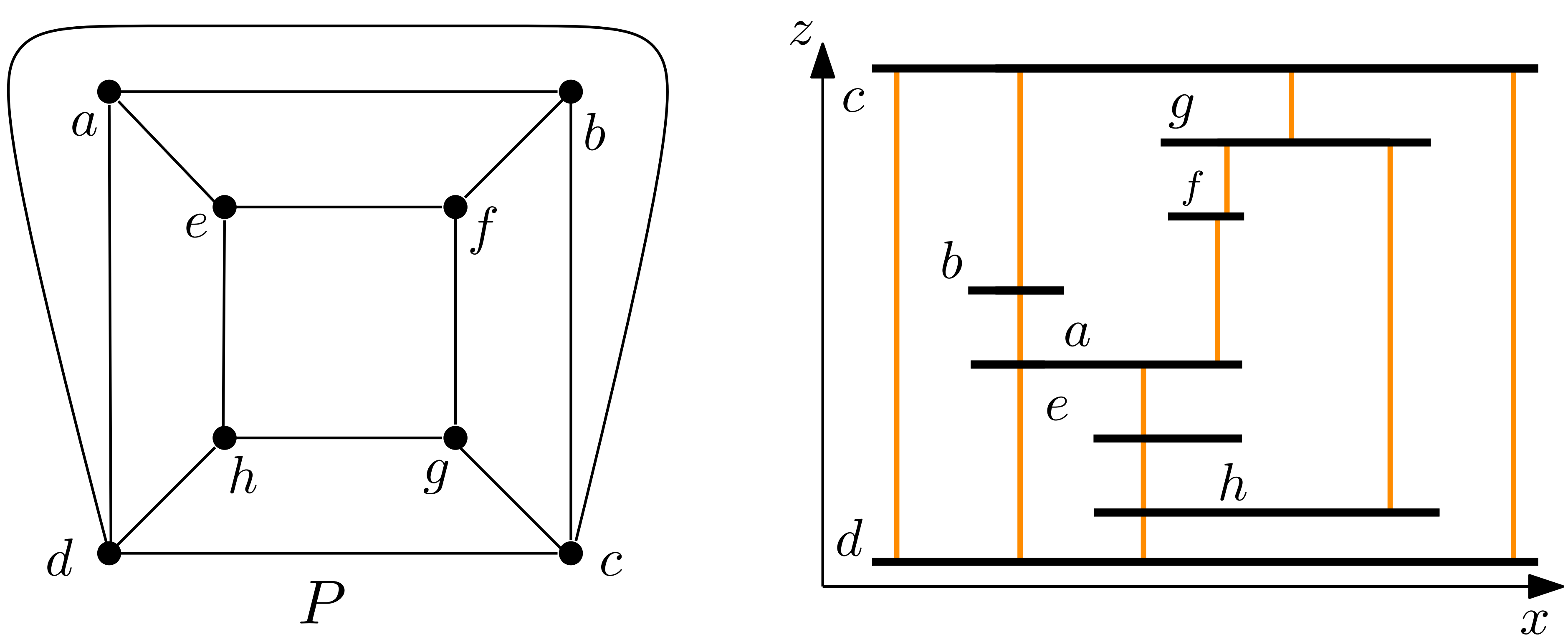
1.b Remove all pairs of crossing edges from G' and obtain a planar (multi)graph P
Apply the algorithm by Tamassia and Tollis [Tamassia-Tollis 1986] to compute a BVR of P



