Drawing bobbin lace graphs

Fundamental cycles for a subclass of periodic graphs

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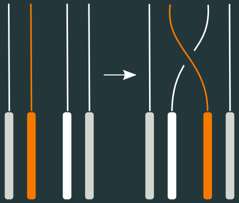
Bobbin lace


Portrait by Frans Purbus the Younger, circa 1600

source: www.metmuseum.org

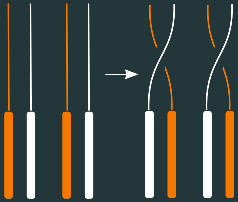
Guipure bobbin lace edging, circa 1620
source: www.sophieploeg.com

Bobbin lace



σ_{2i}

Cross



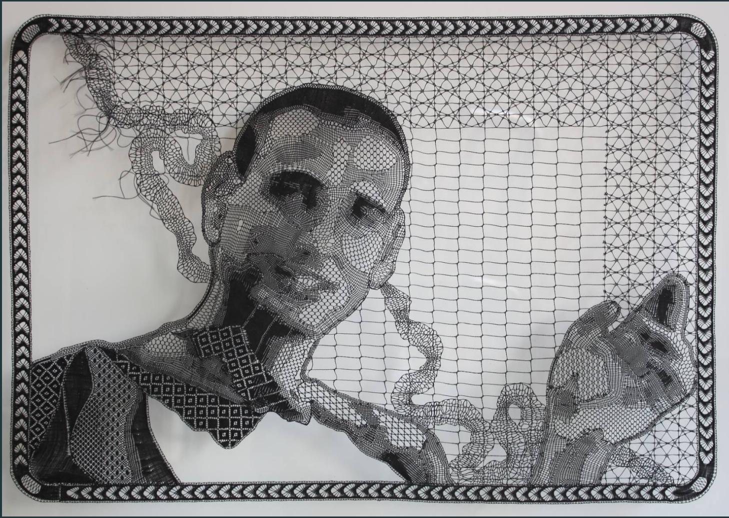
$\sigma_{2i-1}^{-1}\sigma_{2i+1}^{-1}$

Twist

Pin

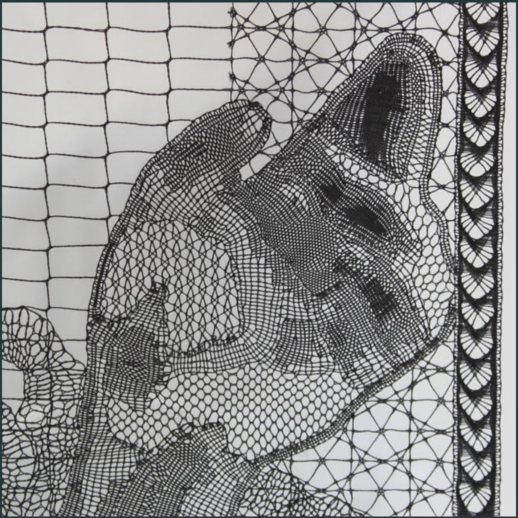
La Dentellière by Véronique Louppe
source: m.ok.ru

Bobbin lace



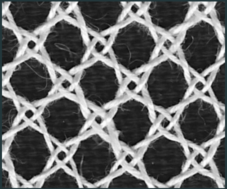
Portrait by Pierre Fouché
source: www.pierrefouche.net

Bobbin lace



Portrait by Pierre Fouché
source: www.pierrefouche.net

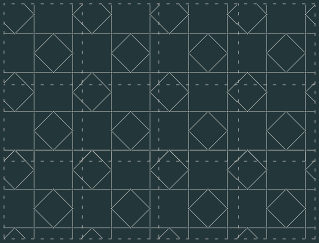
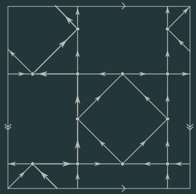
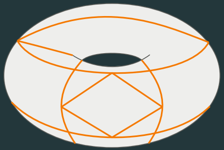
Lace pattern to graph



Conditions on graph embedding

Infinite, simple, acyclic planar digraph: G^∞

Doubly periodic, represent as graph on a torus: G

Conditions on graph embedding

Infinite, simple, acyclic planar digraph: G^∞

Doubly periodic, represent as graph on a torus: G

G lifts to G^∞

G need not be simple BUT

Loops and parallel edges must be non-contractible

Conditions on graph embedding (I., Ruskey 2014)

C1. *G* is a directed 2-2-regular digraph

C2. Rotation system of *G*

toroidal cellular embedding

all facial walks contain at least 3 edges.

