On Smooth Orthogonal and Octilinear Drawings: Relations, Complexity and Kandinsky Drawings

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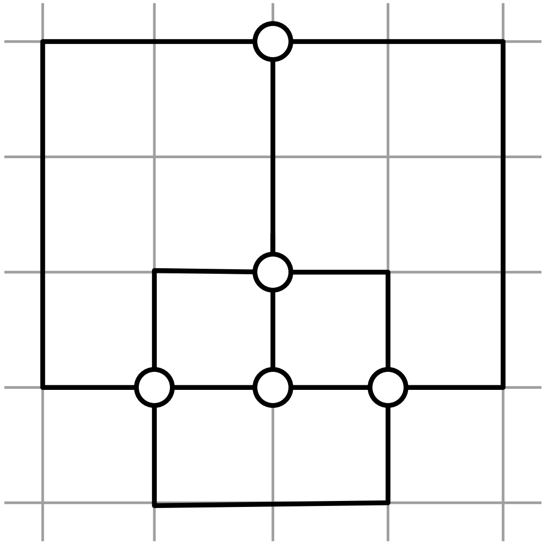
Universität Tübingen, Germany

Smooth Orthogonal and Octilinear Drawings

- Extensions of the orthogonal graph drawing model



Orthogonal
Complexity 3

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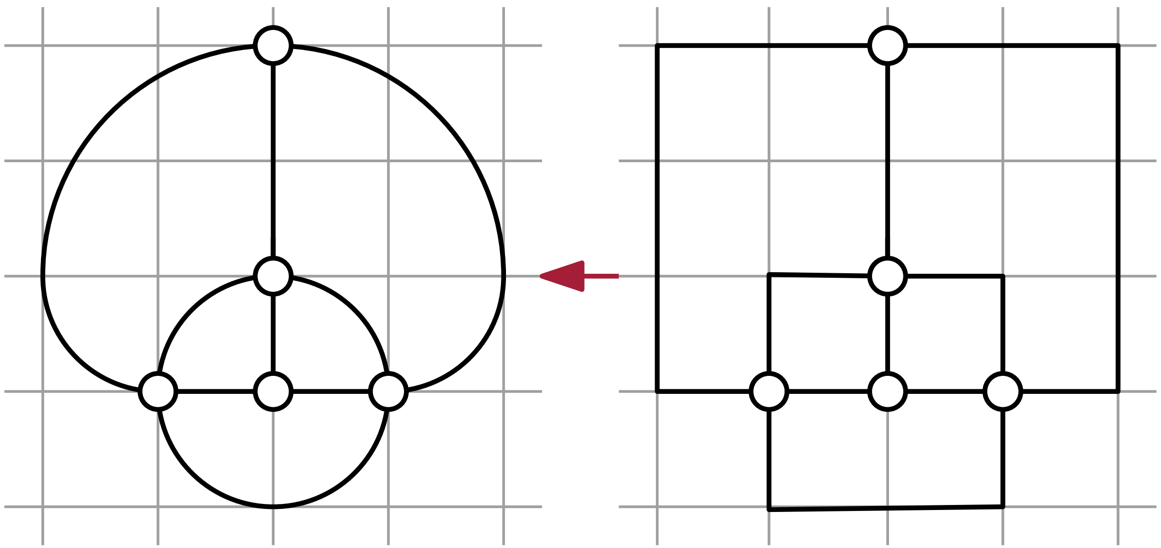

Smooth Orthogonal
Complexity 2

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- Smooth orthogonal: Clarity of orthogonal layouts
  + Aesthetics of Lombardi drawings

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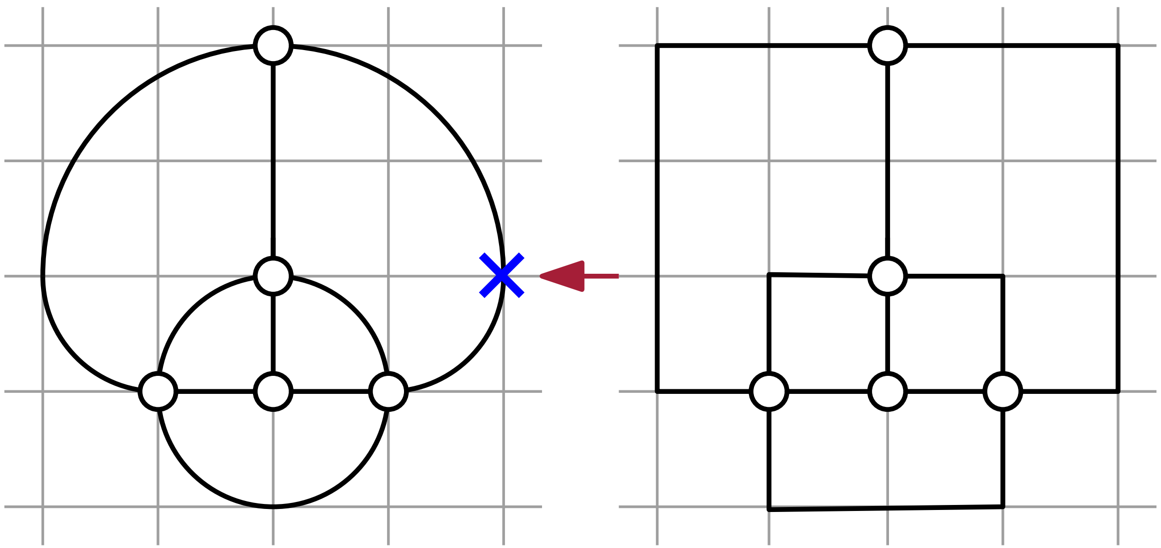

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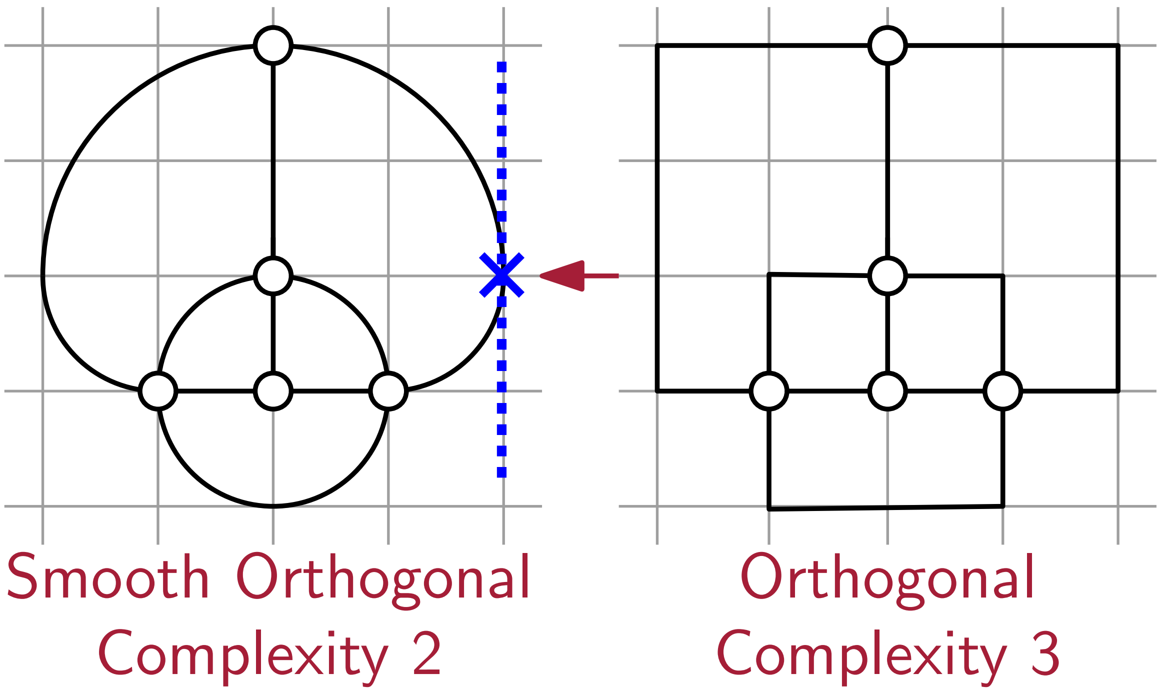

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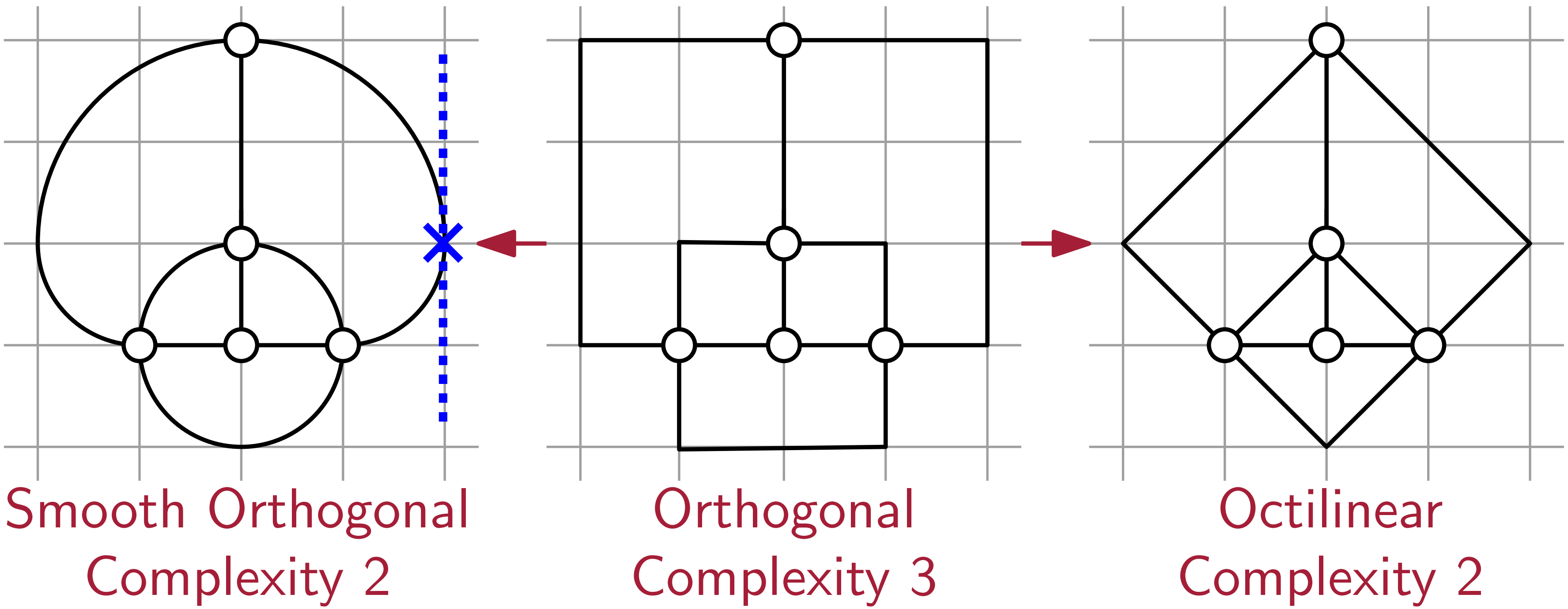

- Smooth orthogonal: Clarity of orthogonal layouts
  + Aesthetics of Lombardi drawings
- Octilinear: Generalization to max-degree 8
  + Metromap applications

Known Results

- **Relations**

- **Complexity**

- **Kandinsky Drawings**

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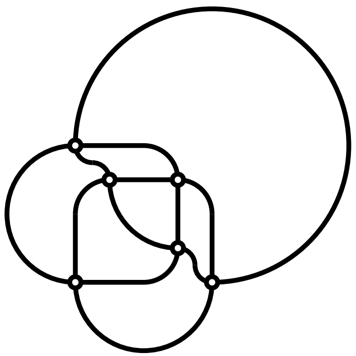
- **Relations**
  - Not all max-degree 4 graphs admit bendless smooth orthogonal/octilinear drawings [Bekos et al. 2013, Bekos et al. 2017]



  - 1 bend per edge suffices for max-degree 4 graphs in both models [Alam et al. 2014, Bekos et al. 2015]
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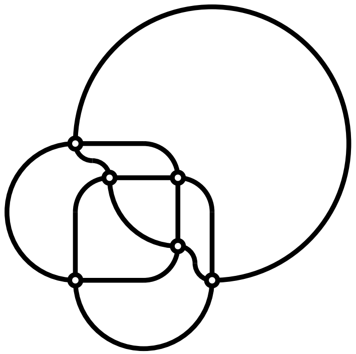

- **Complexity**
  - Bendless octilinear drawing problem $\mathcal{NP}$-hard on max-degree 8 graphs
    [Nöllenburg 2005]
- **Kandinsky Drawings**

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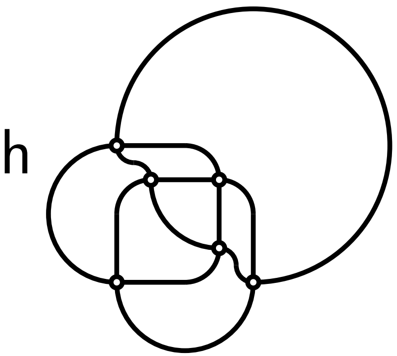

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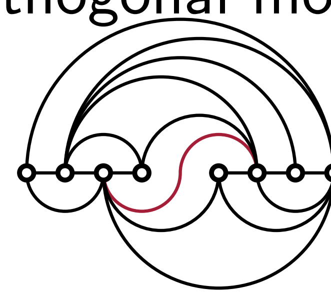
- **Kandinsky Drawings**
  - Book embedding inspired approach for smooth orthogonal model ($< n$ edges with edges of complexity 2)
    [Bekos et al. 2013, Cardinal et al. 2015]



Our Contribution

- **Relations**

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  - Classes of bendless smooth orthogonal drawable (SC_1) and octilinear drawable ($8C_1$) graphs are incomparable
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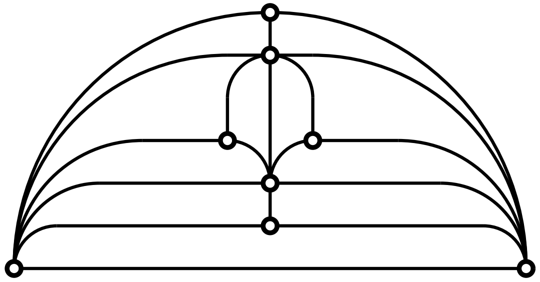
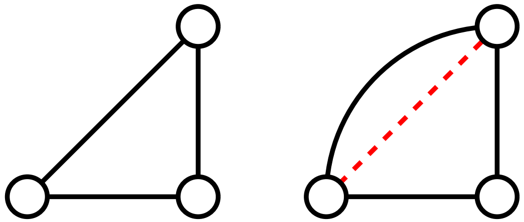
- **Kandinsky Drawings**
  - Smooth orthogonal: Alternative approach producing aesthetically more pleasing drawings
  - Octilinear: First results



Relations

- Bendless smooth orthogonal and octilinear drawings require same endpoint positioning