Step 1: B1VR

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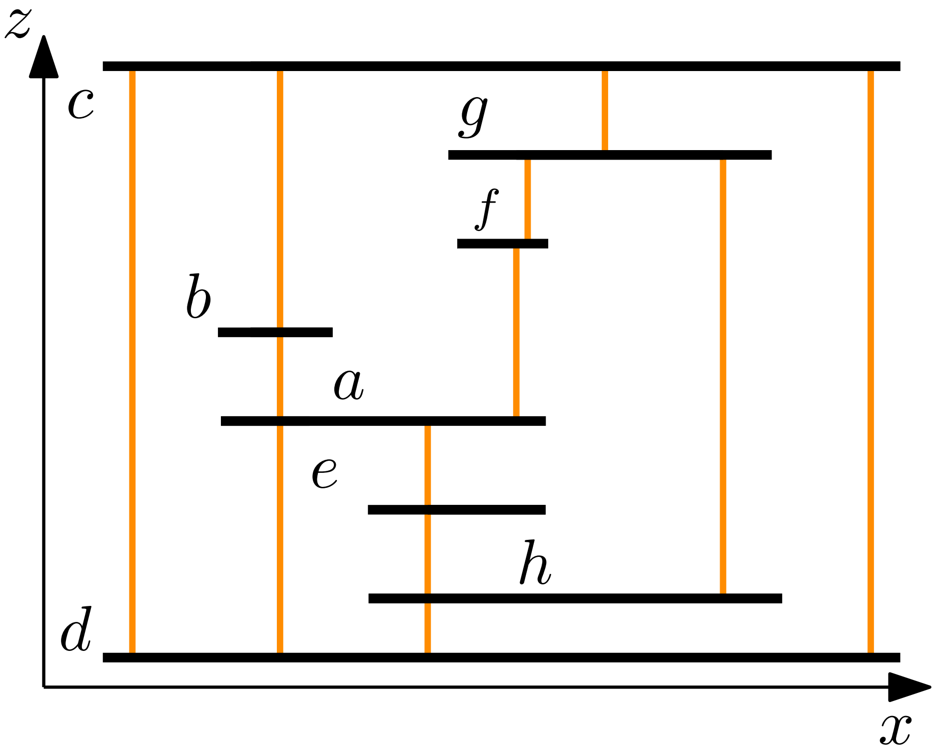
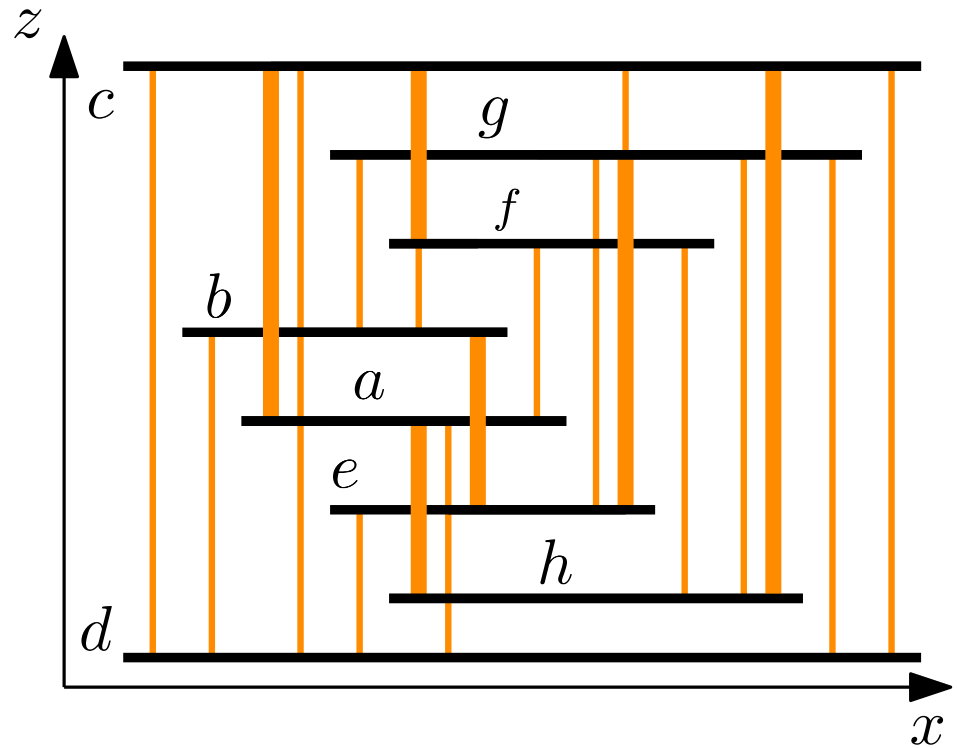
1.c Reinsert all pairs of crossing edges by extending some bars so to introduce new visibilities
The introduced visibilities traverse at most one bar each



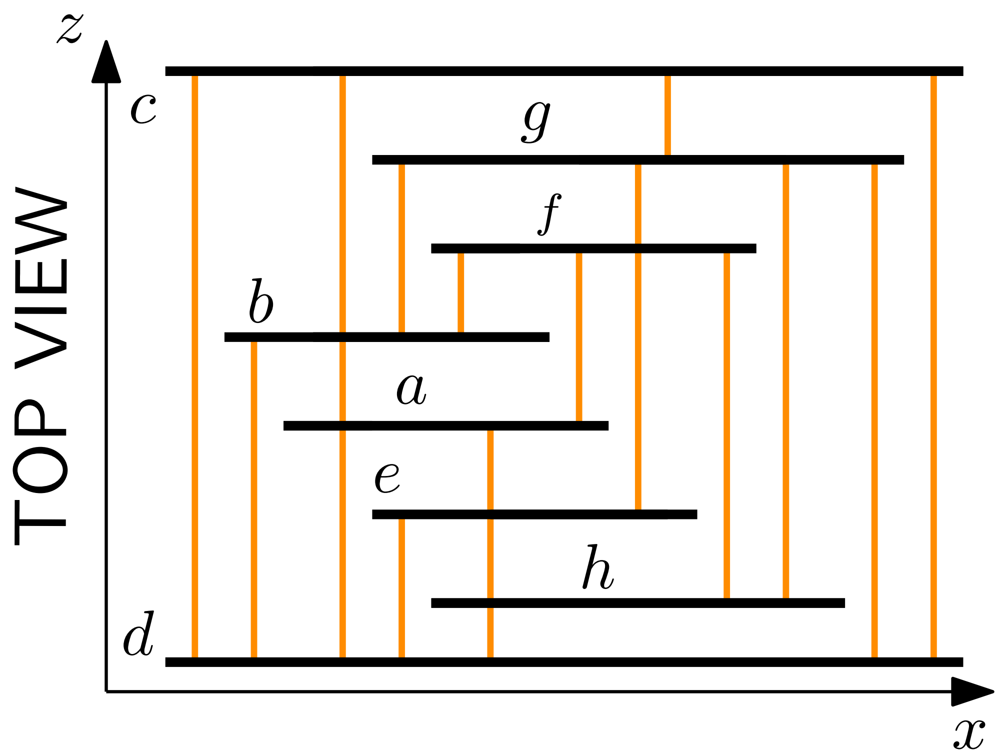


γ_1 Area: $O(n) \times O(n)$

Step 2: 1-visible ZPR

Step 2: Transform the B1VR γ_1 into the ZPR γ

Note: All the visibilities of γ_1 that do not traverse any bar does not need to be moved