C3. All directed circuits of *G* are non-contractible.

Theorem (Biedl, I. 2017)

Conditions (C1-C3) can be tested in linear time.

C1 and C2 fairly obvious

C3: Does embedding admit a contractible circuit?

C3 can be tested in linear time

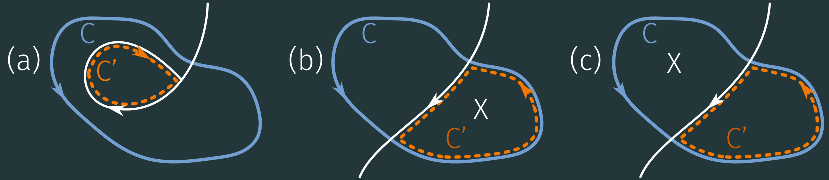
Theorem (Biedl, I. 2017)

Presume (C1, C2) hold.

G has a contractible directed circuit
$\Longrightarrow$ *G has face bounded by contractible directed circuit.*

Let C be a contractible directed circuit

Choose C to maximize # faces “outside” - # faces “inside”



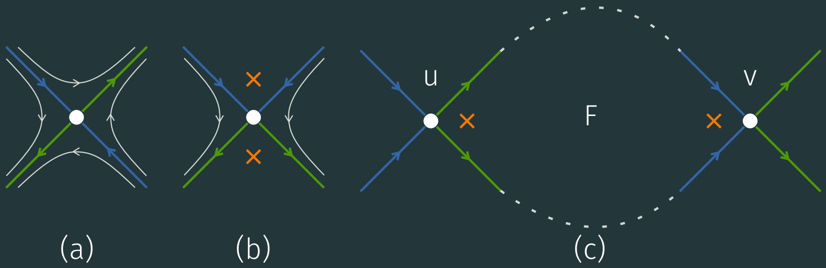
Conservation of threads

source: lacenews.files.wordpress.com

Theorem (I., Ruskey 2014)

Presume (C1-C3) hold.

No contractible cycles
$\implies$ *all vertices, outgoing arcs are rotationally consecutive*



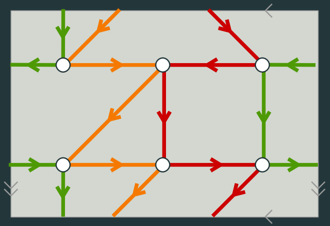
Property resulting from C1-C3

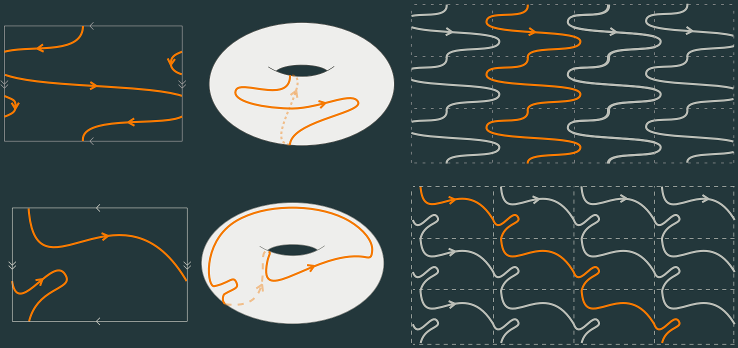
Theorem (I., Ruskey 2014)

Presume (C1-C3) hold.

Partition edges of G into a set of osculating directed circuits.

Partition is unique and can be found in linear time.