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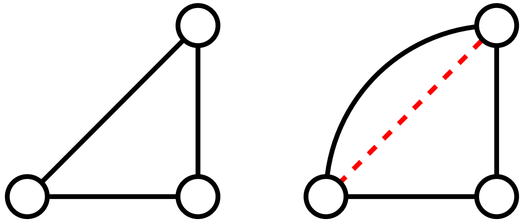
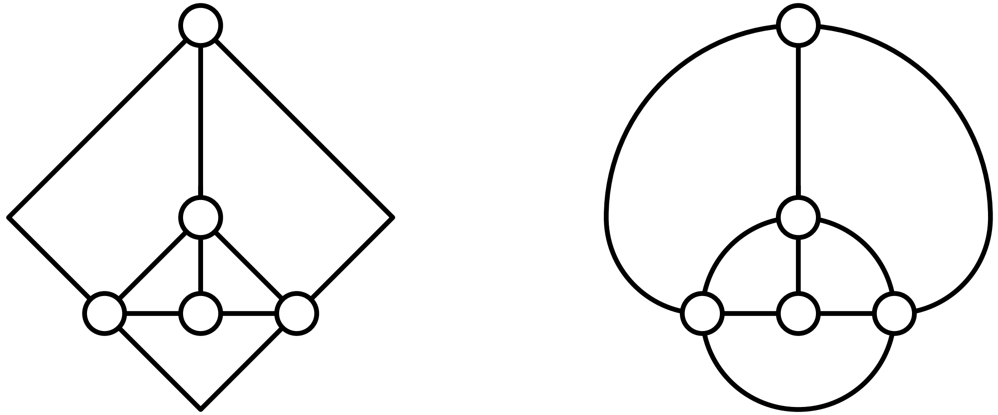


- Idea: Replace arcs with diagonals and vice versa



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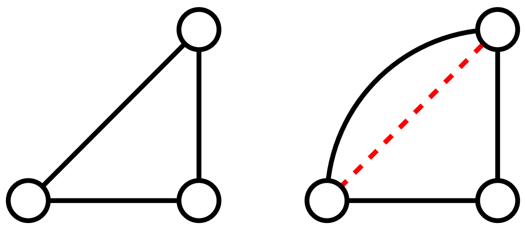

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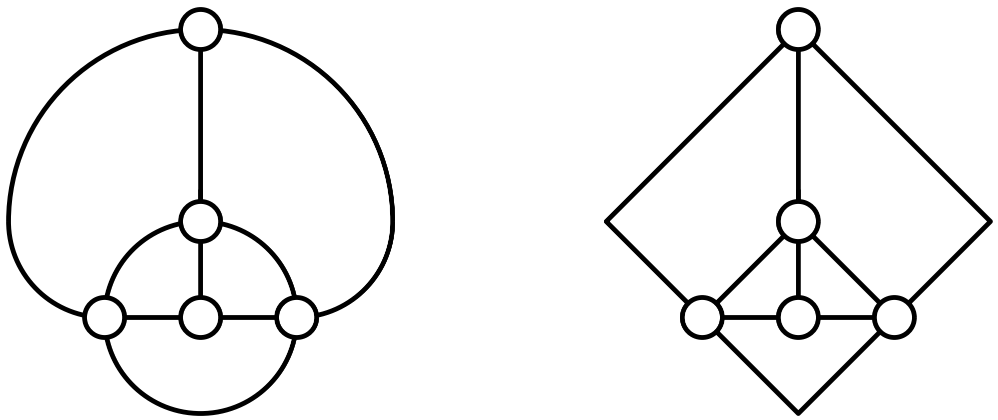

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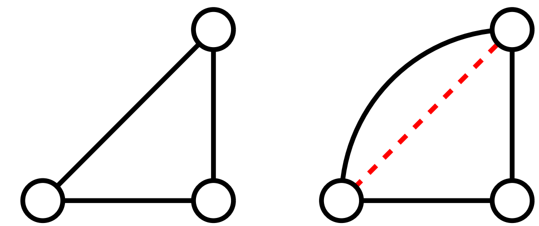

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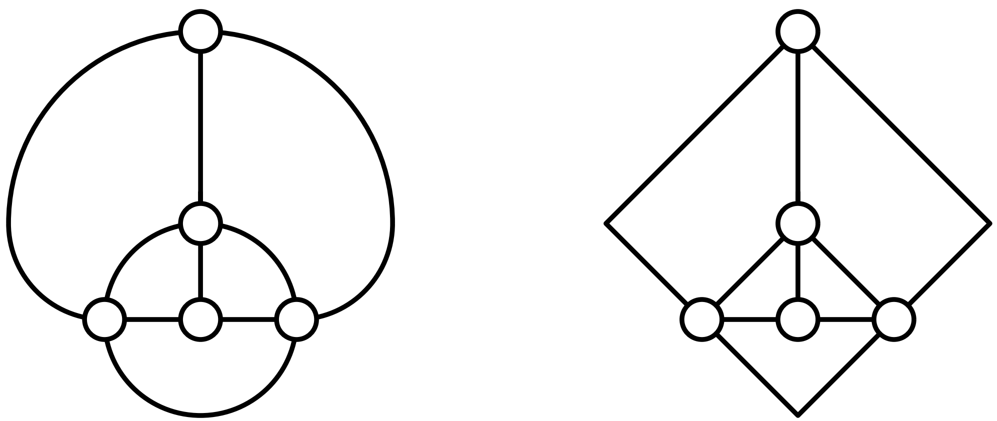

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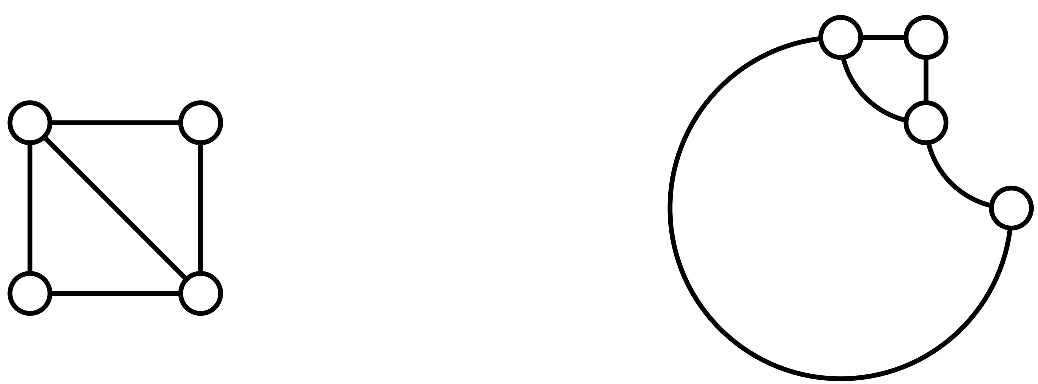

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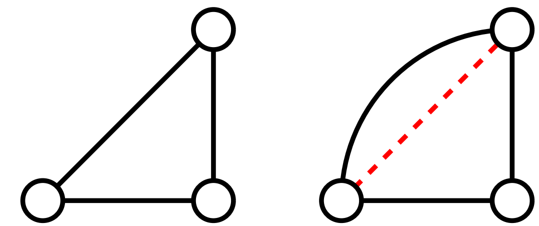

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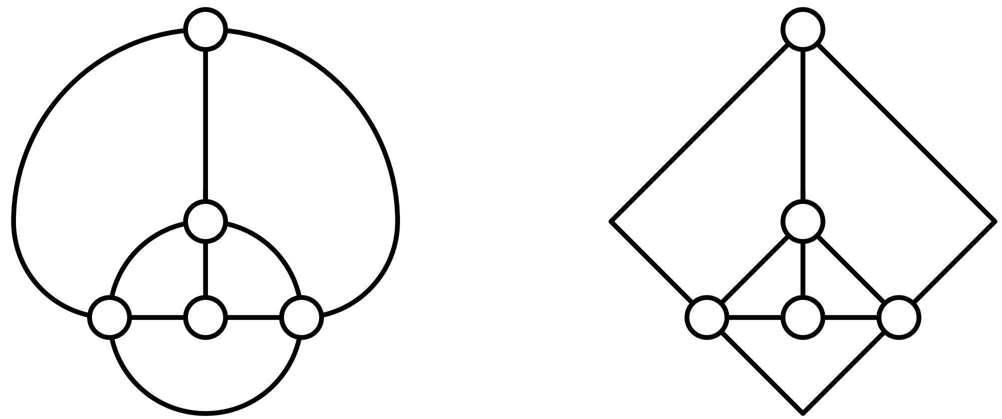

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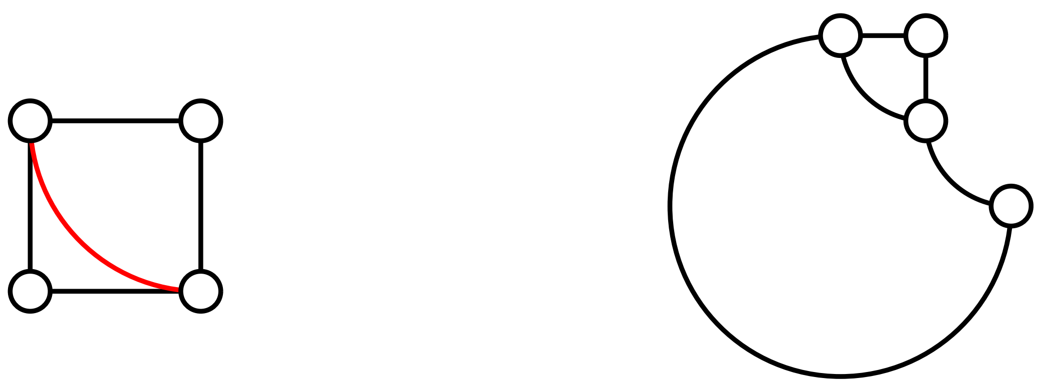

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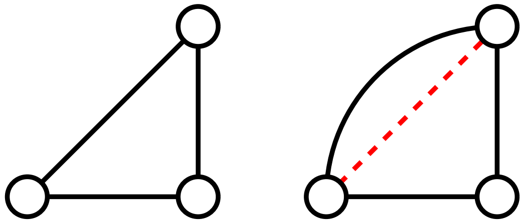

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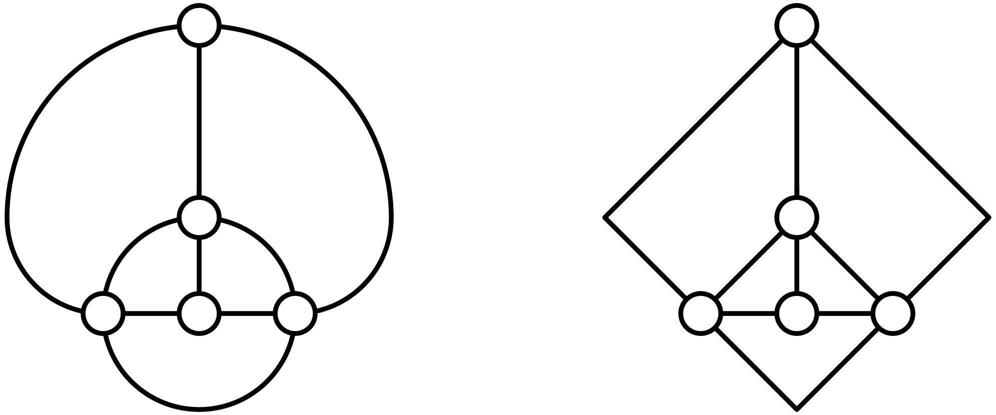

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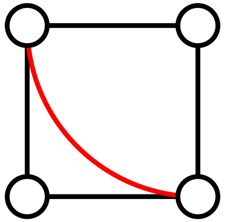