γ_1

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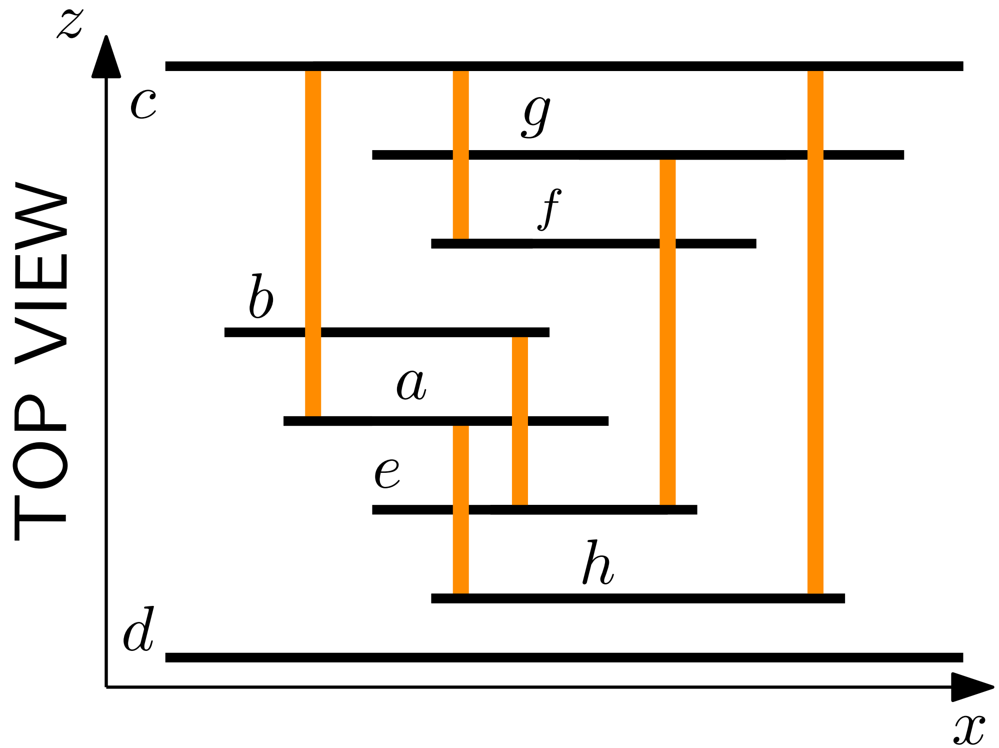
Idea: To realize the other visibilities we set the y-coordinates of the rectangles by using two orientations of (a subset of) the edges of P, called D_1 - for the top sides - and D_2 - for the bottom sides