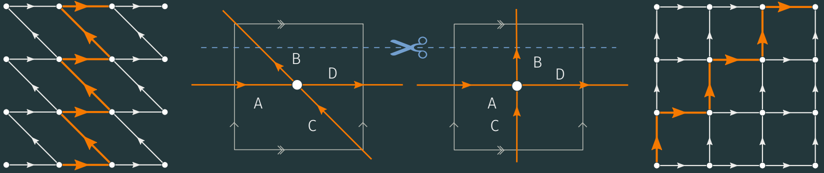
Wrapping index: (m, ℓ)

top: $(1, 0)$

bottom: $(1, 1)$

Usually consider two homeomorphic drawings as equivalent

Dehn twist



C4. There exists:

a meridian M,

a longitude L, and

a partition $\mathcal{P}(G)$ into osculating directed circuits

such that

each circuit in $\mathcal{P}(G)$ is in the $(1, 0)$-homotopy class.

Algebraic intersection:

$$\hat{i}(C_1, C_2) = \#C_1 \text{ crosses } C_2 \textit{ l-to-r} - \#C_1 \text{ crosses } C_2 \textit{ r-to-l}$$

Theorem (well known, e.g. Stillwell)

C_1, C_2 are simple closed curves.

$\hat{i}(C_1, C_2) = 0 \iff$ *same homotopy class.*

Property resulting from C1-C3

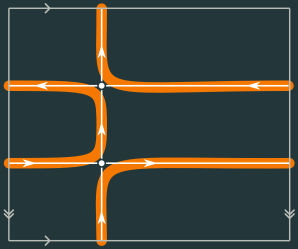
Theorem (Biedl, I. 2017)

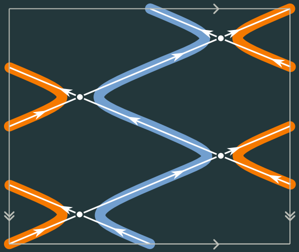
Presume (C1-C3) hold.
All $P \in \mathcal{P}(G)$ belong to the same homotopy class.

Either:

P_0 is a circuit and $|\mathcal{P}(G)| = 1$, or

all P are simple cycles with $\hat{i}(P_i, P_j) = 0$



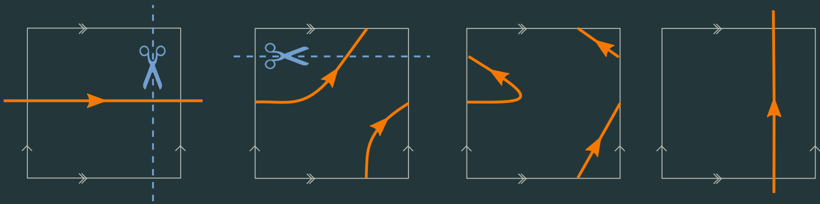


Theorem (Biedl, I. 2017)

Presume (C1-C3) hold.

There exists a drawing of G for which (C4) holds.

Lickorish-twist theorem



Main contribution:

Theorem (Biedl, I. 2017)

Presume G satisfies (C1-C3).

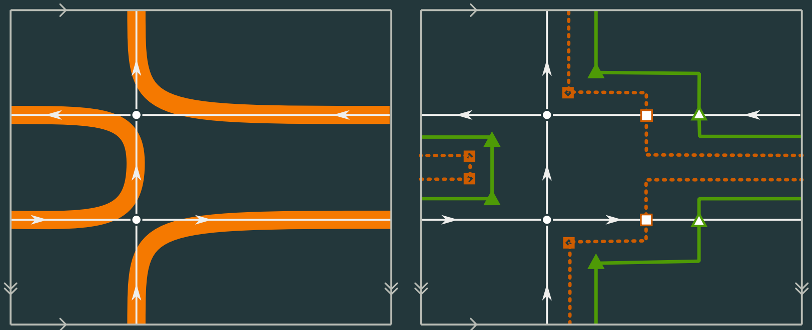
We can draw a graph that satisfies C4 in linear time.

The drawing resides in an $O(n^4) \times O(n^4)$-grid.