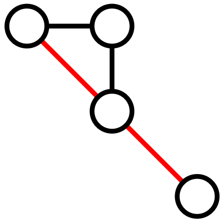

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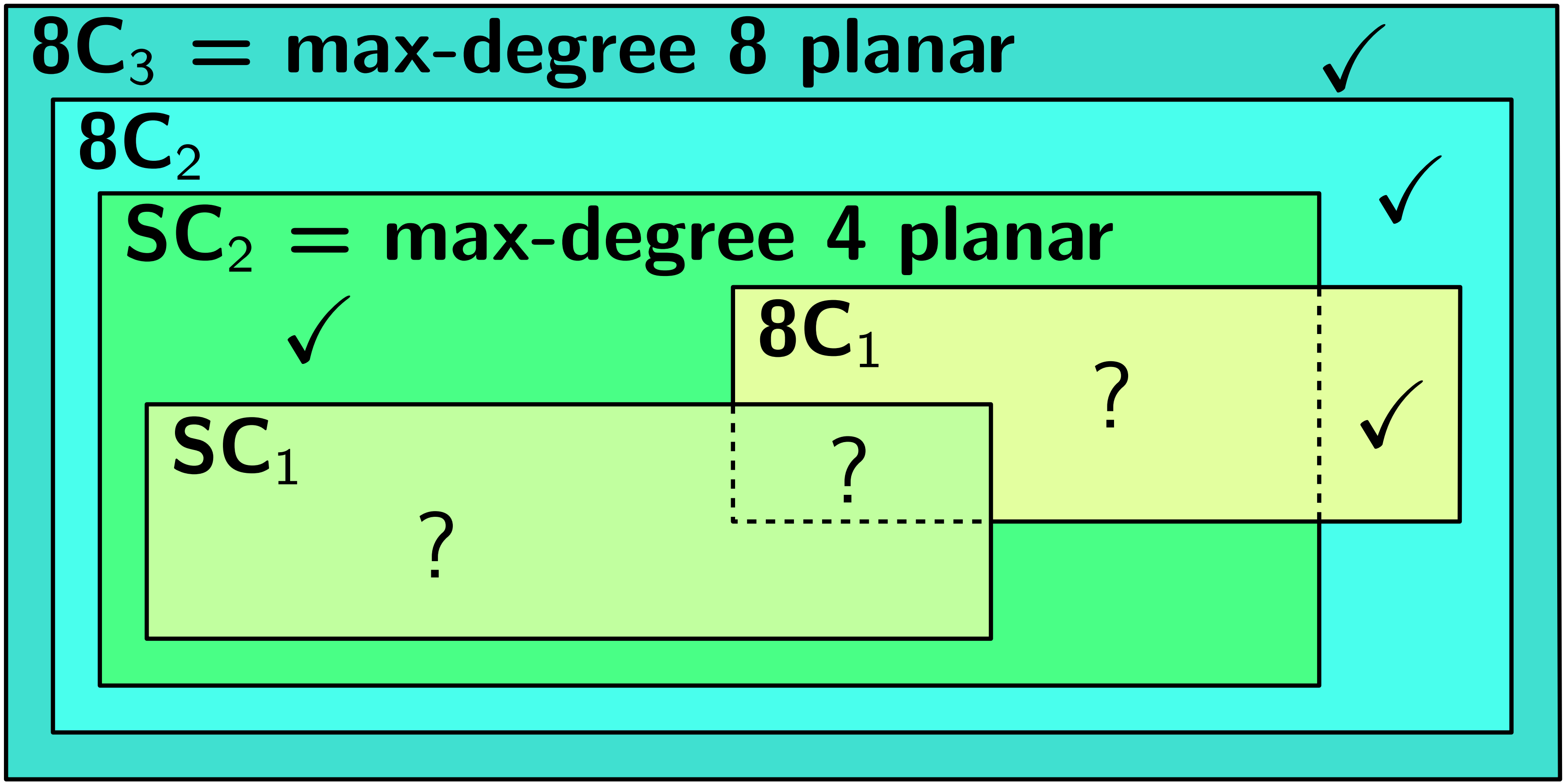
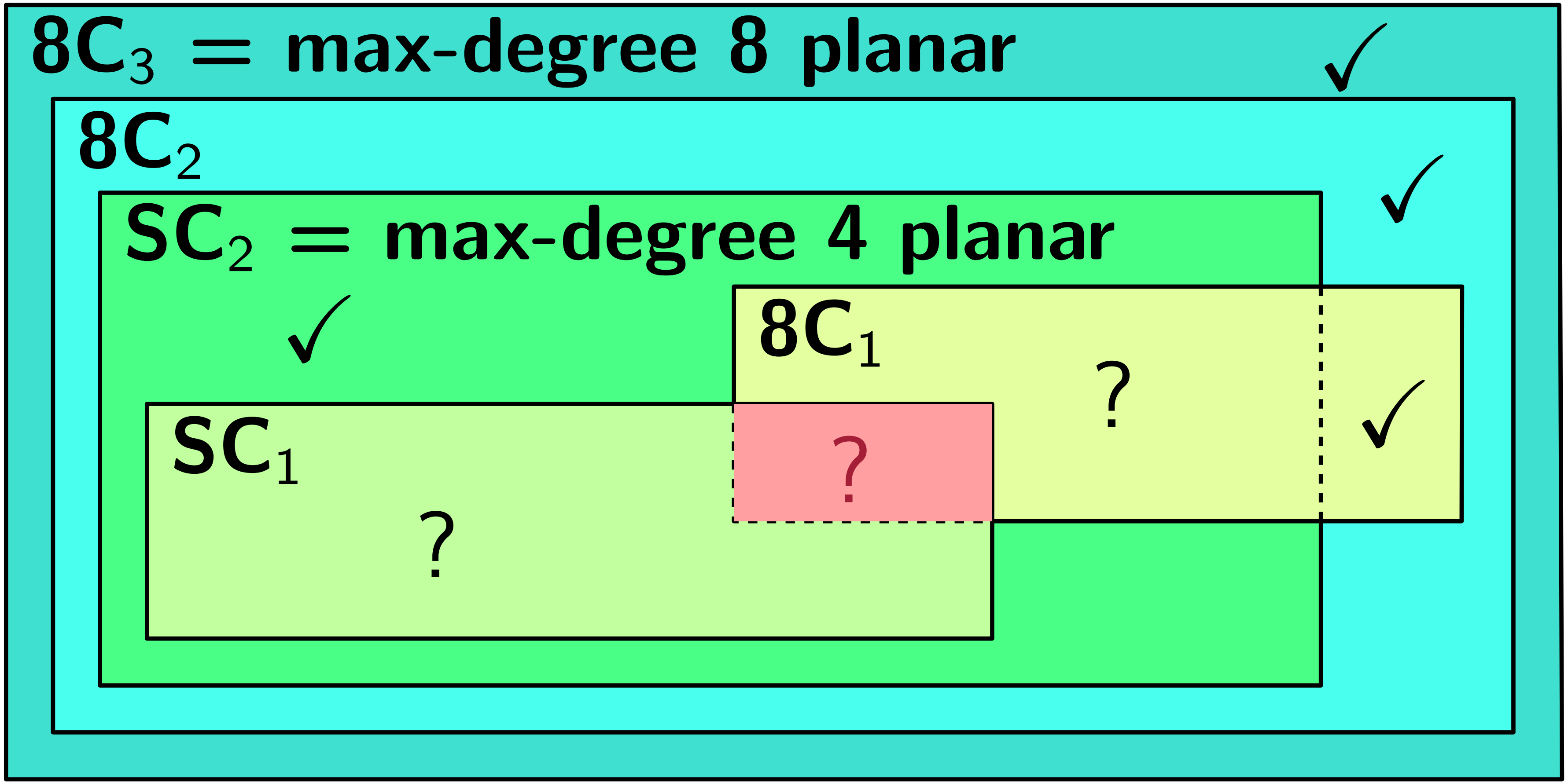


$8C_k$ = Graphs drawable with octilinear complexity k

SC_k = Graphs drawable with smooth orthogonal complexity k

Relations



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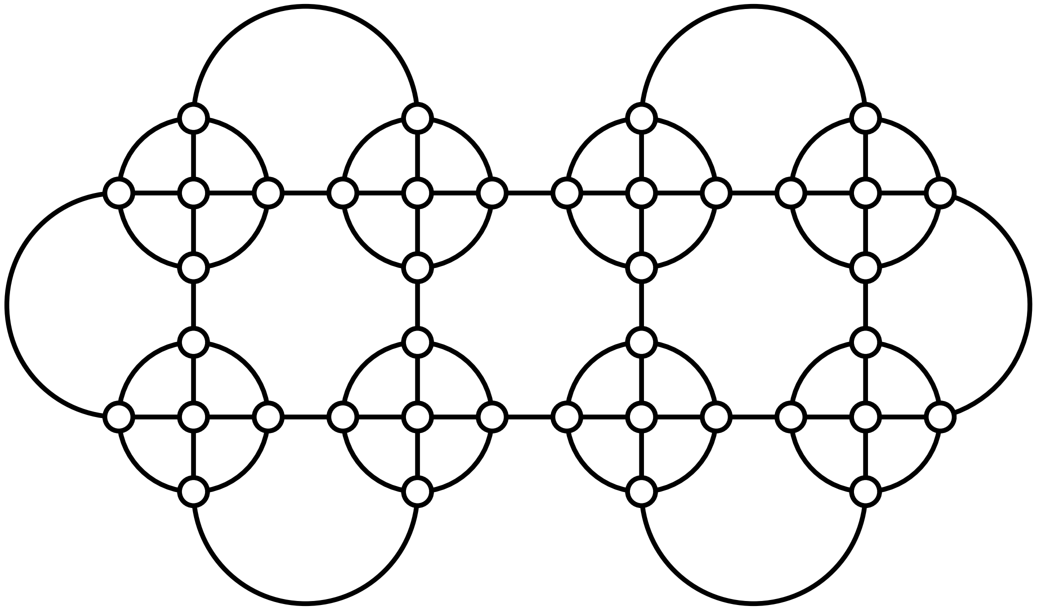
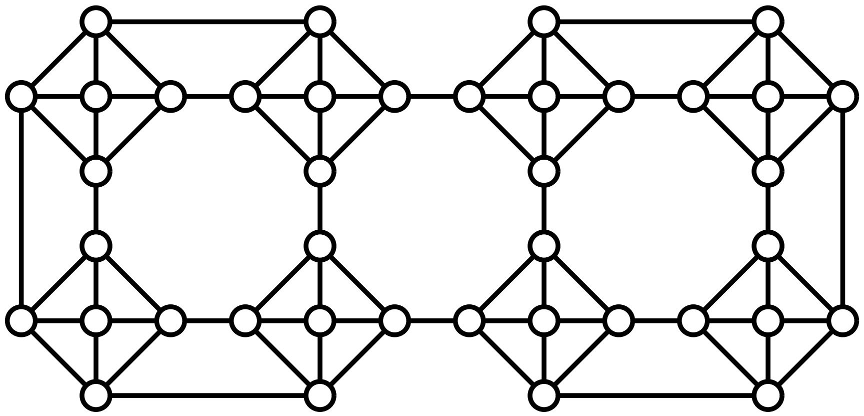
Intersection of SC_1 and $8C_1$

- Infinitely large graph family drawable with both styles:





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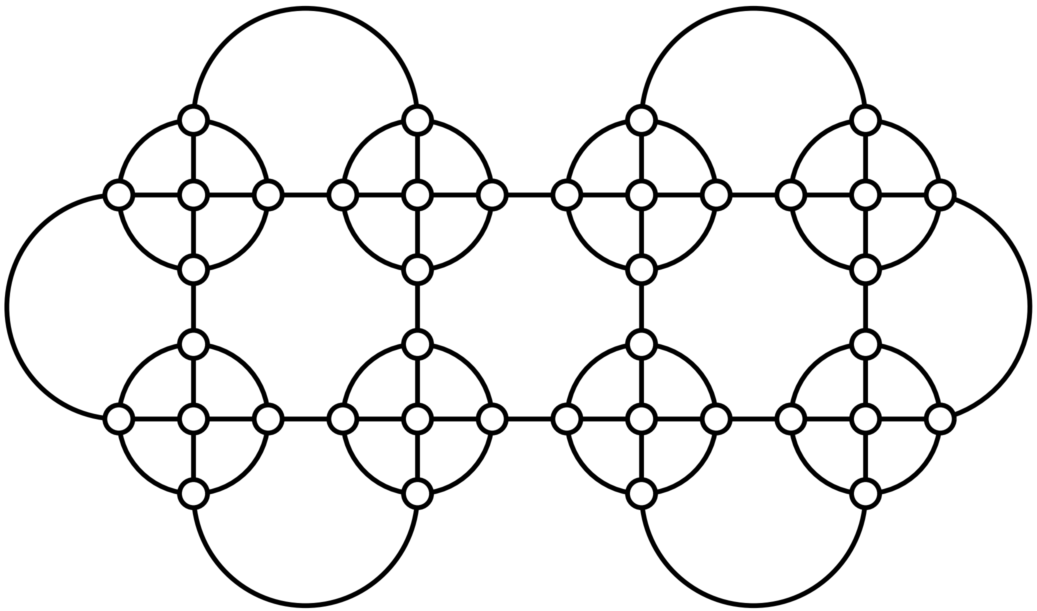
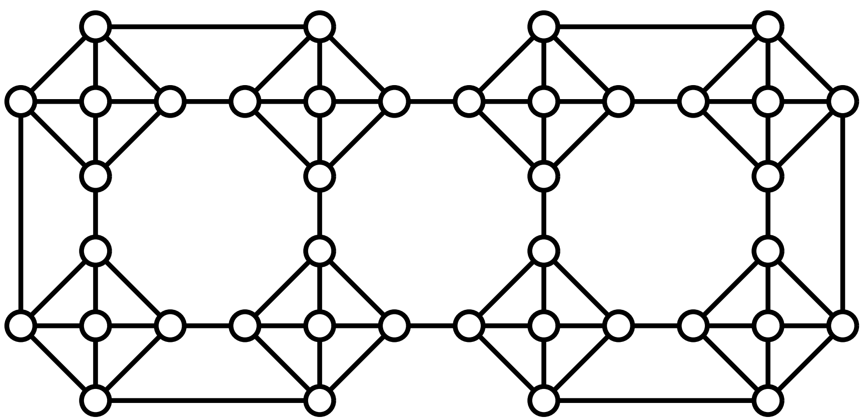
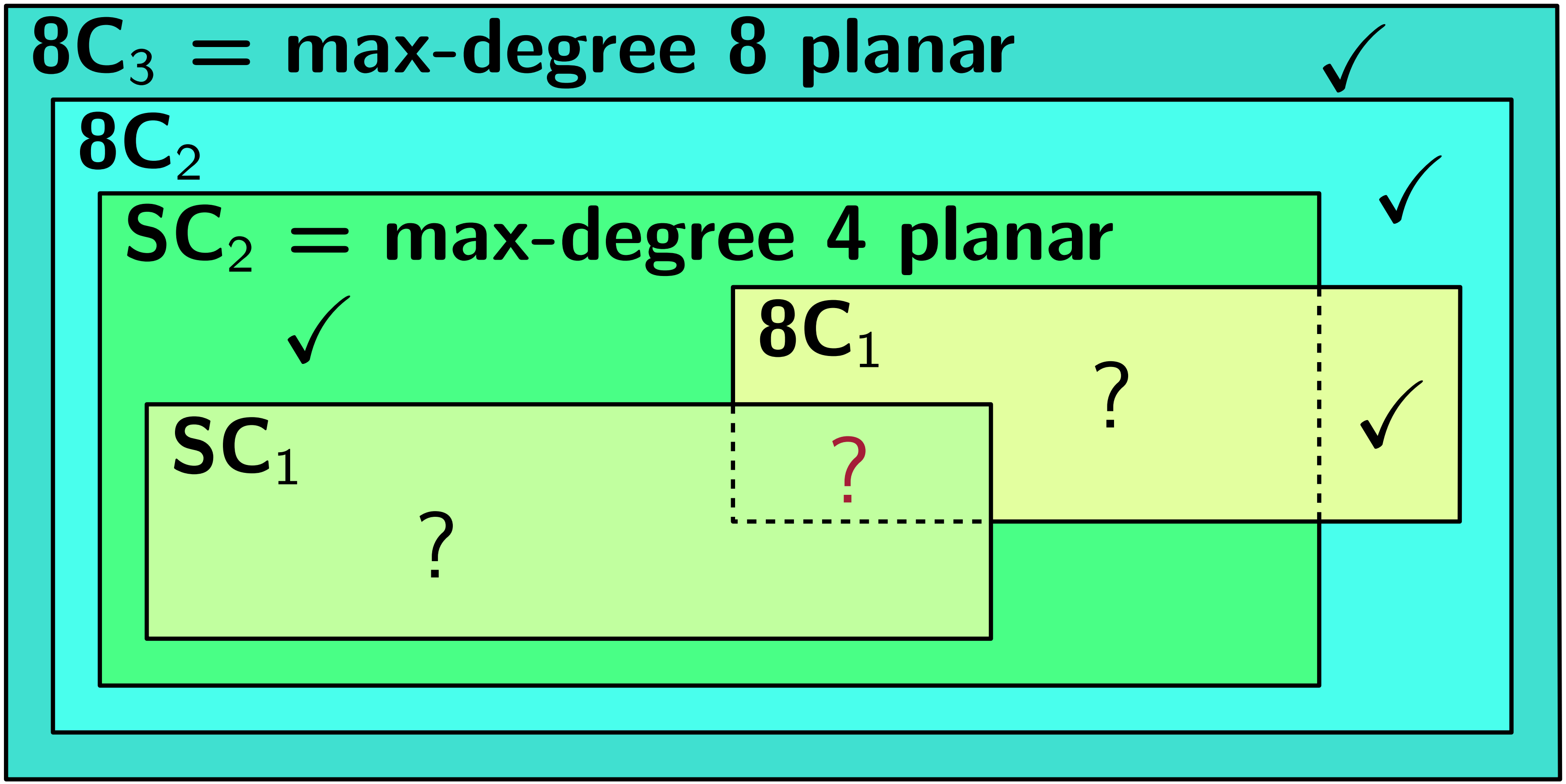