γ_1

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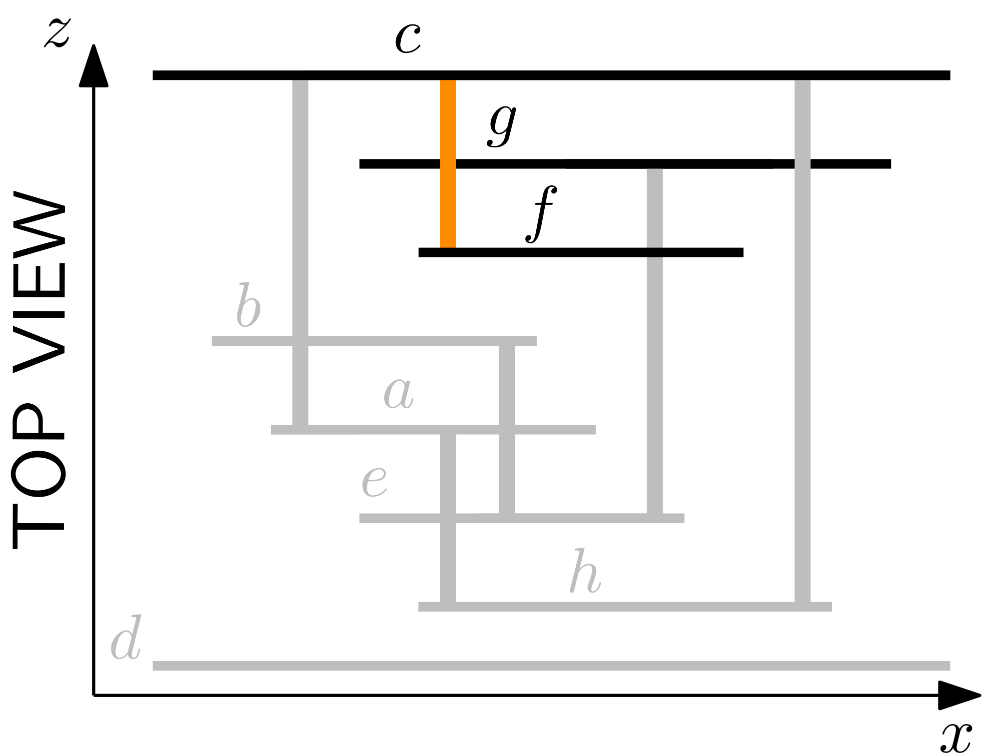
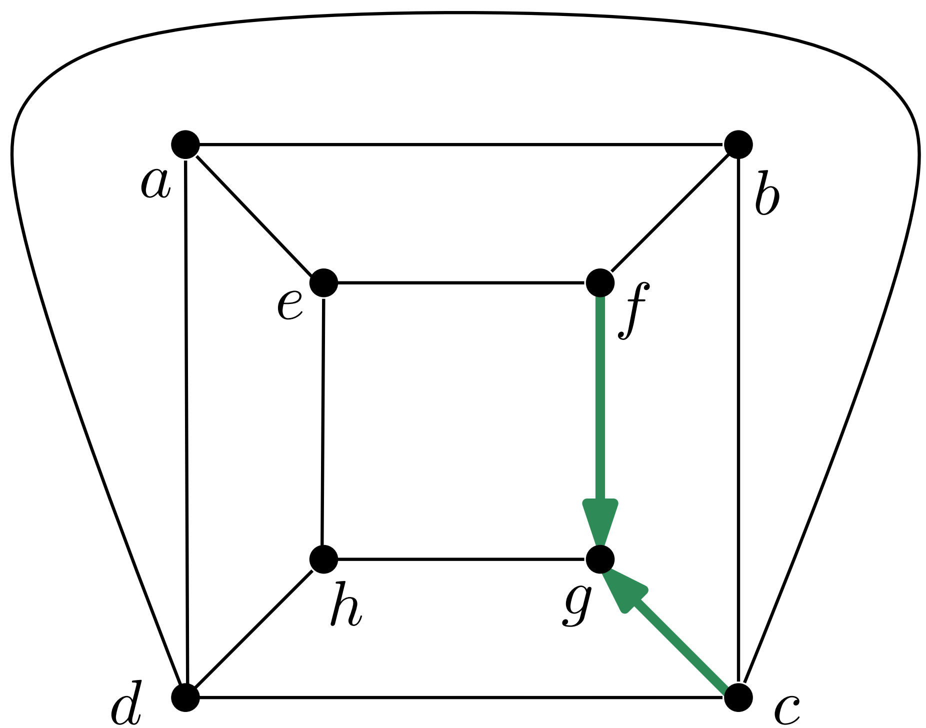
Idea: An edge oriented from u to v in D_1 (D_2) encodes that the top side (bottom side) of u will have y-coordinate greater (smaller) than the one of v





D_1

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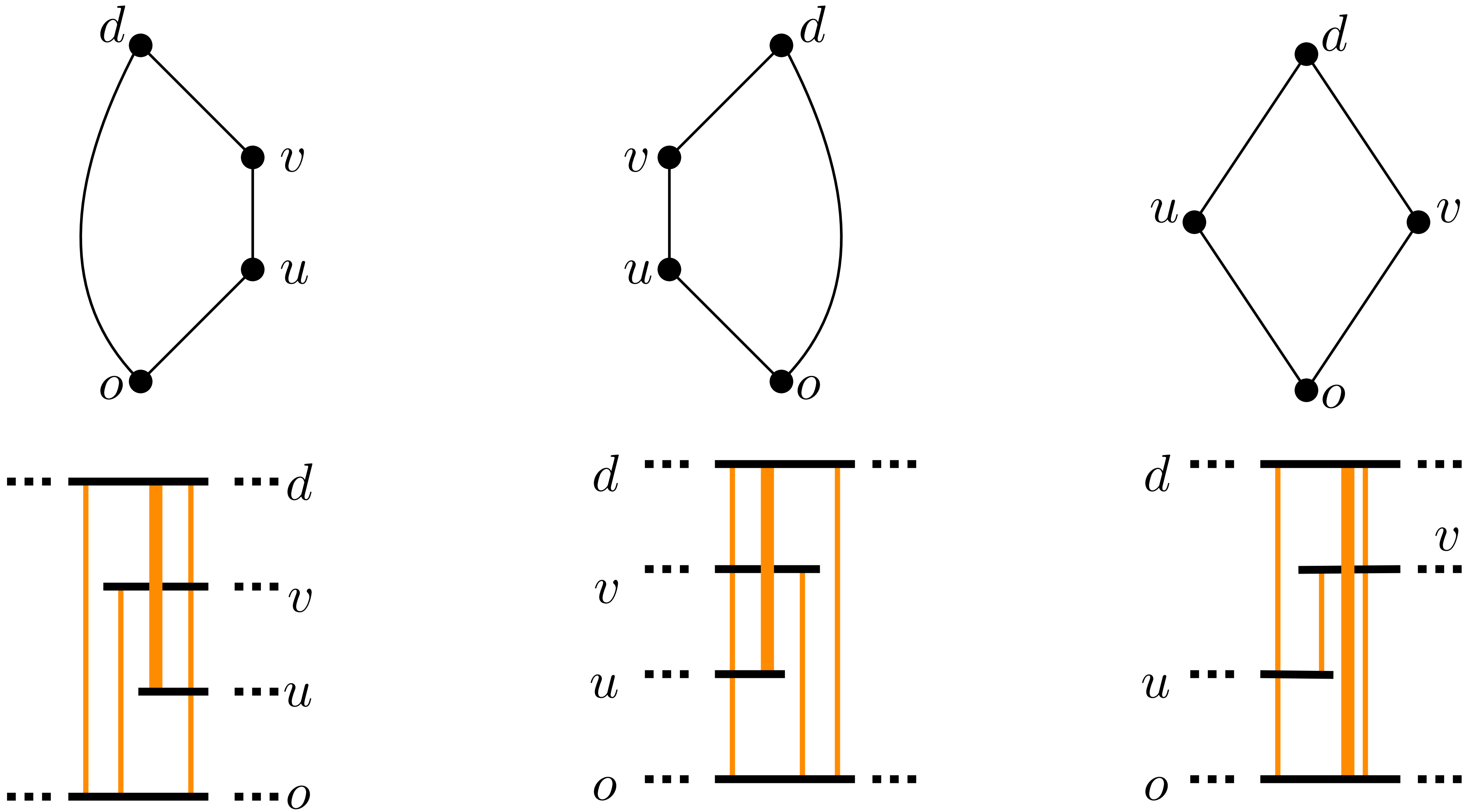
2a: Process the edges of each kite and apply a set of rules to obtain D_1 and D_2