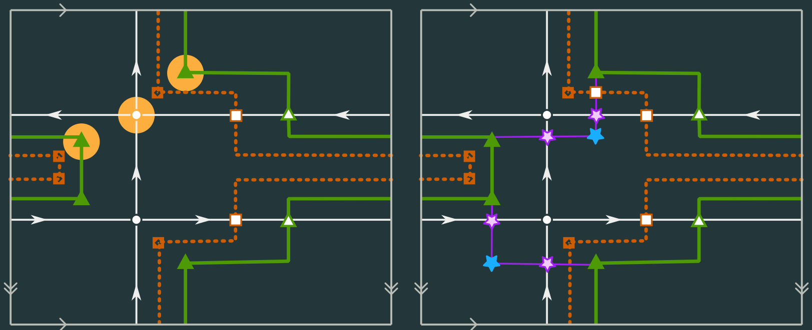
Graph drawing algorithm

- A. Partition G into osculating circuits. Select one, call it P.
- B. Create the offset graph $\mathcal{O}(G)$.
- C. Find a simple cycle M in $\mathcal{O}(G)$ s.t.
  $\hat{i}(M, P) = 0$, and
  M intersects any edge of G at most once.
- D. Find a simple cycle L in $\mathcal{O}(G)$ s.t.
  $\hat{i}(L, P) = \pm 1$, and
  M and L intersect exactly once, and
  L intersects any edge of G at most once.
- E. Use existing torus-drawing techniques to draw G on a rectangle with meridian M and longitude L.