- Family is 4-regular $\rightarrow$ density does not divide classes

Relations



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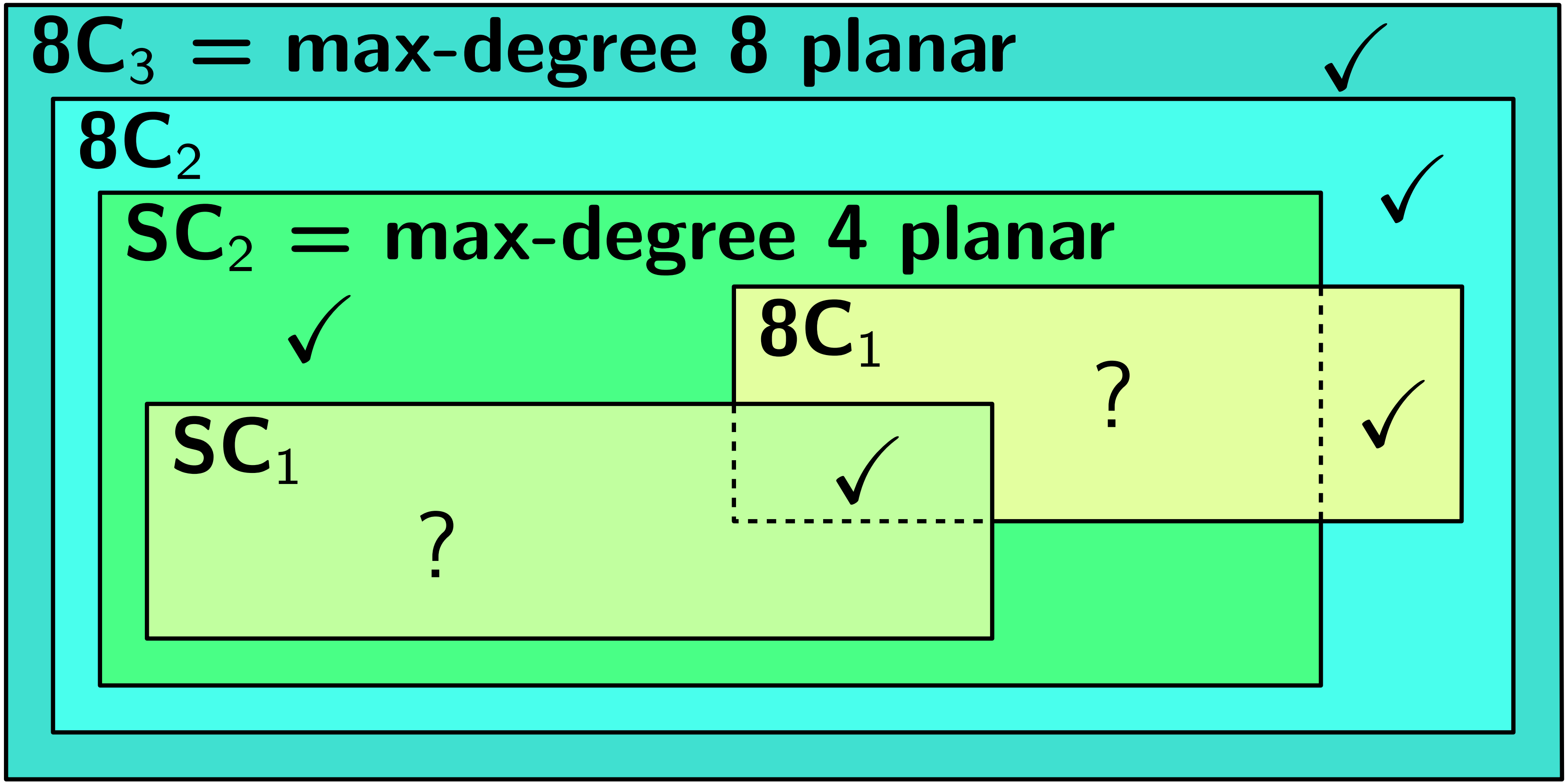

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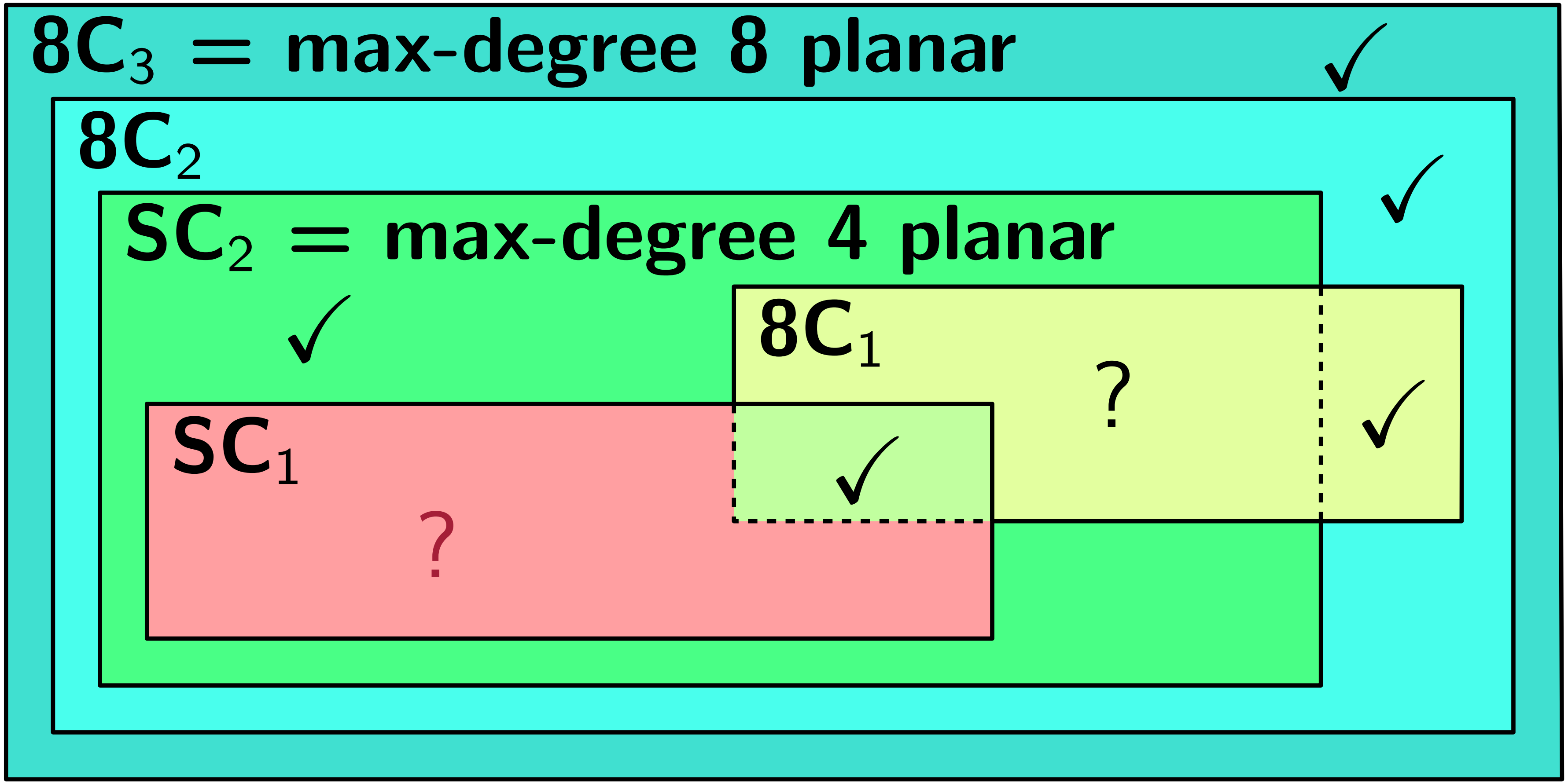

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SC_1 but not $8C_1$

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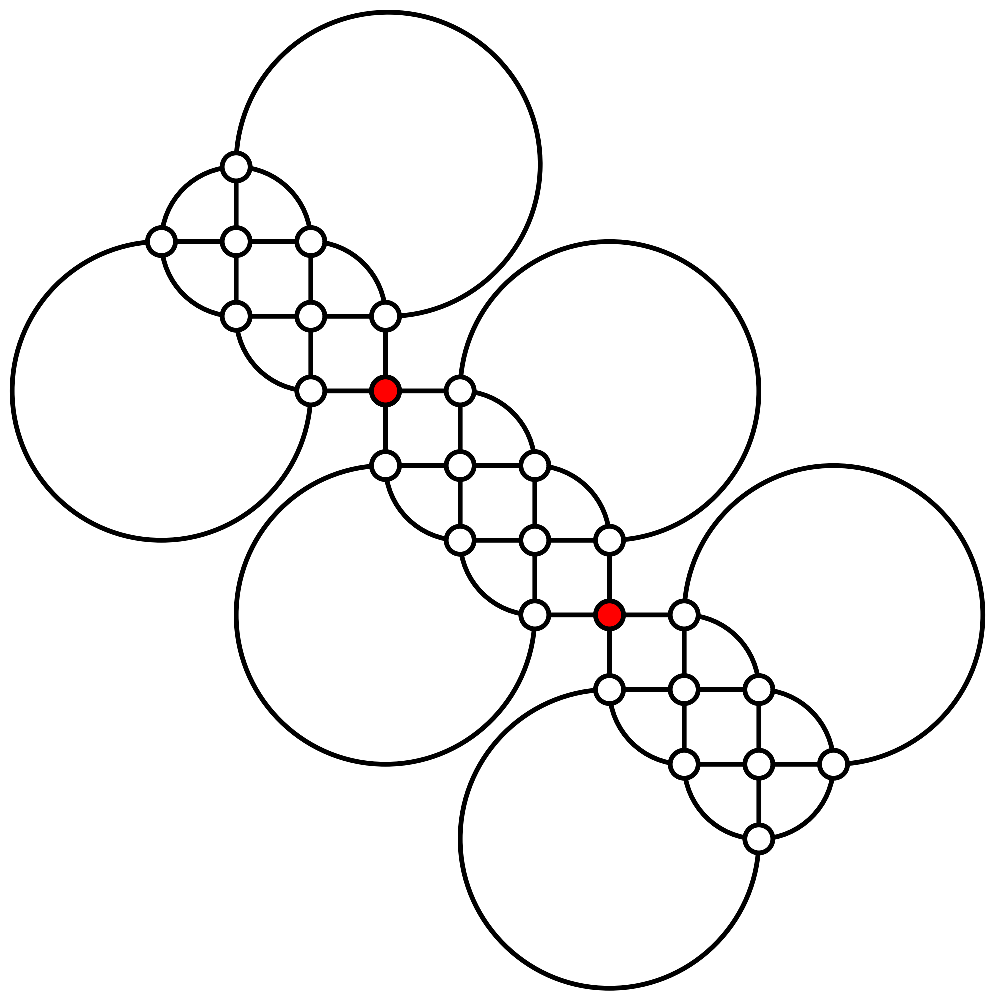

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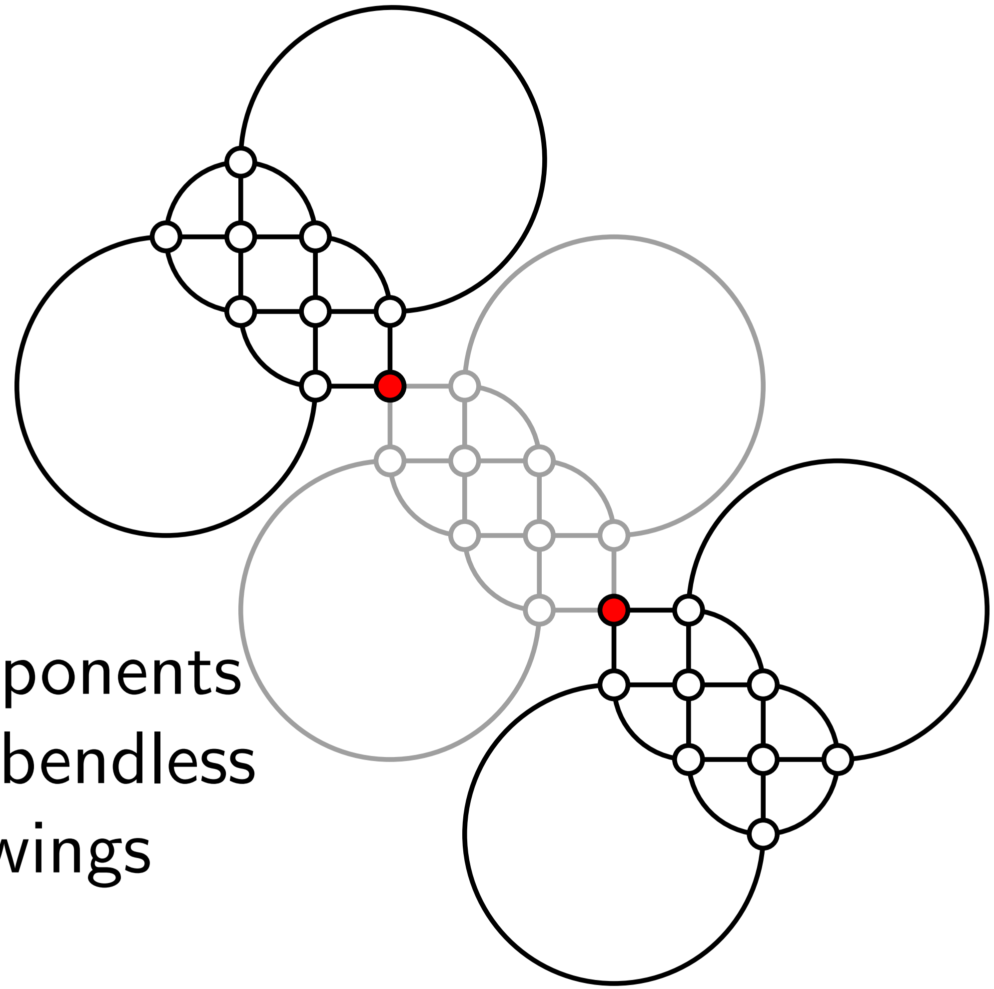
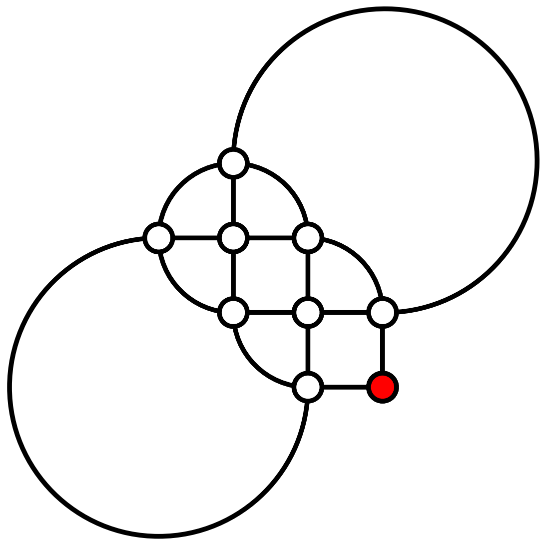
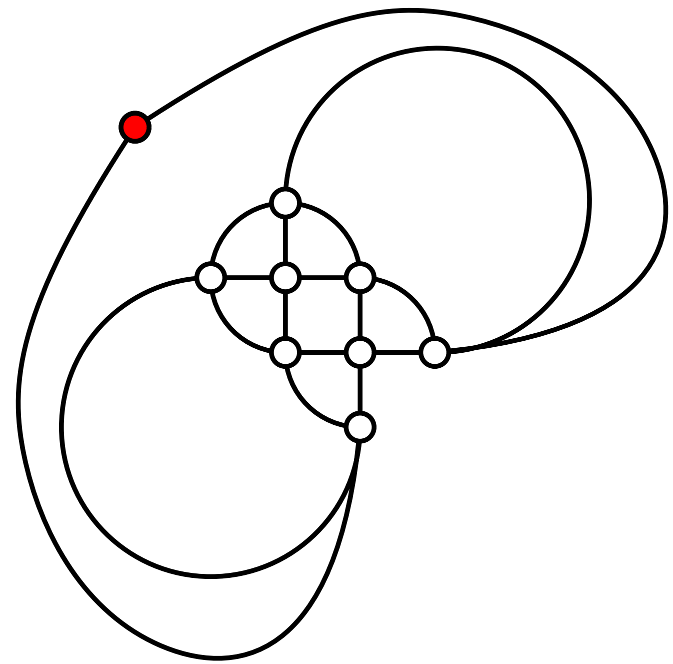