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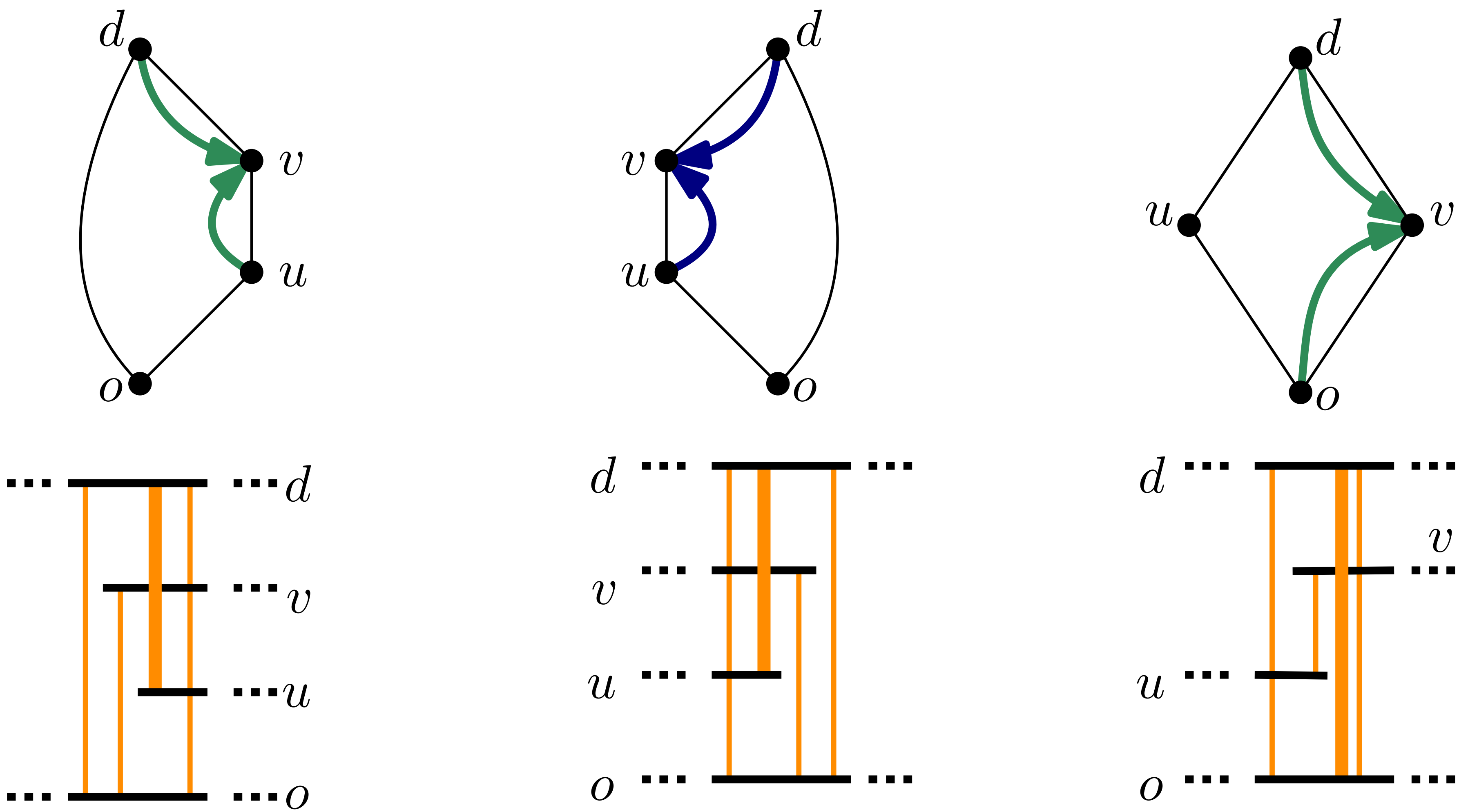

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To prove: No edge directed twice, and both D_1 and D_2 are acyclic