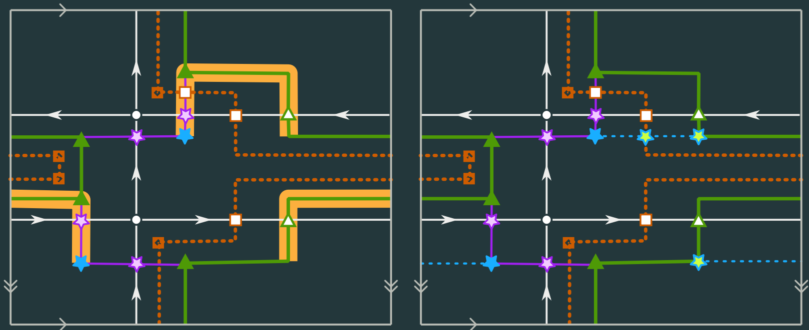
Create offset graph $\mathcal{O}(G)$



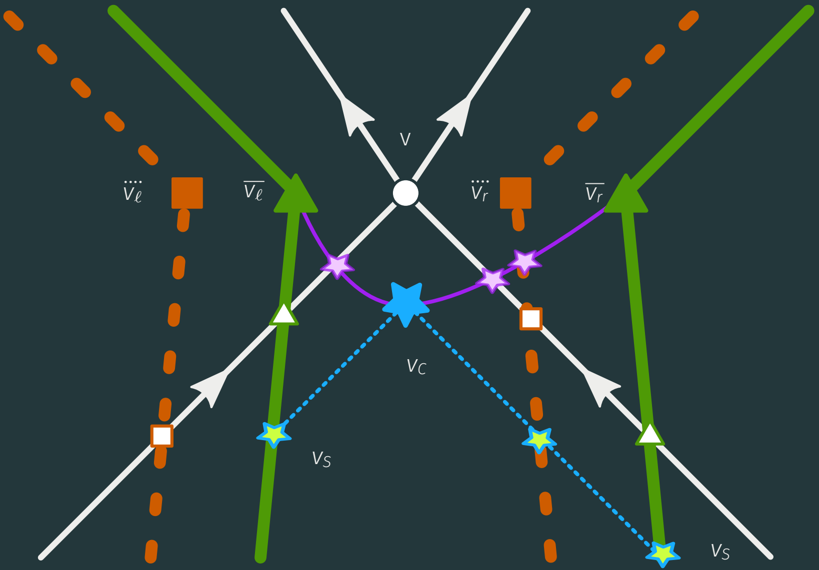
CREATE OFFSET GRAPH $\mathcal{O}(G)$



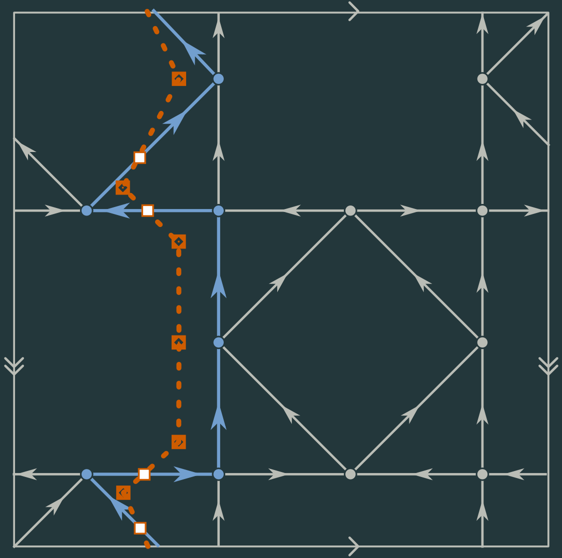
Create offset graph $\mathcal{O}(G)$



Create offset graph $\mathcal{O}(G)$



Find meridian



Find meridian

Select an osculating circuit P from G

Define M to be copy $\ddddot{P}$ in offset graph

$\hat{i}(M, P) = 0$

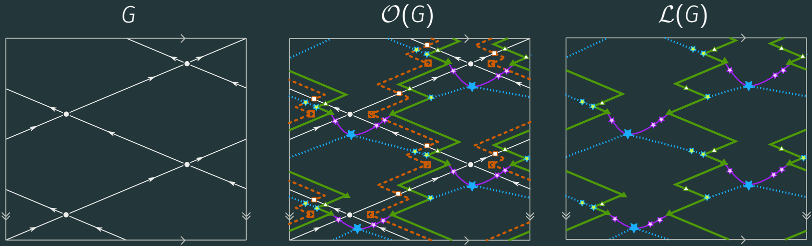
Start and end on same side of P

M intersects every edge of G at most once

By Construction

Define $\mathcal{L}(G)$ subgraph of $\mathcal{O}(G)$

- copies $\overline{P}$ of osculating circuits
- crossover-edges
- shortcut-edges



Theorem (Biedl, I. 2017)

Presume (C1-C3) hold, P is a directed osculating cycle.

There exists a directed walk W in G that

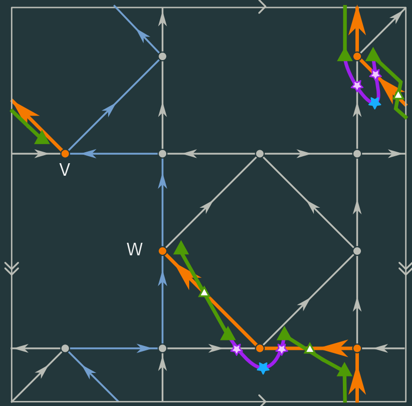
starts at $v \in P$ with left outgoing edge $e_1 \notin P$,

ends at $w \in P$ with right incoming edge $e_k \notin P$,

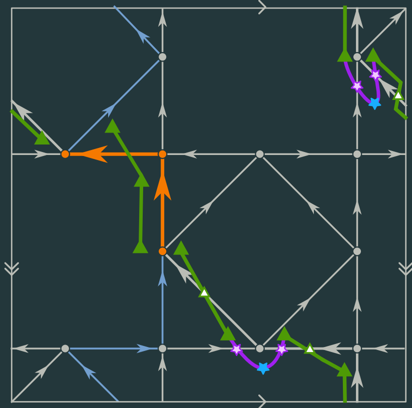
has no transverse intersection or shared edges with P.

Proof not hard but tedious, see paper.

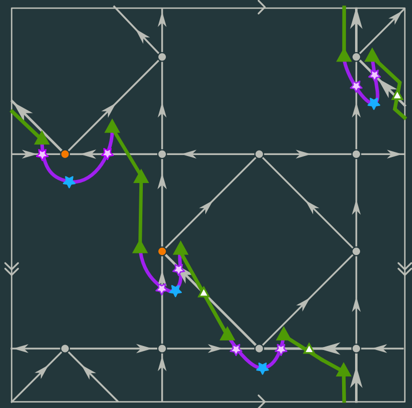
Find L^-: path in $\mathcal{L}(G)$
follows W from $\overline{v_\ell}$ to $\overline{w_r}$



Find L^*: subpath of $\overline{P^*}$
connecting $\overline{w_\ell}$ to $\overline{v_r}$



