# Thrackles: An Improved Upper Bound

**Radoslav Fulek** and János Pach

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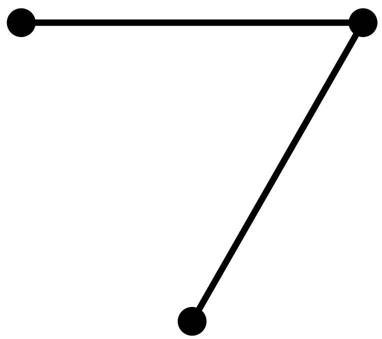

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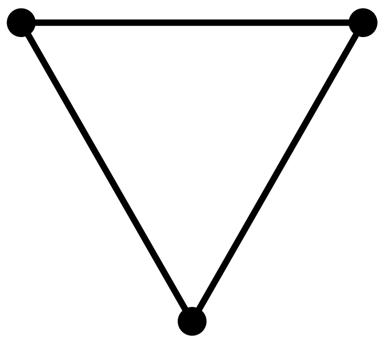

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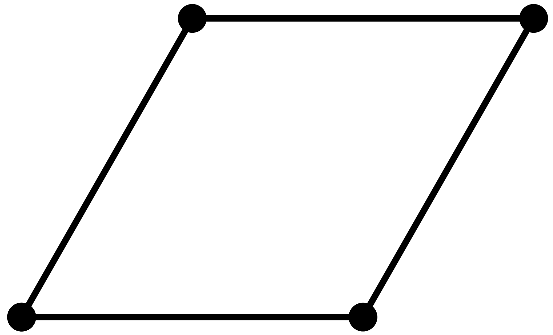

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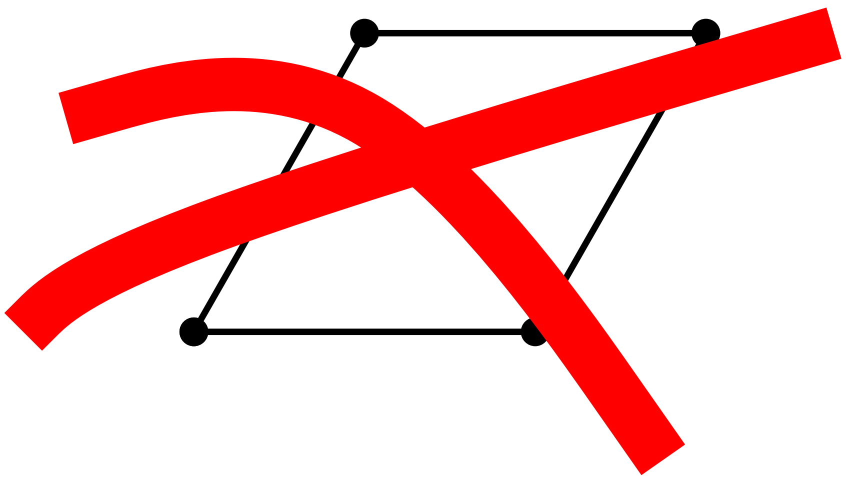

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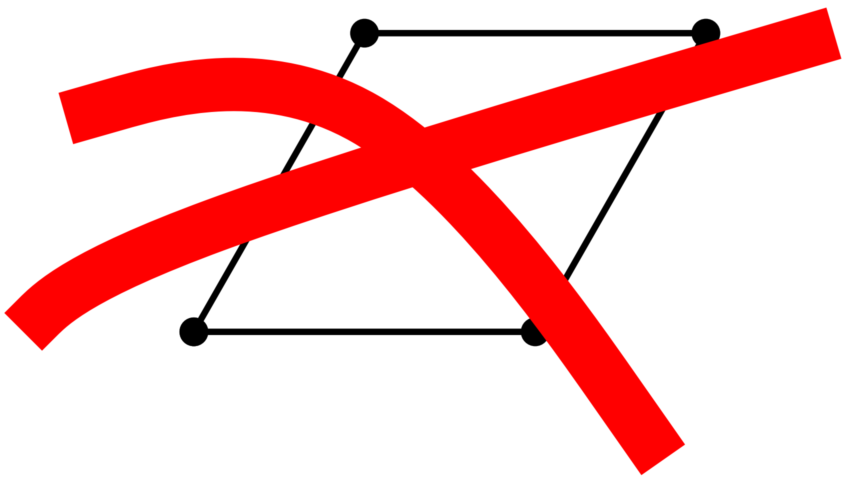