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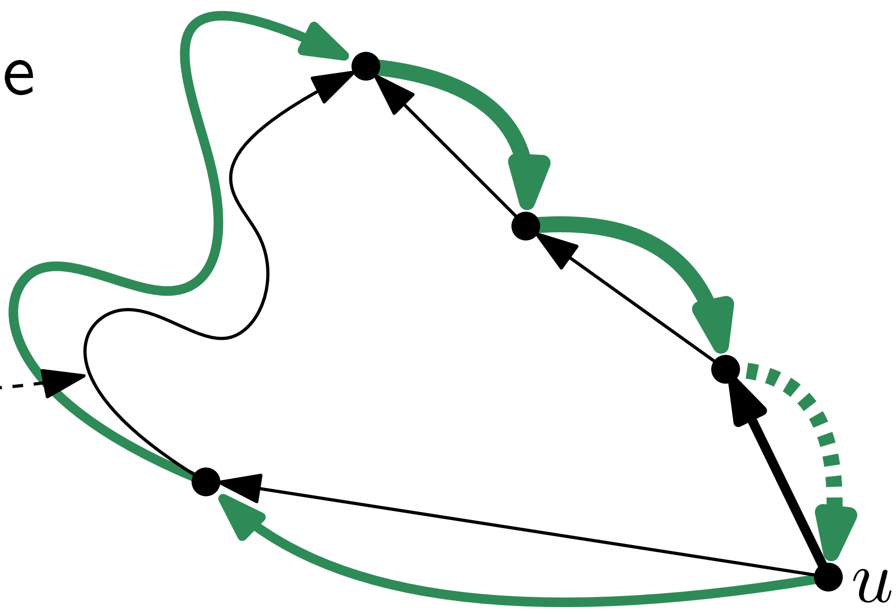
Assuming D_1 or D_2 has a cycle, one can contradict the rules used to construct D_1 or D_2.

Black edges oriented from the bottommost to the topmost endpoint by looking at the B1VR



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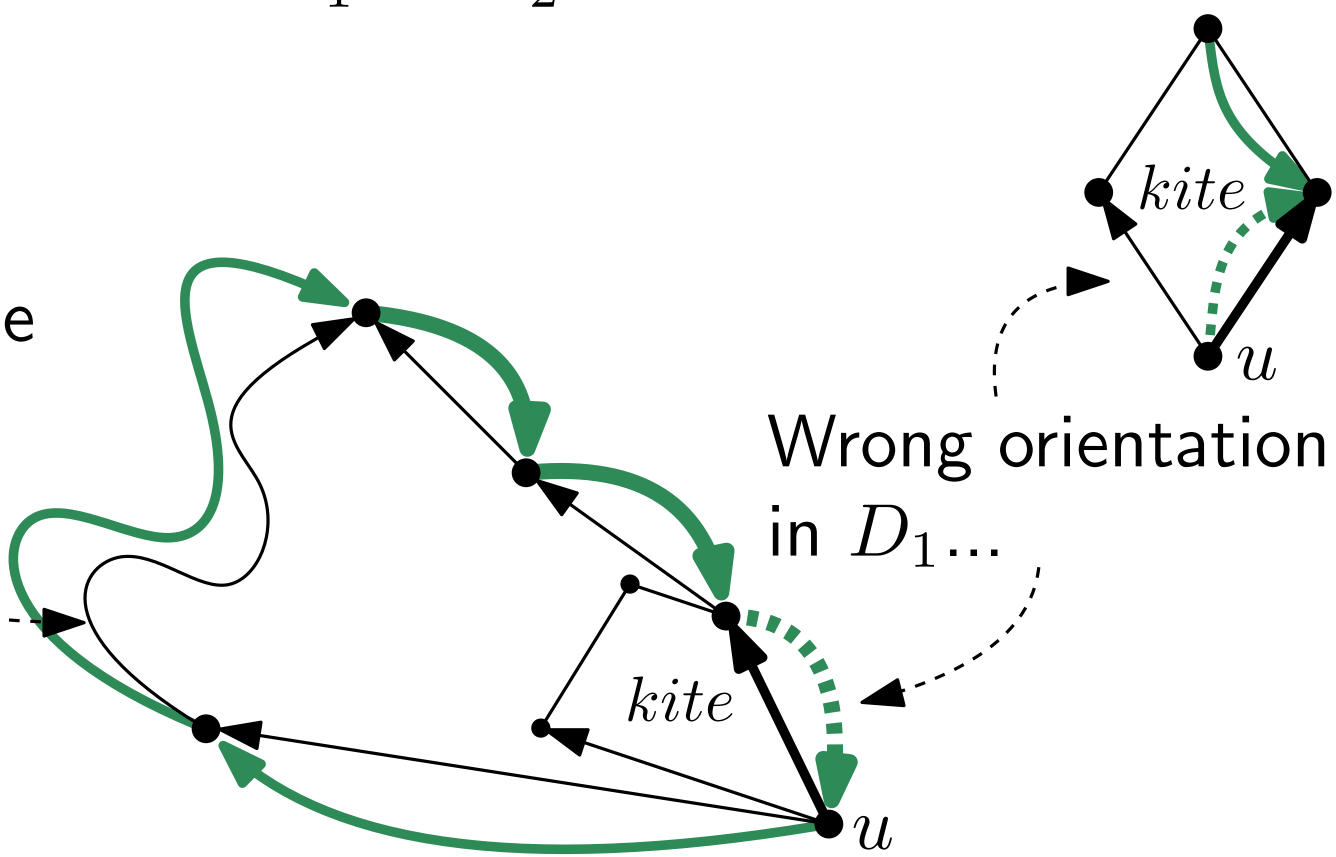
Black edges oriented from the bottommost to the topmost endpoint in the B1VR