Add crossing edges $(\overline{w_r}, \overline{w_\ell})$ and $(\overline{v_r}, \overline{v_\ell})$



Find longitude

$L = L^- + (\overline{w_r}, \overline{w_\ell}) + L^* + (\overline{v_r}, \overline{v_\ell})$

L and M cross exactly once:

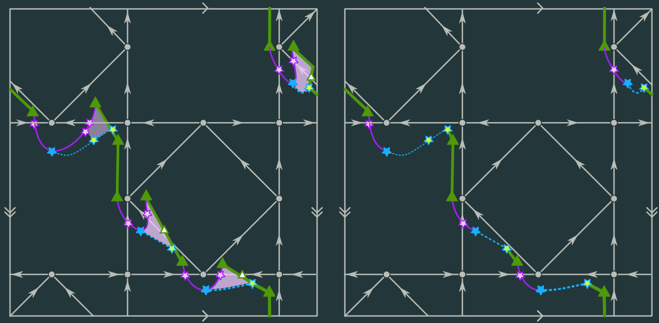
- L and M can only intersect at crossover edges
- Order of vertices at v: $\ddddot{v}_\ell, \overline{v_\ell}, v, \ddddot{v}_r, \overline{v_r}$
- $\ddddot{P}^*$ uses $\ddddot{v}_r$, intersection
- Order of vertices at w: $\ddddot{w}_\ell, \overline{w_\ell}, w, \ddddot{w}_r, \overline{w_r}$
- $\ddddot{P}^*$ uses $\ddddot{w}_\ell$, no intersection

$\hat{i}(\hat{L}, P^*) = \pm 1$:

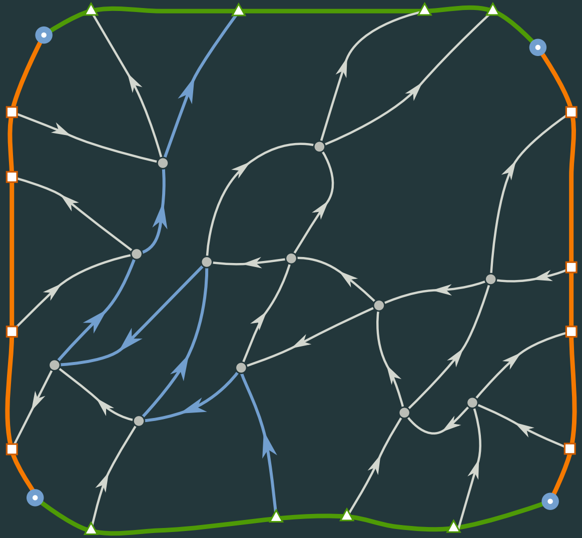
- L^- no transverse intersections with P^*
- L^* starts left side of P^* and ends right side of P^*, net one left to right crossing
- Crossover-edges $(\overline{w_r}, \overline{w_\ell})$ and $(\overline{v_r}, \overline{v_\ell})$ are both right to left crossings

Remove double edge crossings using shortcuts

L intersects any edge of G at most once



Algorithm produces chordless rectangular schema for G



Drawing must be:
(D1.) Planar (D2.) Straight-line (D3.) Integer lattice
(D4.) Straight-frame (D5.) Periodic

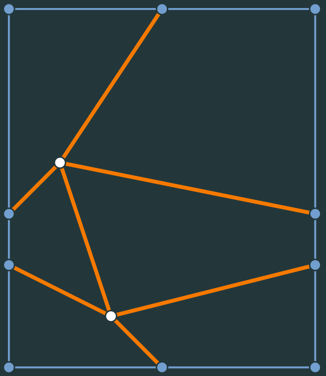
Shift Method

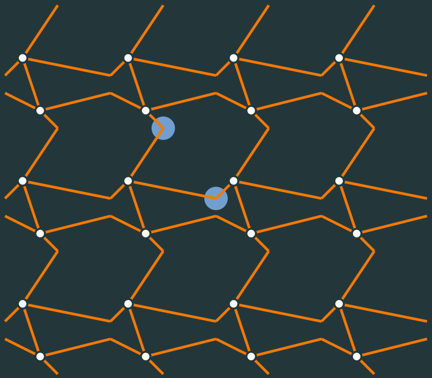
(de Fraysseix, Pach Pollack 1990)
(D1-D3), $O(2n-4) \times O(n-2)$-grid