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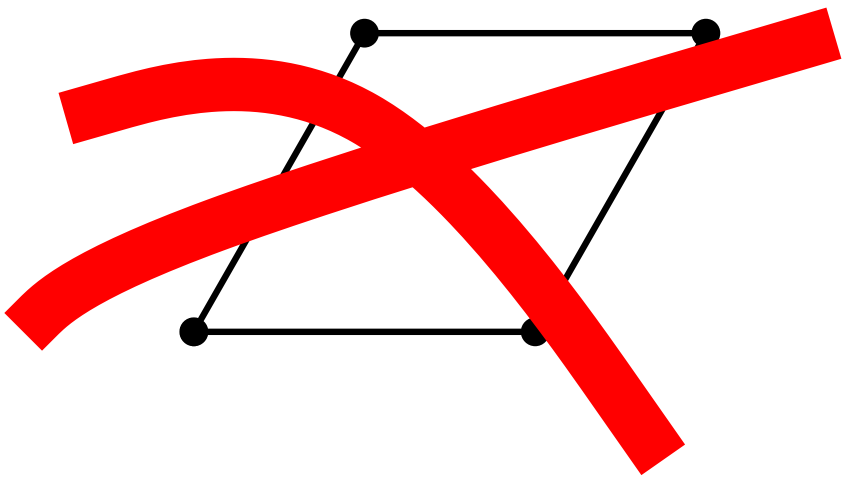

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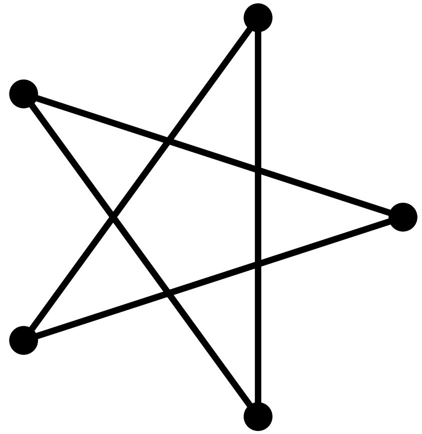

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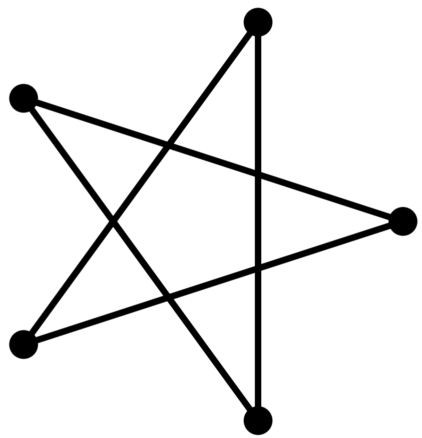

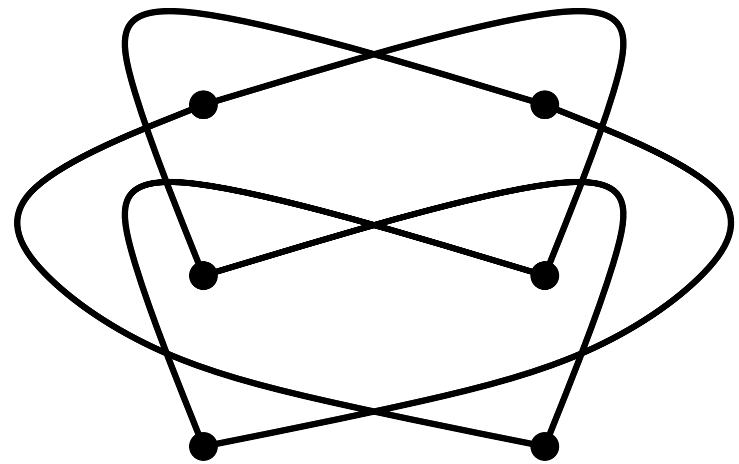

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Our technique is a refinement of the technique of Goddyn and Xu.