- Two end components do not admit bendless octilinear drawings

SC_1 but not $8C_1$

- End components only have one embedding





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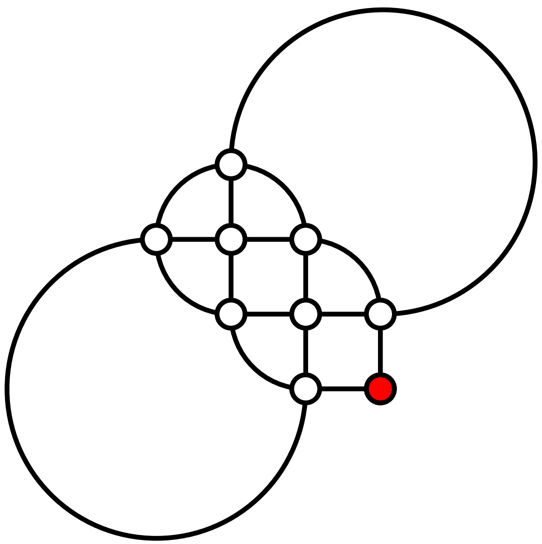

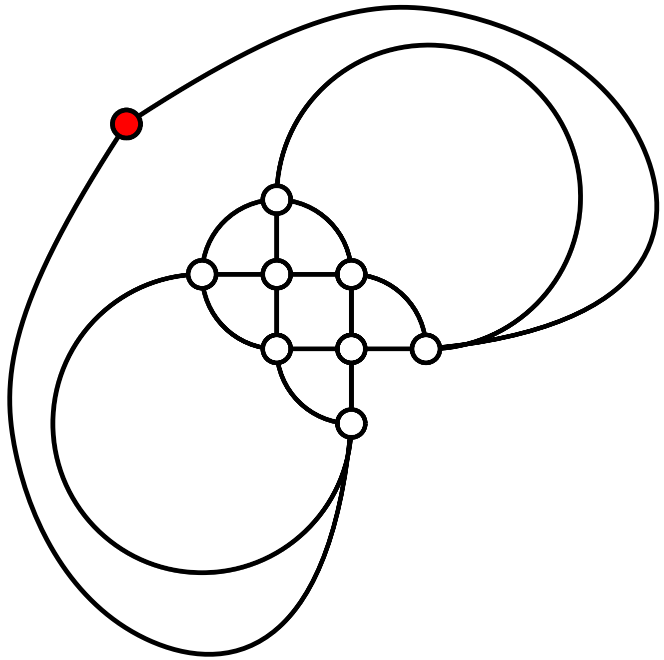

- Properties of this embedding:

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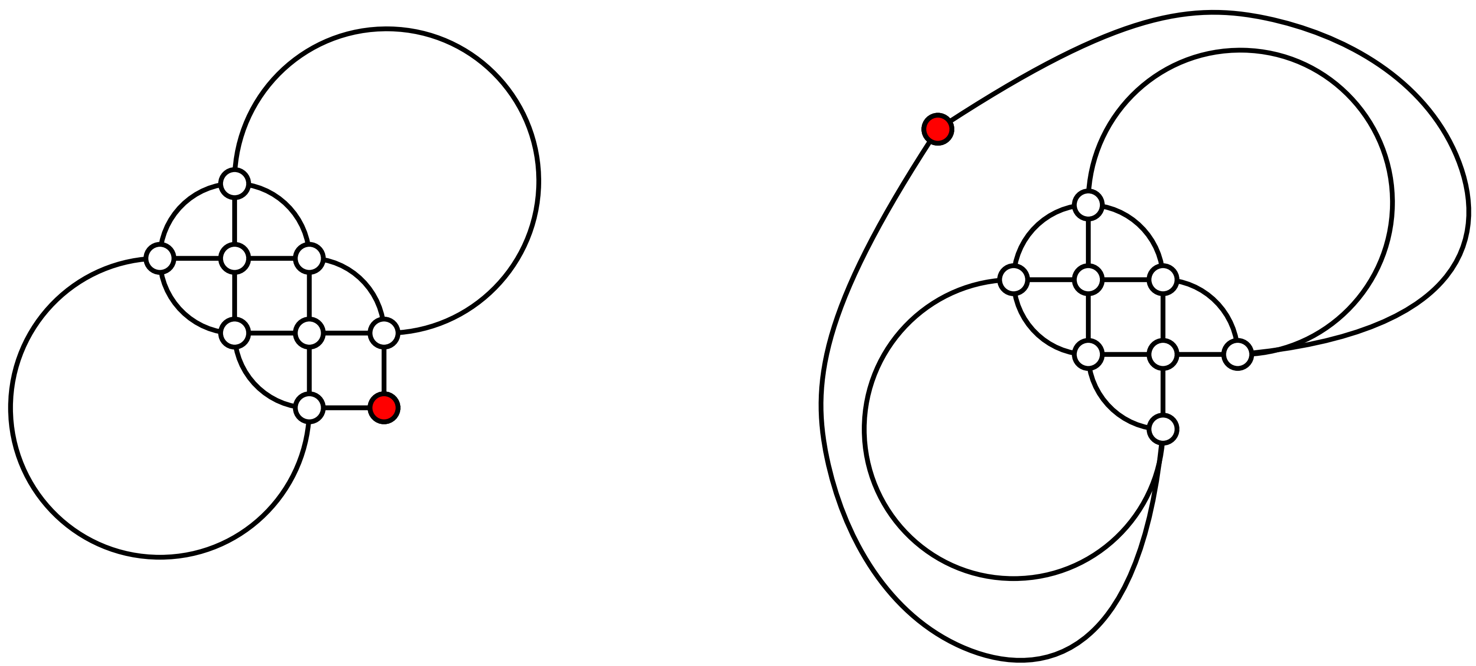

- Properties of this embedding:
  - Each face has length at most 5

SC_1 but not $8C_1$

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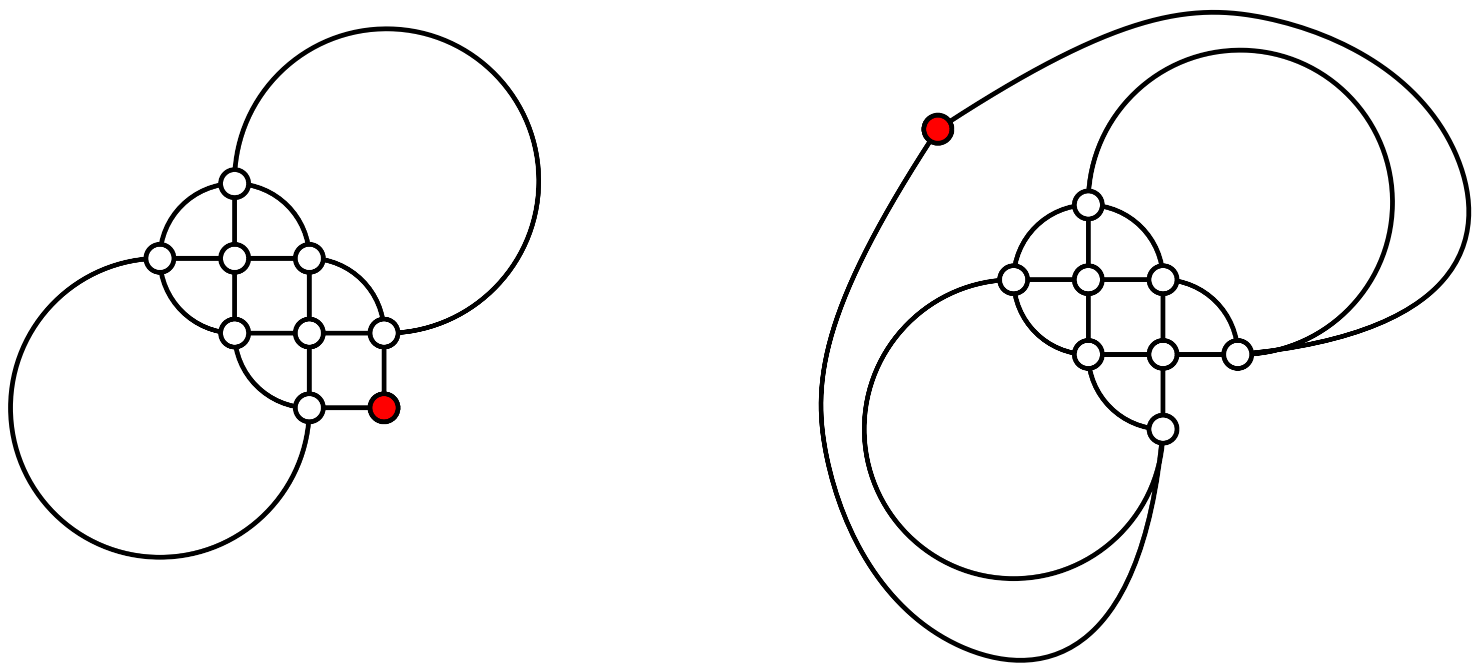
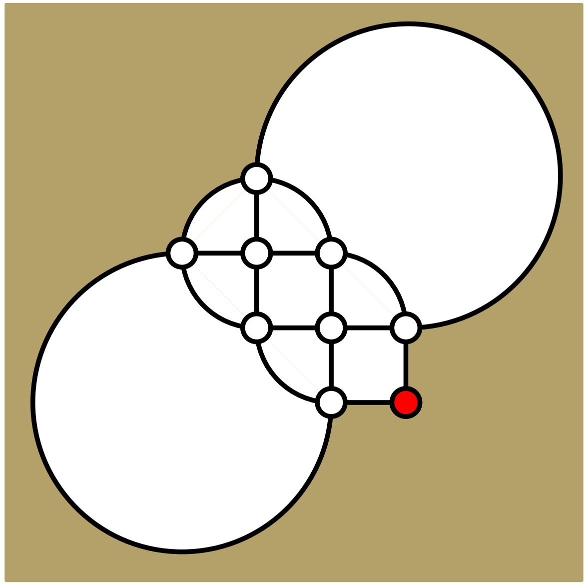


- Properties of this embedding:
  - Each face has length at most 5
  - All but one vertex on the outerface must support two ports to the interior of the drawing

SC_1 but not $8C_1$

- If we try to realize such a drawing, we find, that it is not possible to close the outerface



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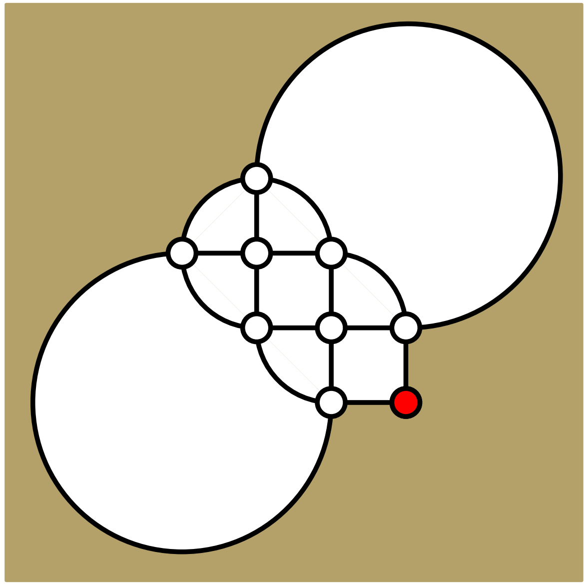

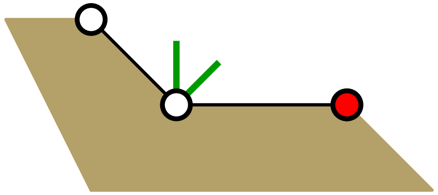

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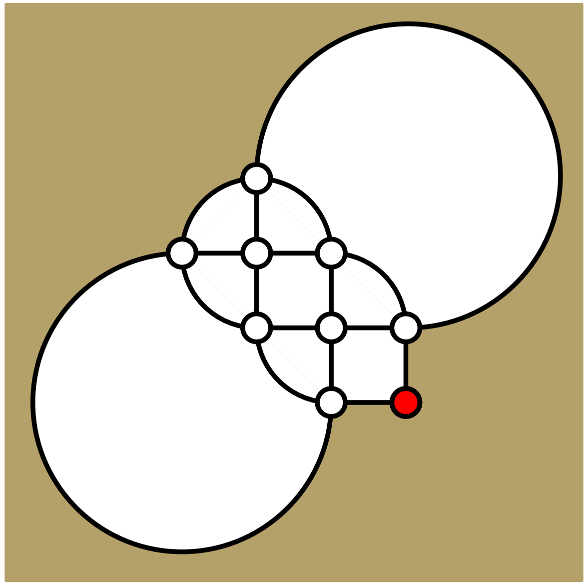

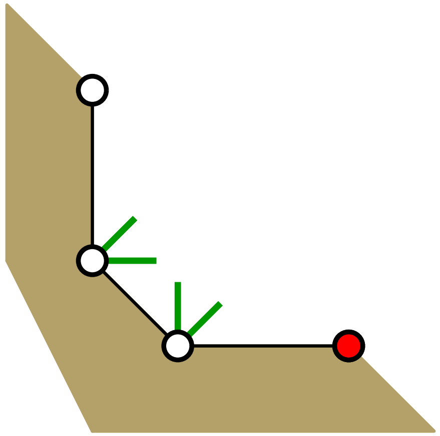

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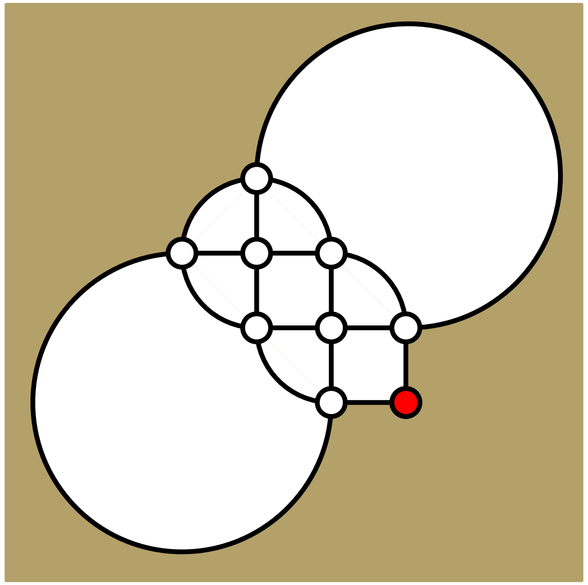