(Duncan, Goodrich, Koborouv 2011)
(D1-D4), $O(n) \times O(n^2)$-grid

(Aleardy, Fusy, Kostrygin 2014)
(D1-D5), $O(n^4) \times O(n^4)$-grid

Open problems





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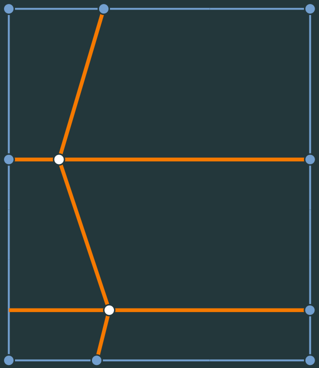

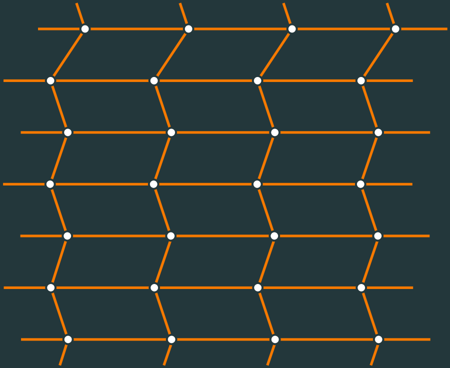

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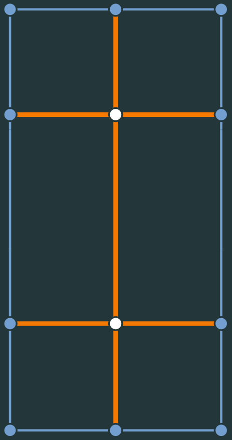
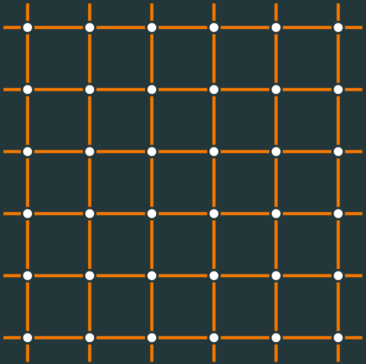
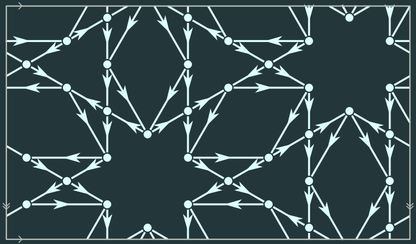
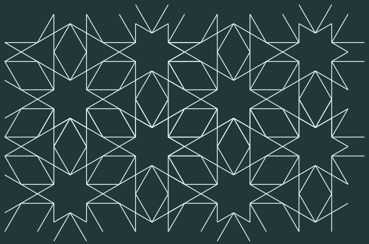
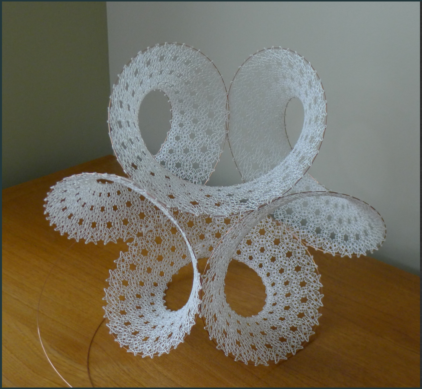
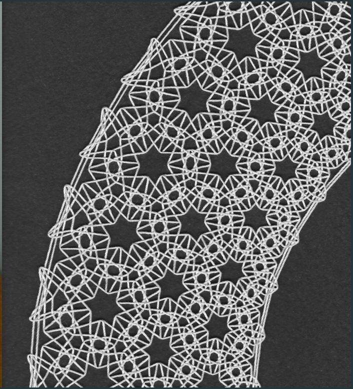








Thank you

http://tesselace.com

Better results