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**Theorem 2** (Lovász, Pach, Szegedy, 1996)**.** *A bipartite graph admits a drawing as a generalized thrackle if and only if it is planar.*

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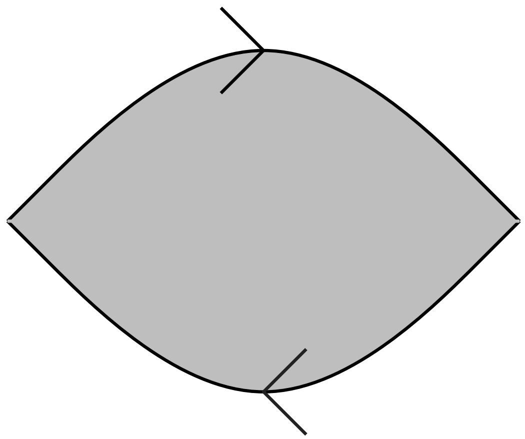

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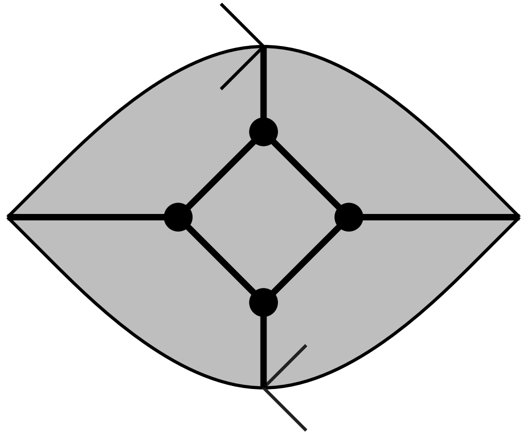

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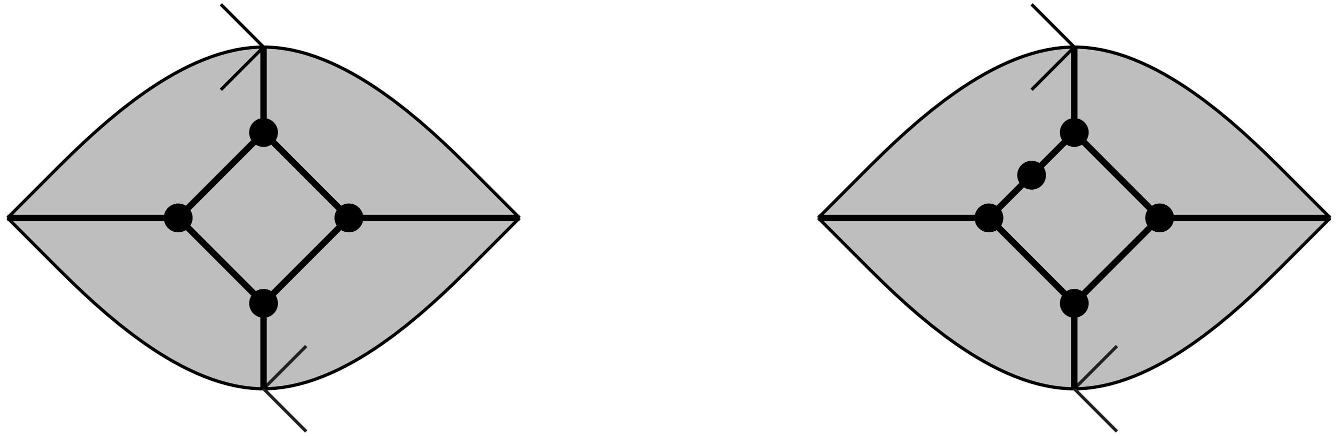

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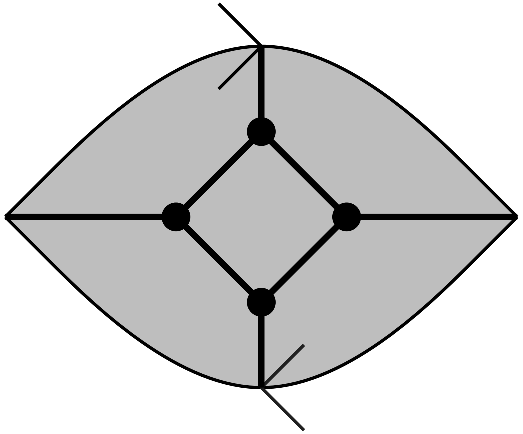