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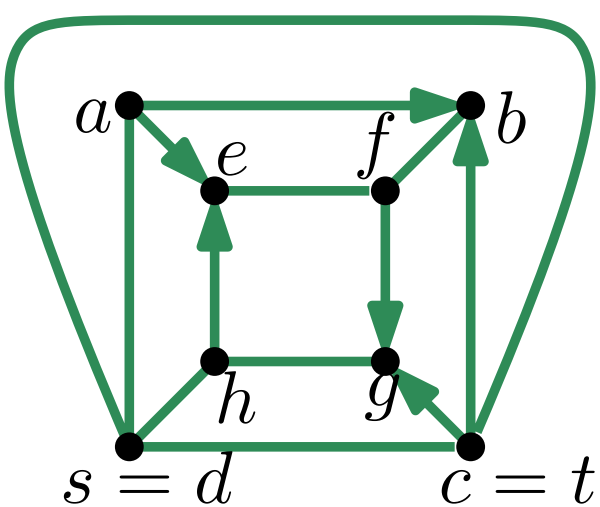
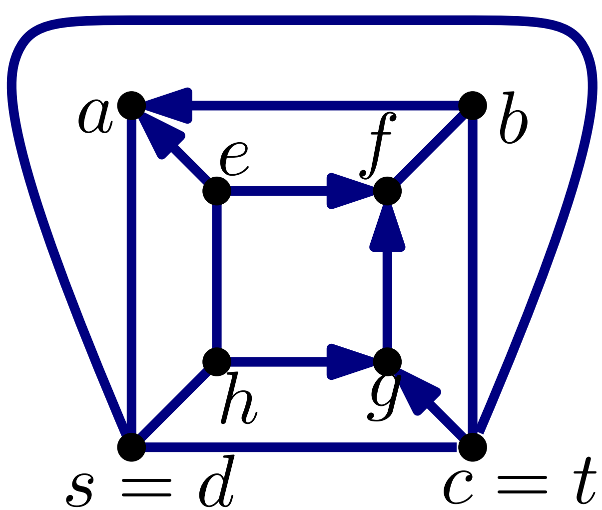
2b. Compute a topological ordering of both D_1 and D_2 (each ordering might consists of several components). This gives two total orderings (after possible concatenations) σ_1 and σ_2 of the vertices of G

$$\sigma_1 = \{c, a, h, f, b, e, g, d\}$$
$$\sigma_2 = \{h, c, g, e, b, f, a, d\}$$





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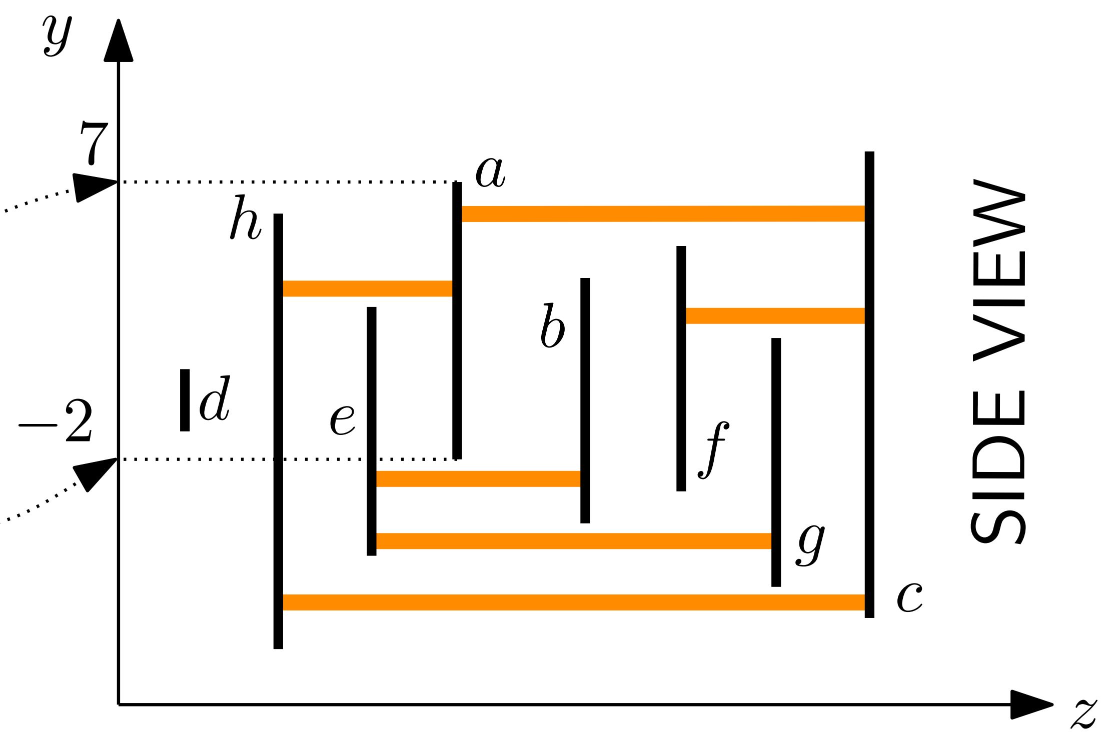
2c. Set the y-coordinate of the top side of the rectangle representing the i-th vertex in σ_1 equal to $n - i + 1$
Set the y-coordinate of the bottom side of the rectangle representing the i-th vertex in σ_2 equal to $i - n - 1$

