Obstacle Numbers of Planar Graphs

John Gimbel, Patrice Ossona de Mendez, Pavel Valtr
(GD 2017)

Visibility representation with obstacles

Scenario (in the plane):
$O_1, \ldots, O_k$ connected obstacles (simple polygons, say)
P = finite set of points in general position

Visibility representation with obstacles

Scenario (in the plane):
$O_1, \ldots, O_k$ connected obstacles (simple polygons, say)
P = finite set of points in general position

consider geometric graph (= straight-line graph) (V, E):
$V = P$
E = visibility segments (for P)

Visibility representation with obstacles

Scenario (in the plane):
$O_1, \dots, O_k$ connected obstacles (simple polygons, say)
P = finite set of points in general position

consider geometric graph (= straight-line graph) (V, E):
$V = P$
E = visibility segments (for P)

it is the (visibility) representation of the underlying abstract graph

Obstacle numbers

Definition (ALPERT–KOCH–LAISON 2010)
G graph
obstacle number of G:
$\mathrm{obs}(G) :=$ minumum number of obstacles in a representation of G

Obstacle numbers

Definition (ALPERT–KOCH–LAISON 2010)
G graph
obstacle number of G:
$\mathrm{obs}(G) :=$ minumum number of obstacles in a representation of G

Definition (new)
G planar graph
planar obstacle number of G:
$\mathrm{pobs}(G) :=$ minumum number of obstacles in a planar representation of G

Obstacle numbers

Definition (ALPERT–KOCH–LAISON 2010)

G graph

obstacle number of G:

$\mathrm{obs}(G) :=$ minumum number of obstacles in a representation of G

Definition (new)

G planar graph

planar obstacle number of G:

$\mathrm{pobs}(G) :=$ minumum number of obstacles in a planar representation of G

(correct due to Fáry's theorem)

Bounding the obstacle numbers

Definition:
$\mathrm{obs}(n) := \max\{\mathrm{obs}(G) : G \text{ graph on } n \text{ vertices}\}$

Bounding the obstacle numbers

Definition:

$\mathrm{obs}(n) := \max\{\mathrm{obs}(G) : G \text{ graph on } n \text{ vertices}\}$

$\mathrm{pobs}(n) := \max\{\mathrm{pobs}(G) : G \text{ planar graph on } n \text{ vertices}\}$

Bounding the obstacle numbers

Definition:
$\mathrm{obs}(n) := \max\{\mathrm{obs}(G) : G \text{ graph on } n \text{ vertices}\}$

$\mathrm{pobs}(n) := \max\{\mathrm{pobs}(G) : G \text{ planar graph on } n \text{ vertices}\}$

Best known bounds on $\mathrm{obs}(n)$:

$$\Omega(n/(\log\log n)^2) \leq \mathrm{obs}(n) \leq 2n\log n$$

[DUJMOVIĆ–MORIN 2013] [BALKO–CIBULKA–VALTR 2015]

Bounding the obstacle numbers

Definition:
$\mathrm{obs}(n) := \max\{\mathrm{obs}(G) : G \text{ graph on } n \text{ vertices}\}$

$\mathrm{pobs}(n) := \max\{\mathrm{pobs}(G) : G \text{ planar graph on } n \text{ vertices}\}$

Best known bounds on $\mathrm{obs}(n)$:

$$\Omega(n/(\log\log n)^2) \leq \mathrm{obs}(n) \leq 2n\log n$$

[DUJMOVIĆ–MORIN 2013] [BALKO–CIBULKA–VALTR 2015]

OPEN (ALPERT et al. 2010): $\mathrm{obs}(n) = O(n)$??

Bounding the obstacle numbers

$\mathrm{obs}(n) := \max\{\mathrm{obs}(G) : G \text{ graph on } n \text{ vertices}\}$
$\mathrm{pobs}(n) := \max\{\mathrm{pobs}(G) : G \text{ planar graph on } n \text{ vertices}\}$
$\Omega(n/(\log\log n)^2) \leq \mathrm{obs}(n) \leq 2n\log n$
OPEN (ALPERT et al. 2010): $\mathrm{obs}(n) = O(n)$??