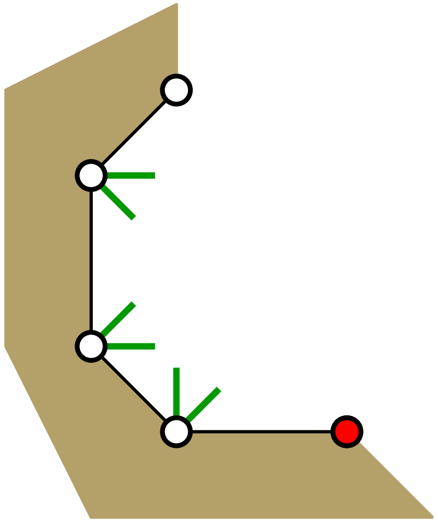

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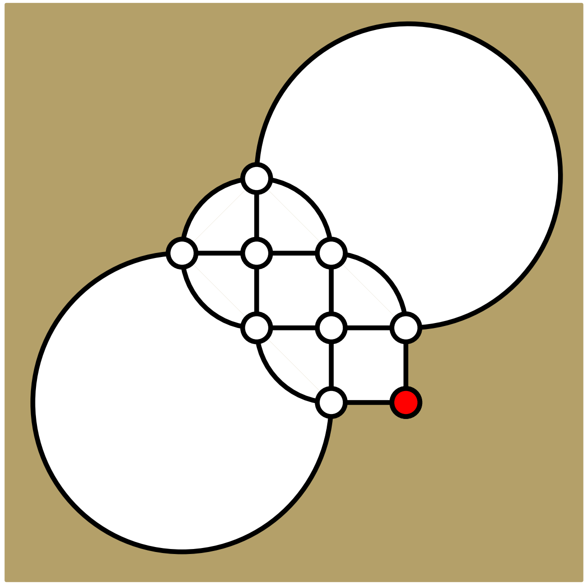

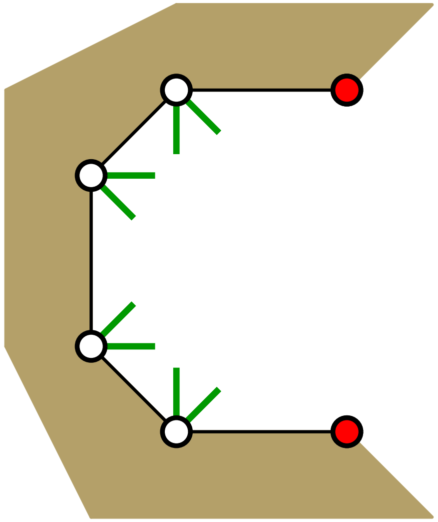

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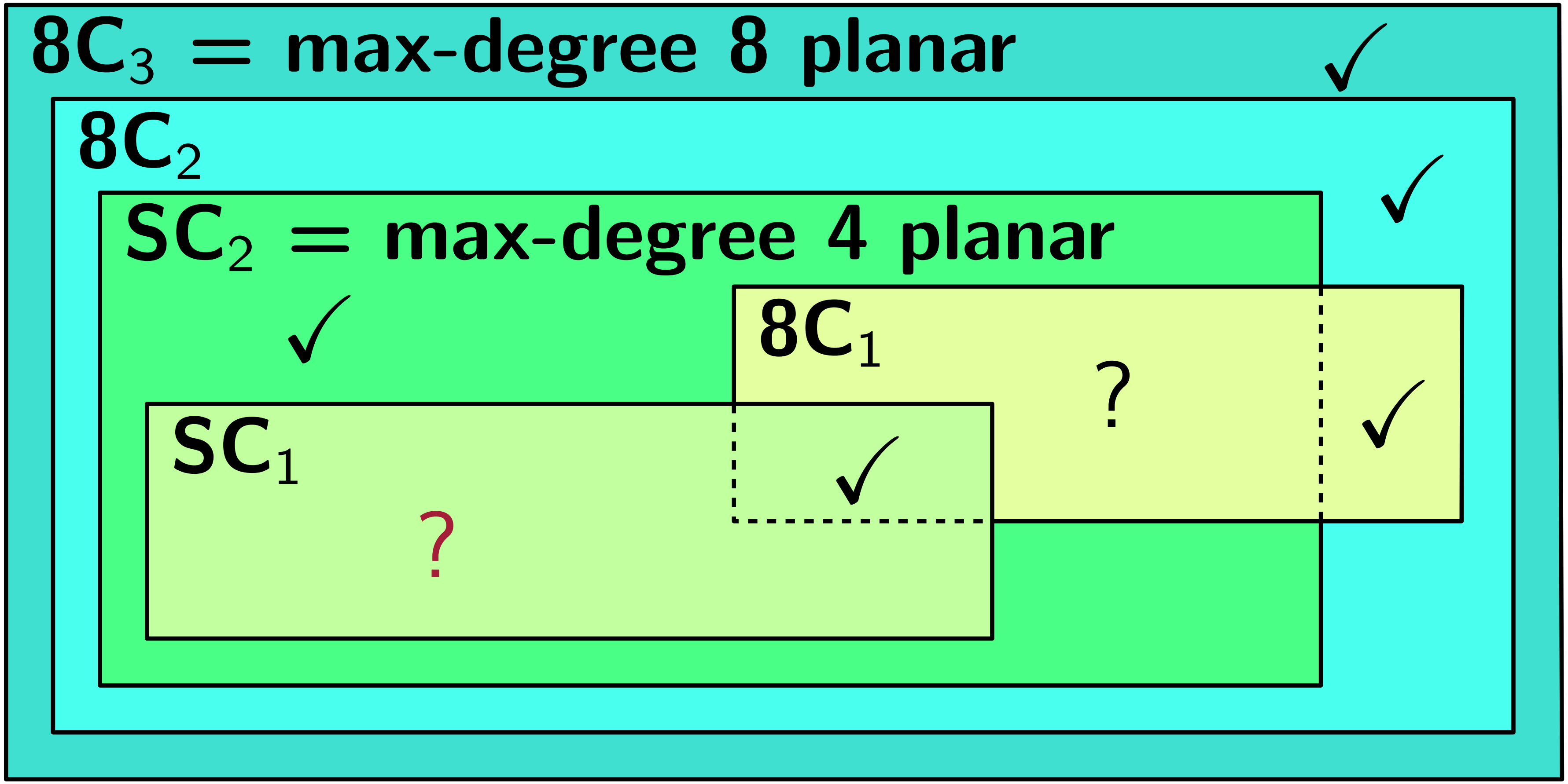

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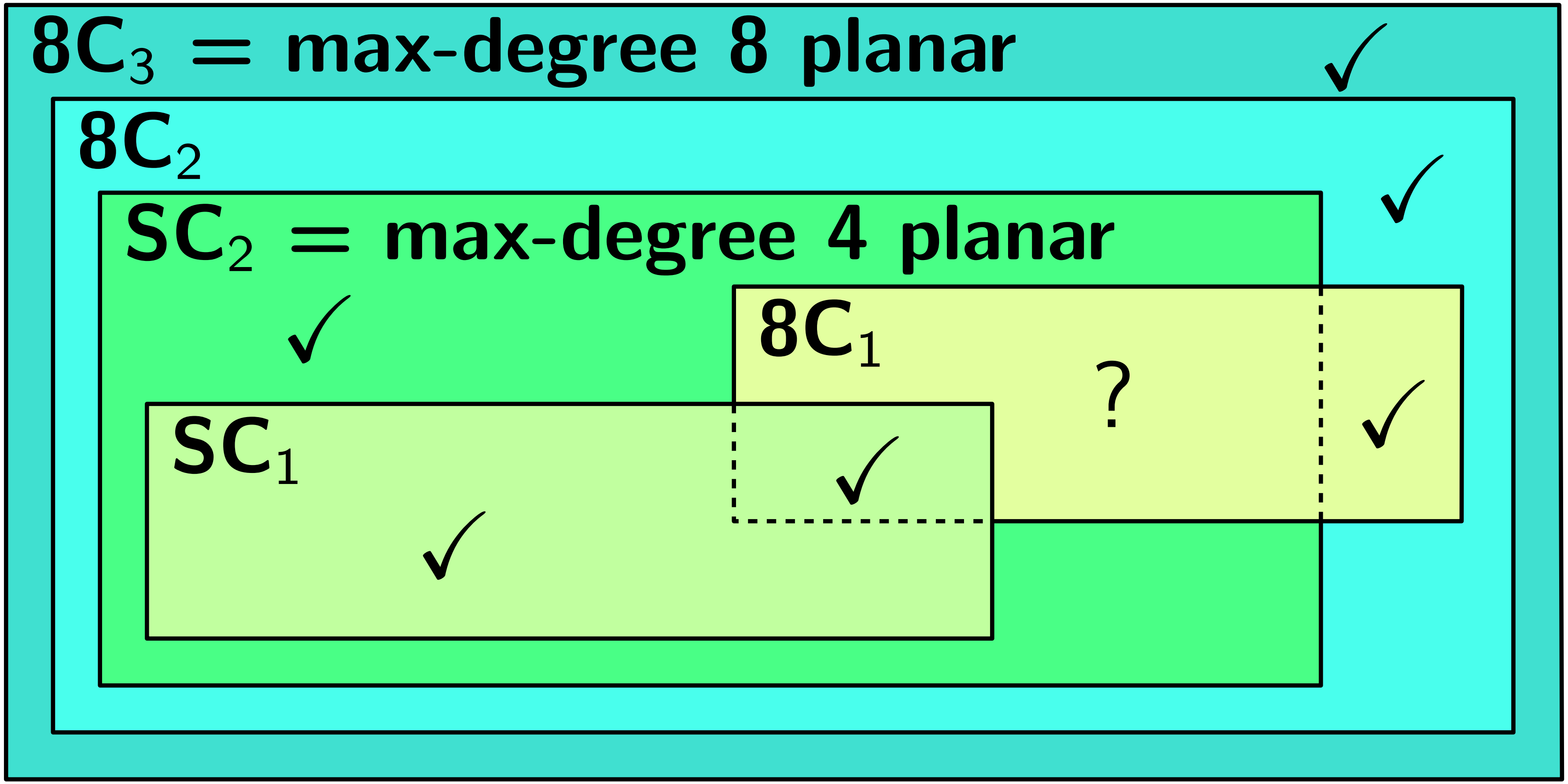

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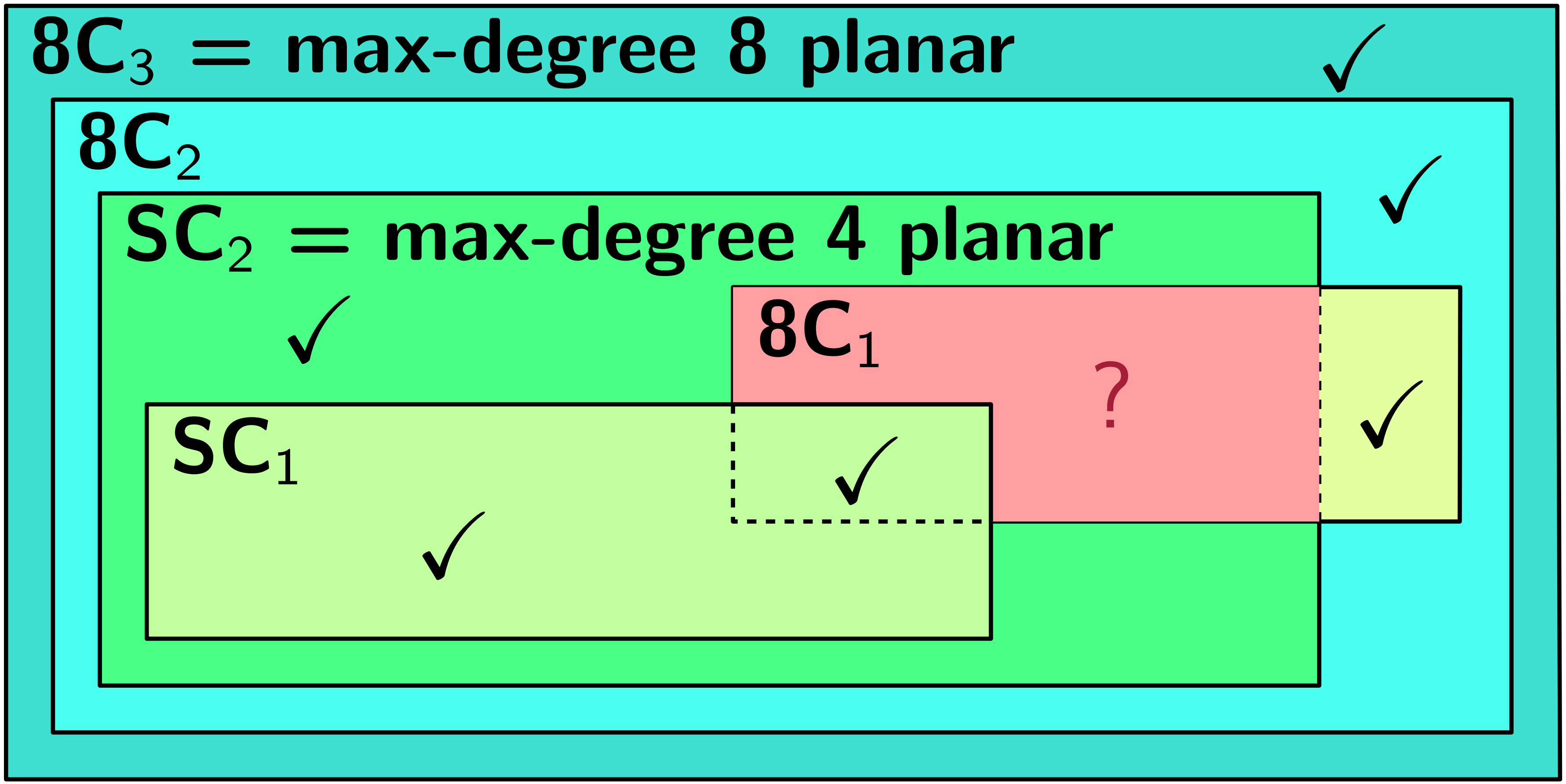