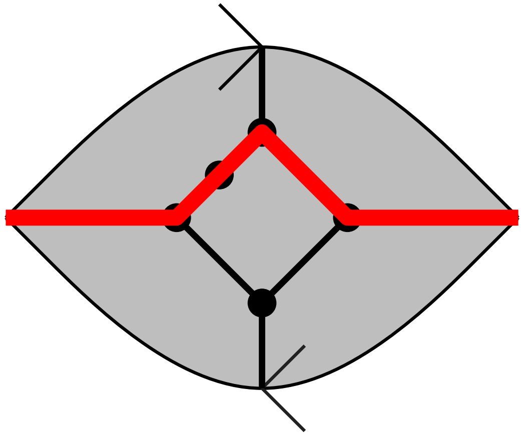

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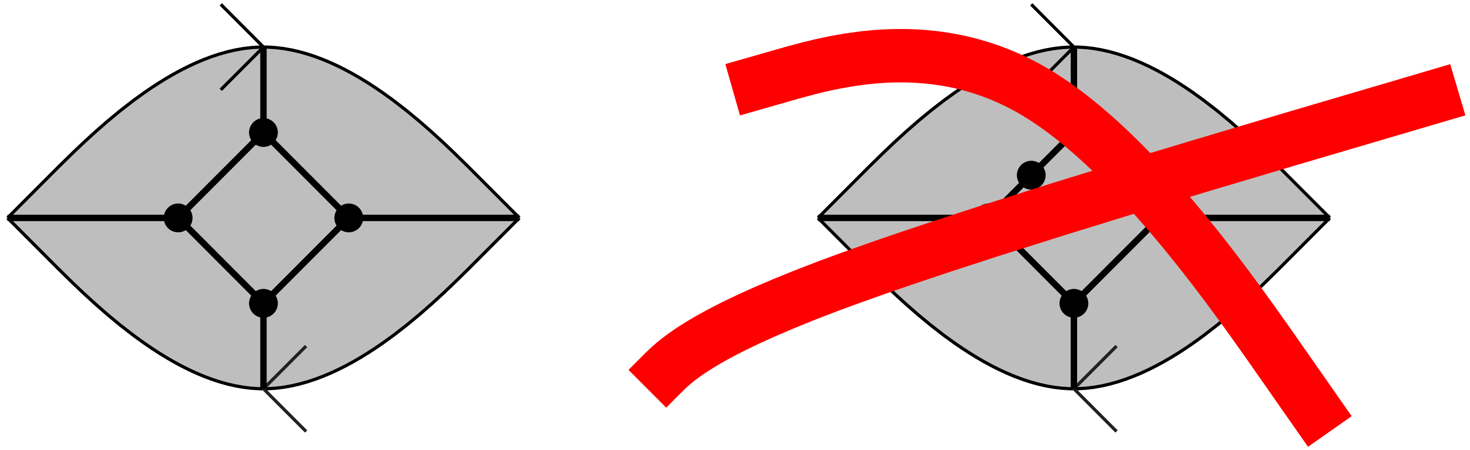

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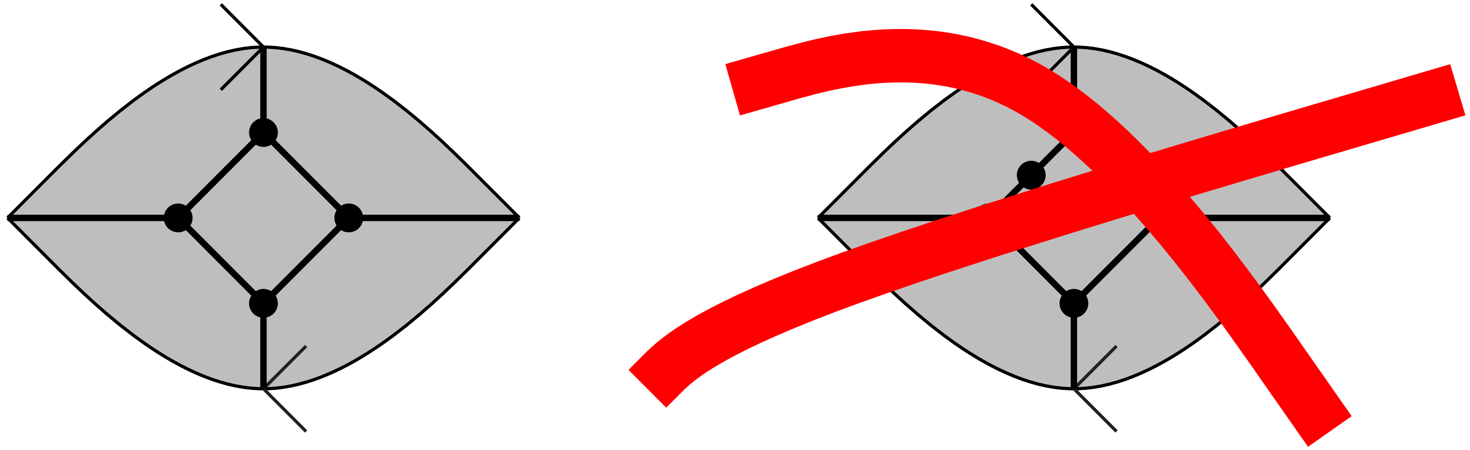