Bounding the obstacle numbers

$\mathrm{obs}(n) := \max\{\mathrm{obs}(G) : G \text{ graph on } n \text{ vertices}\}$
$\mathrm{pobs}(n) := \max\{\mathrm{pobs}(G) : G \text{ planar graph on } n \text{ vertices}\}$
$\Omega(n/(\log\log n)^2) \leq \mathrm{obs}(n) \leq 2n \log n$
OPEN (ALPERT et al. 2010): $\mathrm{obs}(n) = O(n)$??

Observation: $\mathrm{pobs}(n) \leq 2n - 4$

Bounding the obstacle numbers

$\mathrm{obs}(n) := \max\{\mathrm{obs}(G) : G \text{ graph on } n \text{ vertices}\}$
$\mathrm{pobs}(n) := \max\{\mathrm{pobs}(G) : G \text{ planar graph on } n \text{ vertices}\}$
$\Omega(n/(\log\log n)^2) \leq \mathrm{obs}(n) \leq 2n\log n$
OPEN (ALPERT et al. 2010): $\mathrm{obs}(n) = O(n)$??

Observation: $\mathrm{pobs}(n) \leq 2n - 4$
(Hint: place an obstacle in each face)

Bounding the obstacle numbers

$\mathrm{obs}(n) := \max\{\mathrm{obs}(G) : G \text{ graph on } n \text{ vertices}\}$
$\mathrm{pobs}(n) := \max\{\mathrm{pobs}(G) : G \text{ planar graph on } n \text{ vertices}\}$
$\Omega(n/(\log\log n)^2) \leq \mathrm{obs}(n) \leq 2n \log n$
OPEN (ALPERT et al. 2010): $\mathrm{obs}(n) = O(n)$??

Observation: $\mathrm{pobs}(n) \leq 2n - 4$
(Hint: place an obstacle in each face)

(Main) Theorem: $\mathrm{pobs}(n) = n - 3$ (for $n \geq 4$)

Other new results

Theorem: G a PURE 2-DIR graph.
Then $\mathrm{obs}(G) \leq 1$

Other new results

Theorem: G a PURE 2-DIR graph.
Then $\mathrm{obs}(G) \leq 1$

Theorem (corollary): G bipartite planar.
Then $\mathrm{obs}(G) \leq 1$

Proof of the main result (idea)

(Main) Theorem: $\mathrm{pobs}(n) = n - 3$ (for $n \geq 4$)

Proof of the main result (idea)

(Main) Theorem: $\mathrm{pobs}(n) = n - 3$ (for $n \geq 4$)

Proof (idea):

First case: G is a triangulation

(a) find a Fáry drawing of G and a 2-coloring of its inner faces s.t. for each color, placing obstacles in faces of the chosen color gives a visibility representation of G

Proof of the main result (idea)

(Main) Theorem: $\text{pobs}(n) = n - 3$ (for $n \geq 4$)

Proof (idea):

First case: G is a triangulation

(a) find a Fáry drawing of G and a 2-coloring of its inner faces s.t. for each color, placing obstacles in faces of the chosen color gives a visibility representation of G (KEY PART – NOT EASY!)

Proof of the main result (idea)

(Main) Theorem: $\mathrm{pobs}(n) = n - 3$ (for $n \geq 4$)

Proof (idea):

First case: G is a triangulation

(a) find a Fáry drawing of G and a 2-coloring of its inner faces s.t. for each color, placing obstacles in faces of the chosen color gives a visibility representation of G (KEY PART – NOT EASY!)

(b) choose the less frequent color: $\leq \lfloor (2n-5)/2 \rfloor$ faces/obstacles

Proof of the main result (idea)

(Main) Theorem: $\mathrm{pobs}(n) = n - 3$ (for $n \geq 4$)

Proof (idea):

First case: G is a triangulation

(a) find a Fáry drawing of G and a 2-coloring of its inner faces s.t. for each color, placing obstacles in faces of the chosen color gives a visibility representation of G (KEY PART – NOT EASY!)

(b) choose the less frequent color: $\leq \lfloor (2n-5)/2 \rfloor$ faces/obstacles

Second case: G arbitrary planar

(a) triangulate G

(b) proceed as in step (a) in the first case

(c) remove the edges added in step (a)

(d) put obstacles in triangular faces of less frequent color

END (OF PROOF IDEA)