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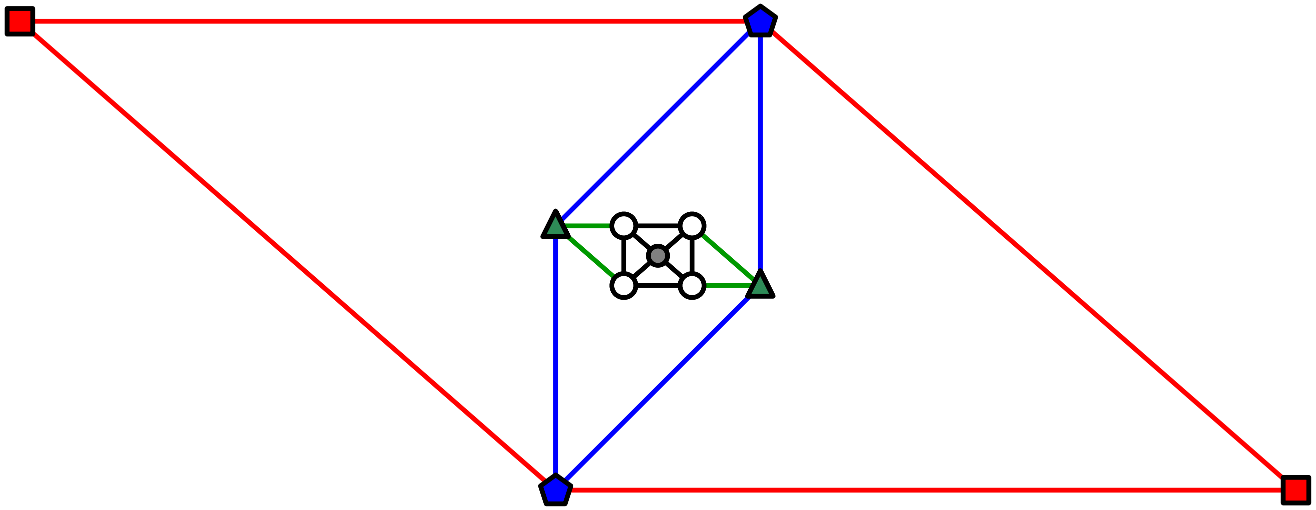

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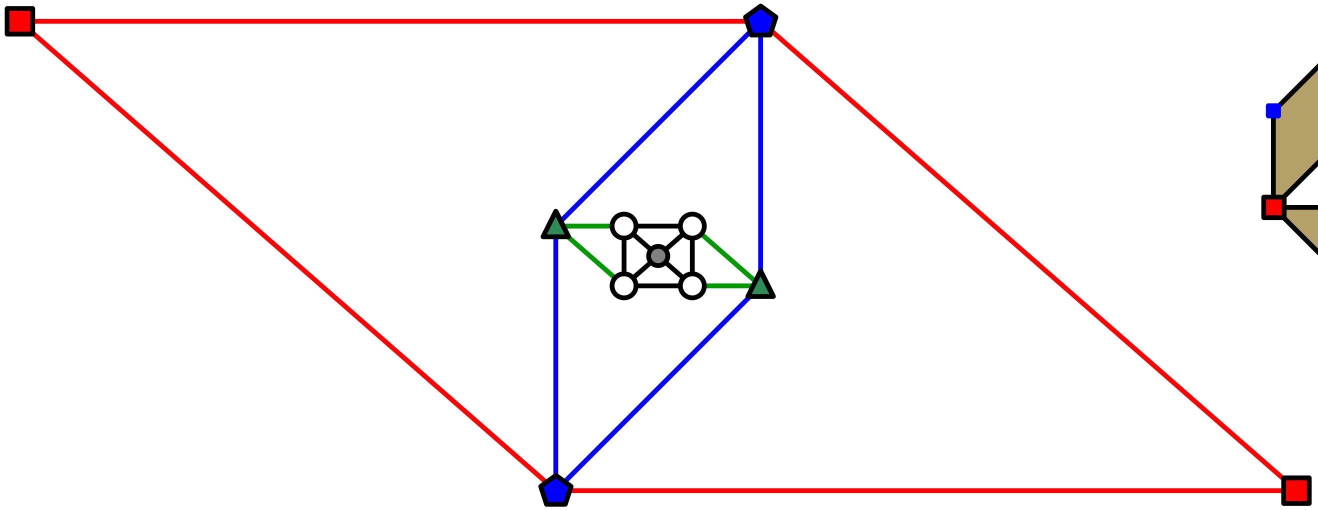

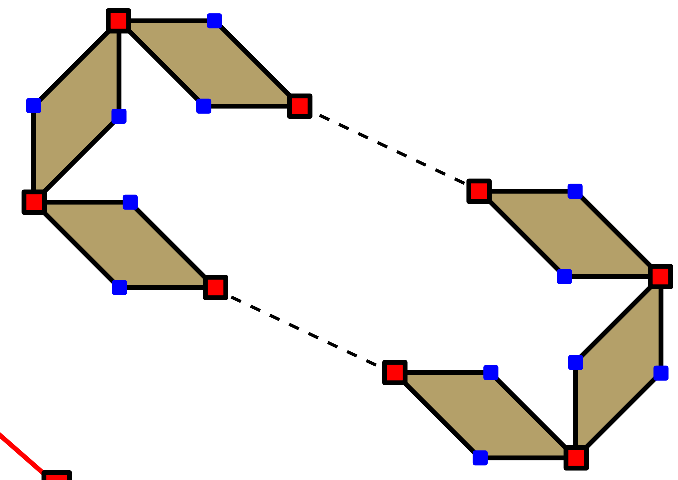

- Multiple copies of a basic component in a cycle

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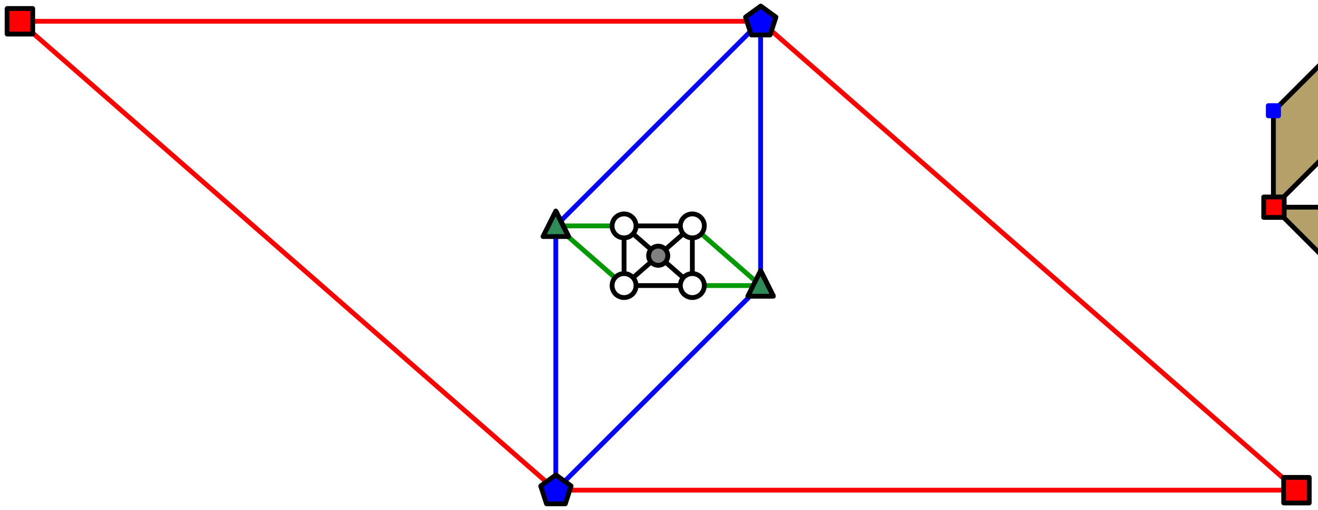

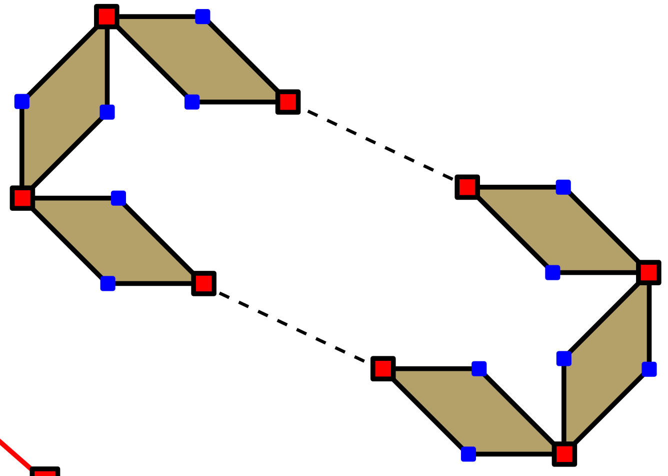

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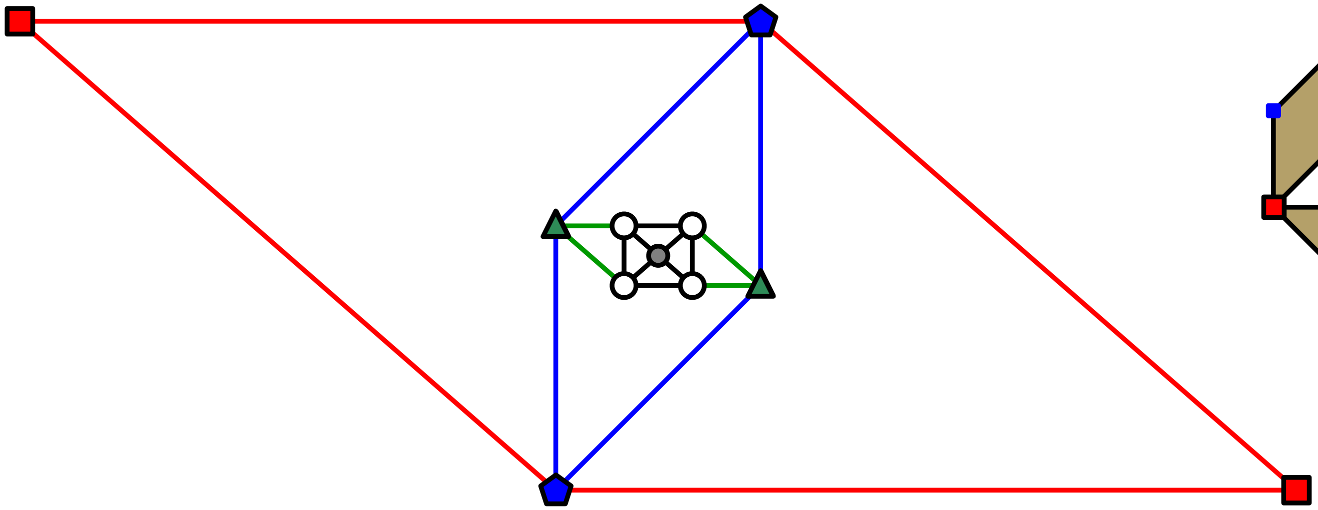

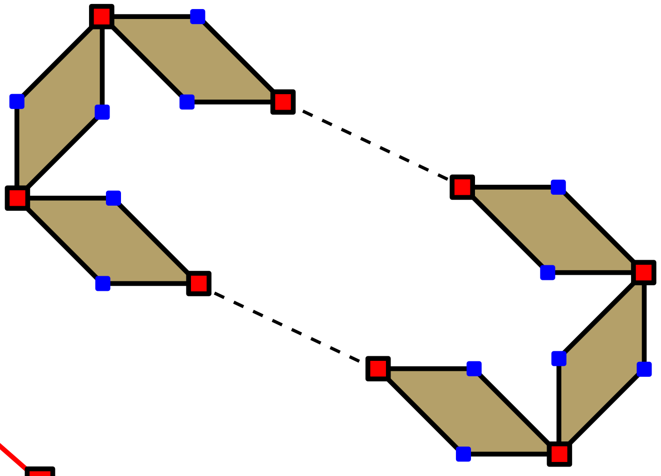

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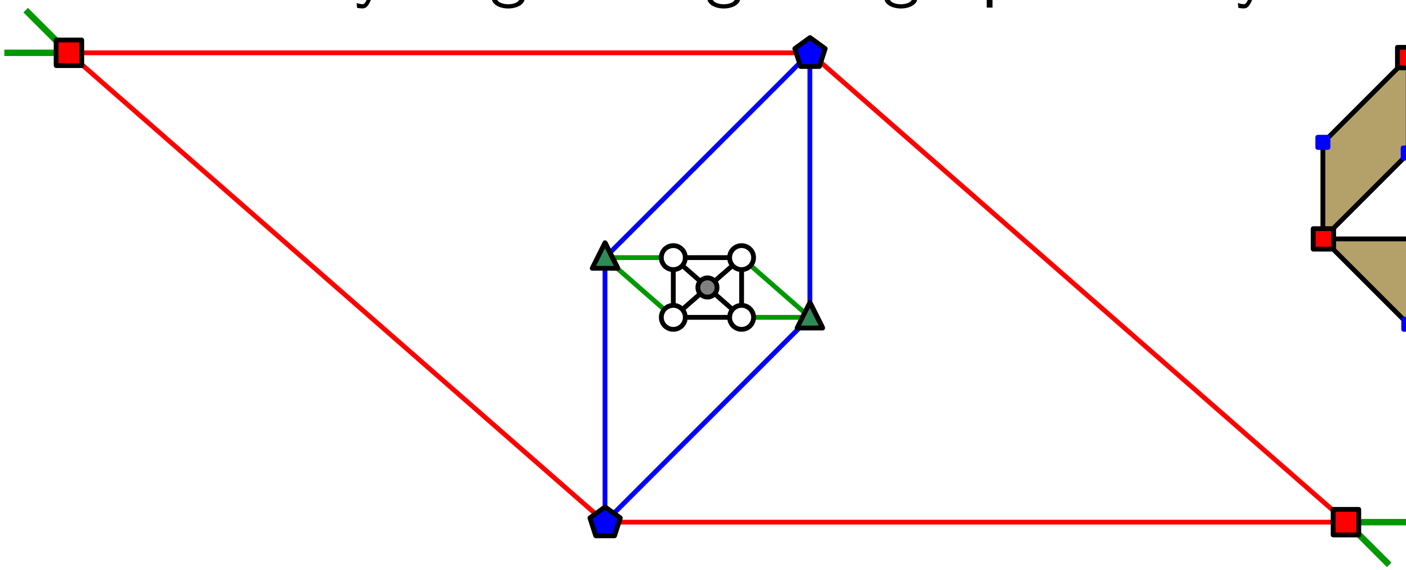