**Theorem 3** (LPS, 1996 and Cairns and Nikolayevsky, 2009)). *A graph $G$ is a generalized thrackle if and only if $G$ admits a parity embedding in the projective plane. In particular, any bipartite thrackle can be embedded in the plane.*

## Proof of the $1.4n$-bound

**Theorem 4** (Pach, Lovász and Szegedy, 1996 and Cairns and Nikolayevsky, 2009). *A graph $G$ is a generalized thrackle if and only if $G$ admits a parity embedding in the projective plane. In particular, any bipartite thrackle can be embedded in the plane.*

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It can be shown that the total charge $2|E(G)|$ is at least $7|F|$, where $F$ is the set of faces, and hence by Euler's formula $7|E(G)| \leq 7|V(G)| + 2|E(G)|$.

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We consider the parity embedding of $G$ and put the charge of $|f|$ on every face $f$.

It can be shown that the total charge $2|E(G)|$ is at least $\left(7 + \frac{1}{48}\right)|F|$, where $F$ is the set of faces, and hence by Euler's formula $\left(7 + \frac{1}{48}\right)|E(G)| \leq \left(7 + \frac{1}{48}\right)|V(G)| + 2|E(G)|$.

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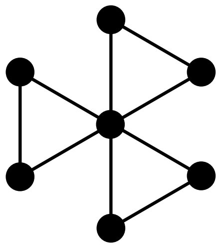

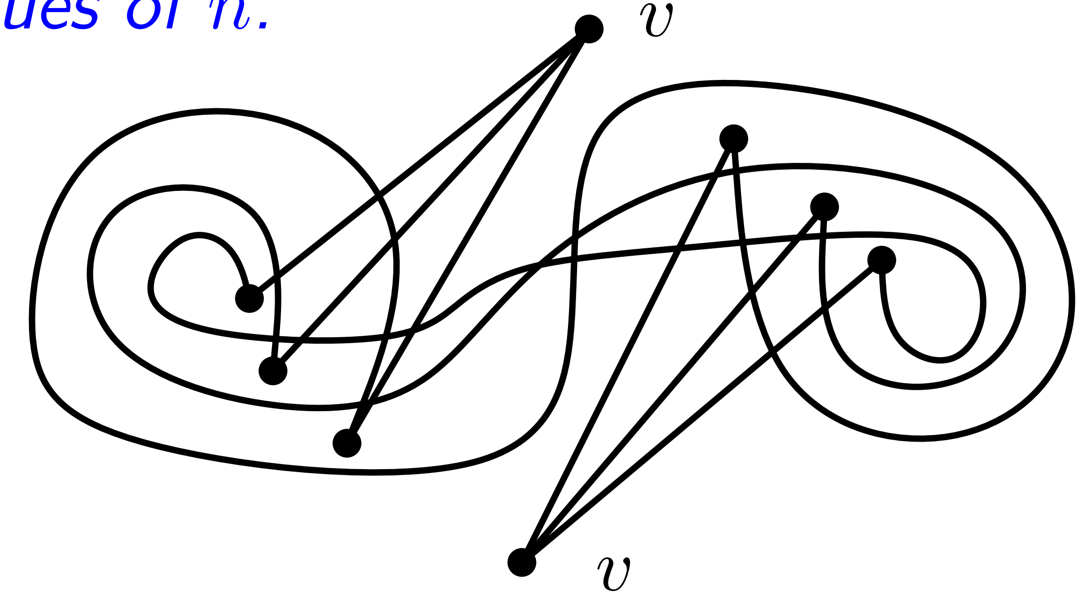

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**Question.** Is is true that a simple topological graph with no three pairwise disjoint edges can have at most $O(n)$ vertices?

Pach and G. Tóth showed that a simple topological graph with no $k$ pairwise disjoint edges can have at most $O(n \mathrm{poly}(\log n))$.