$\sigma_1 = \{c, a, h, f, b, e, g, d\}$
$\sigma_2 = \{h, c, g, e, b, f, a, d\}$

$top(\sigma_1(2) = a) = 7$

$bottom(\sigma_2(7) = a) = -2$

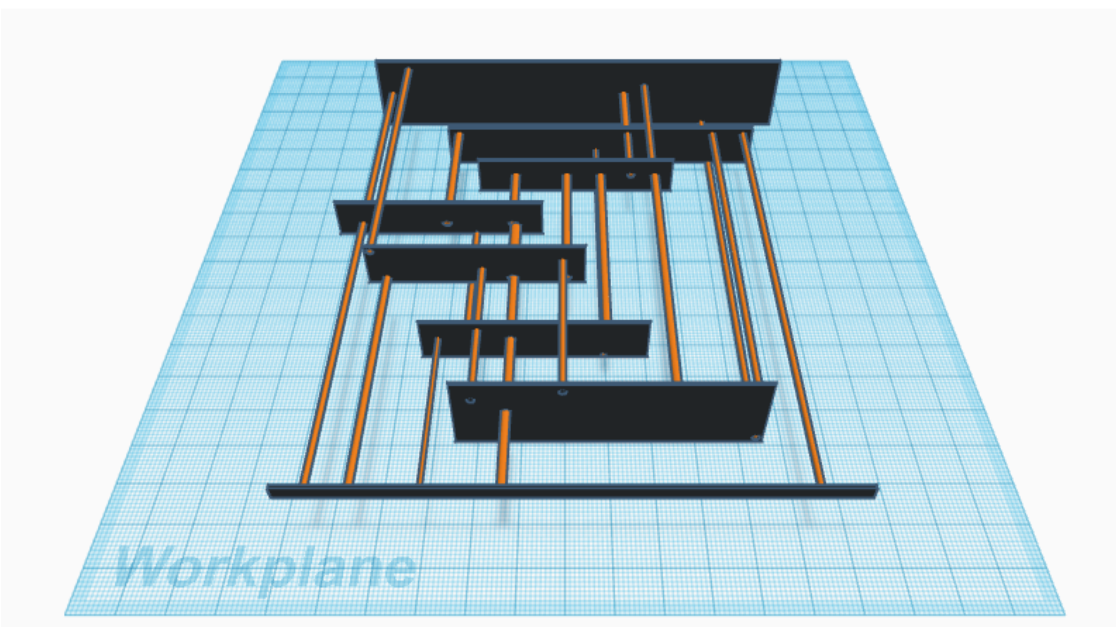


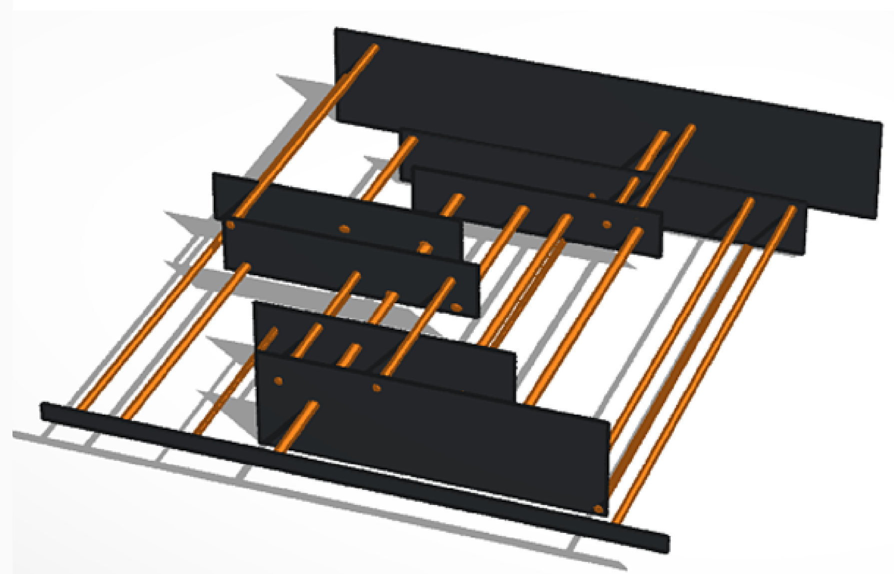
End!

Output: A 1-visible ZPR γ of G

Each step takes $O(n)$ time

The height of each rectangle is at most $2n$, hence γ takes $O(n) \times O(n) \times O(n)$ volume





Open problems

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3. Does every 1-planar graph admit a 2.5D box visibility representation (i.e., vertices are axis-aligned boxes whose bottom faces lie on a same plane, and visibilities are both vertical and horizontal)?

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THANKS FOR YOUR ATTENTION!