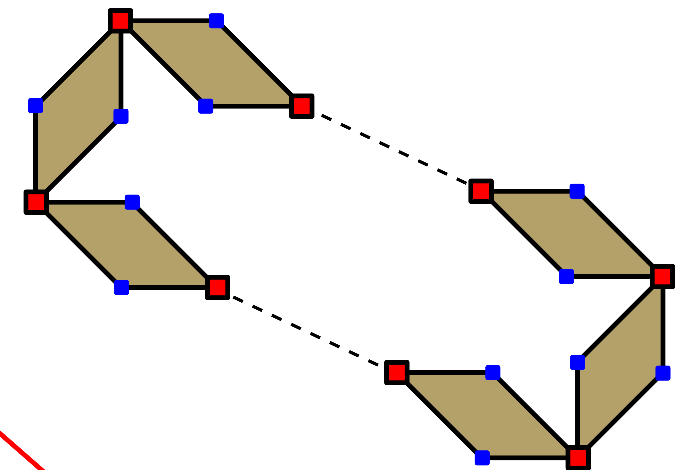

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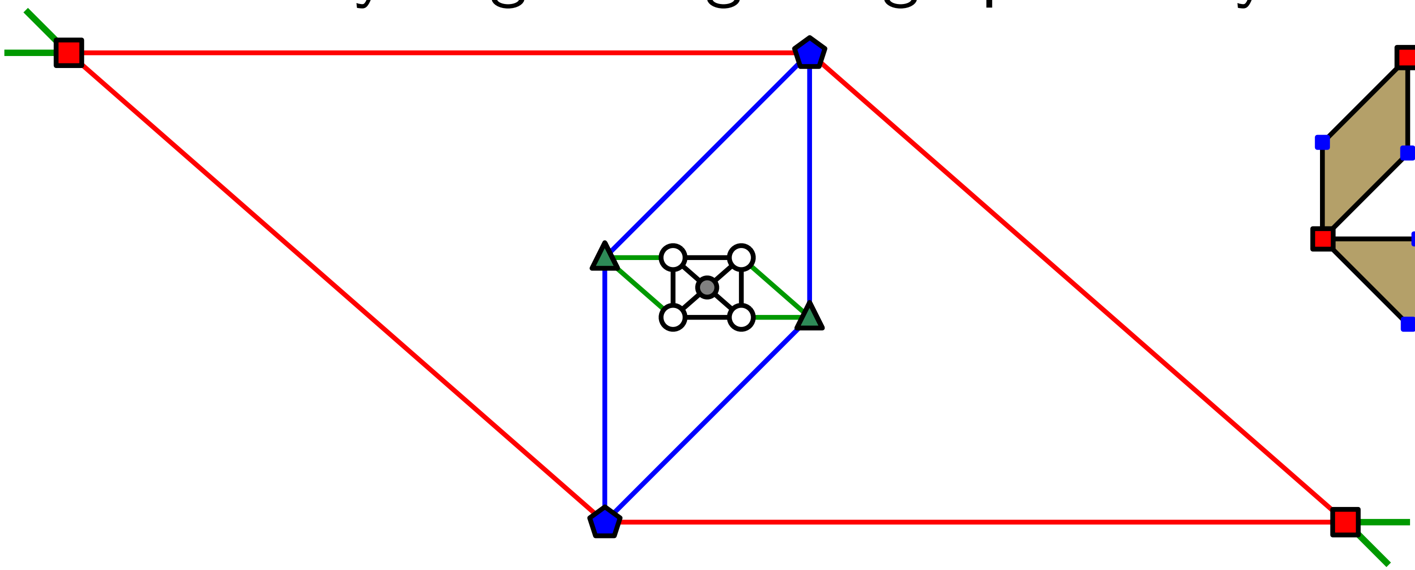

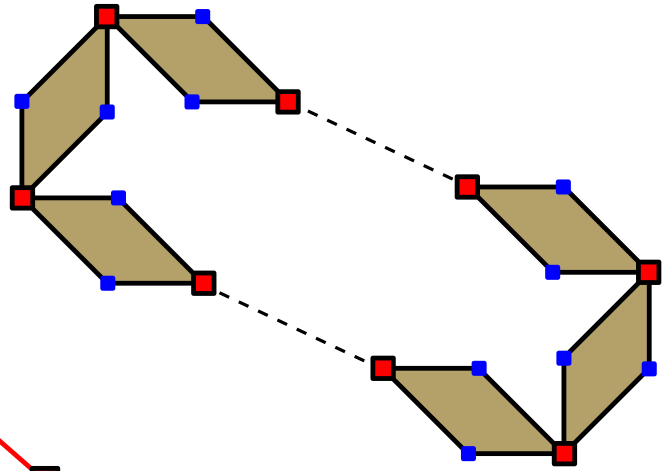

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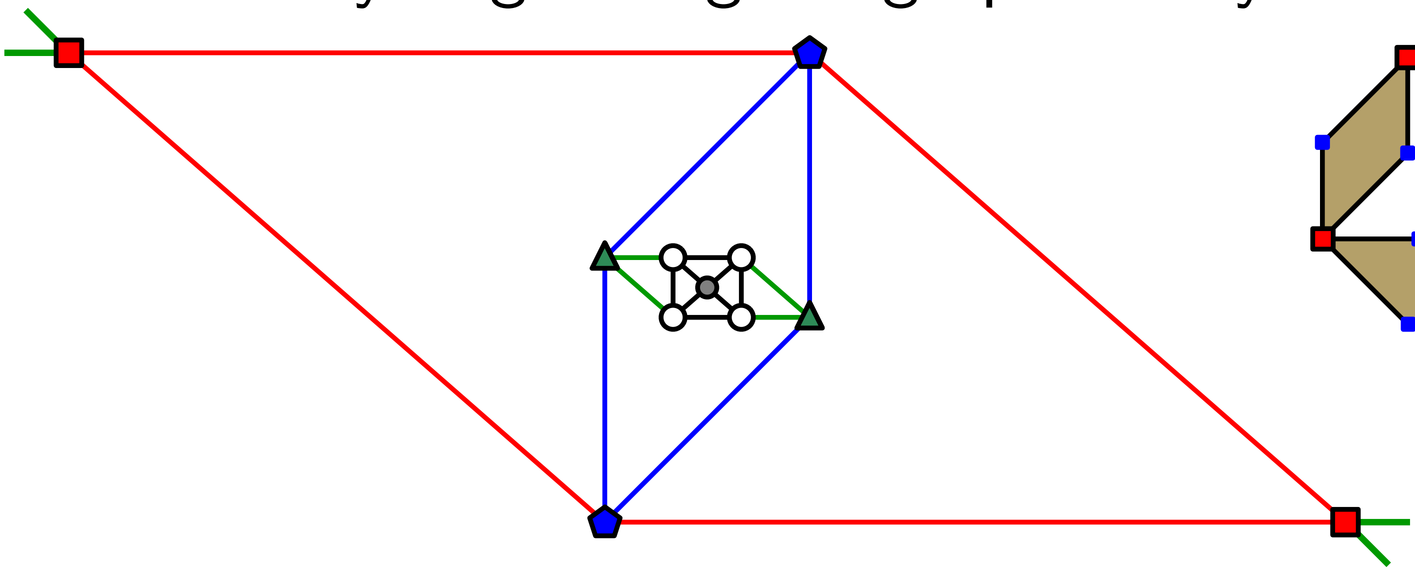

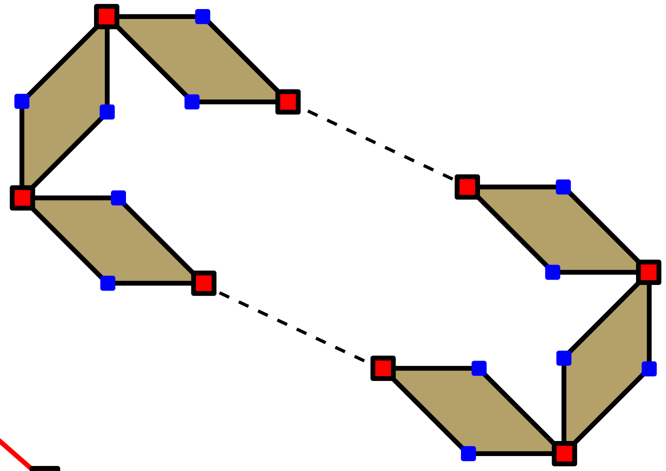

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- Smooth Orthogonal Representation Realizability
  - Input: angles between edges and edge segments along edges

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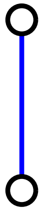

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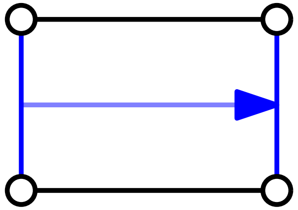

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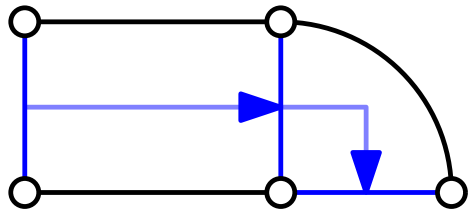

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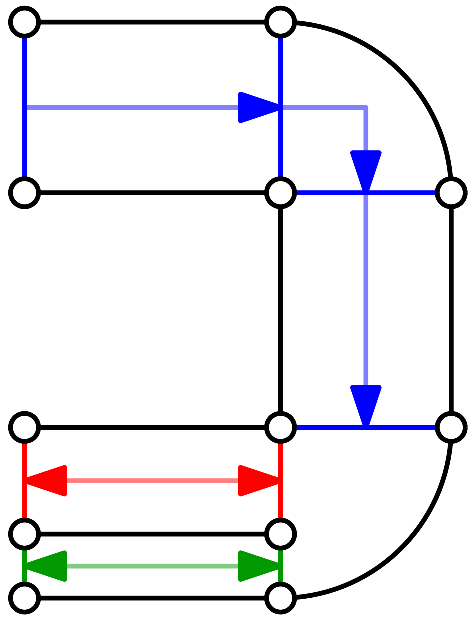

Auxiliary Gadgets

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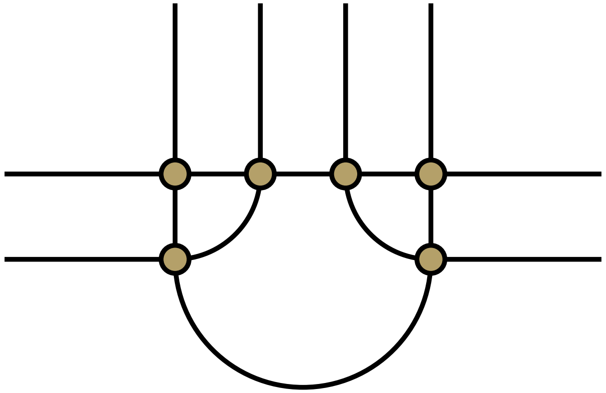

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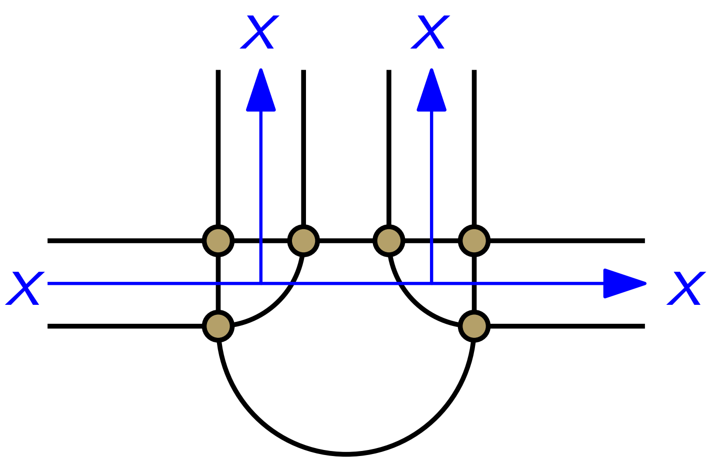

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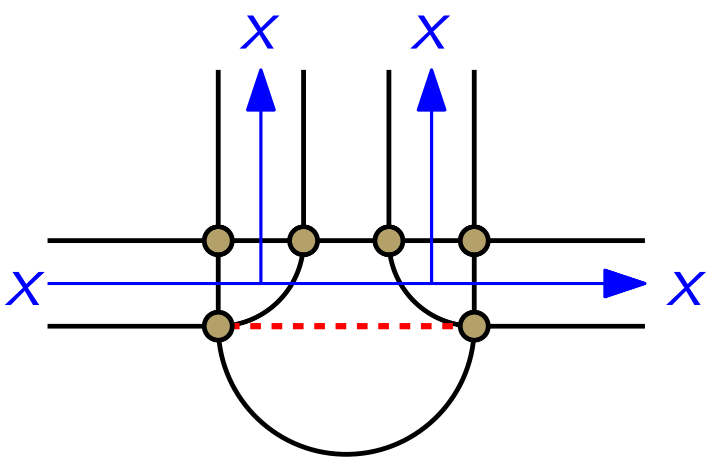

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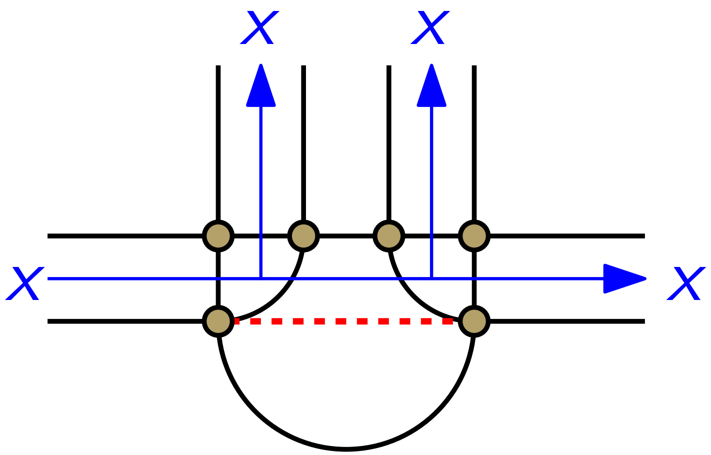

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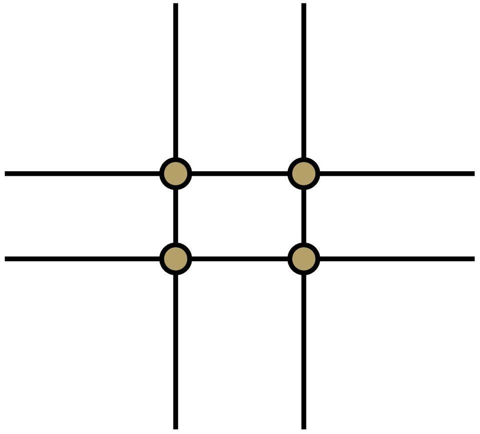

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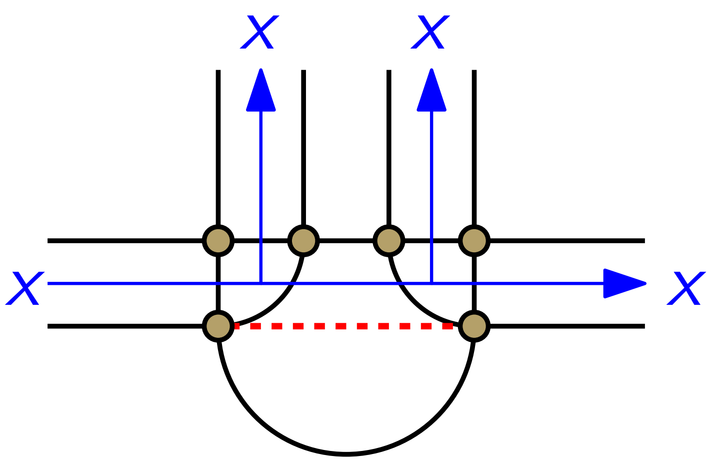

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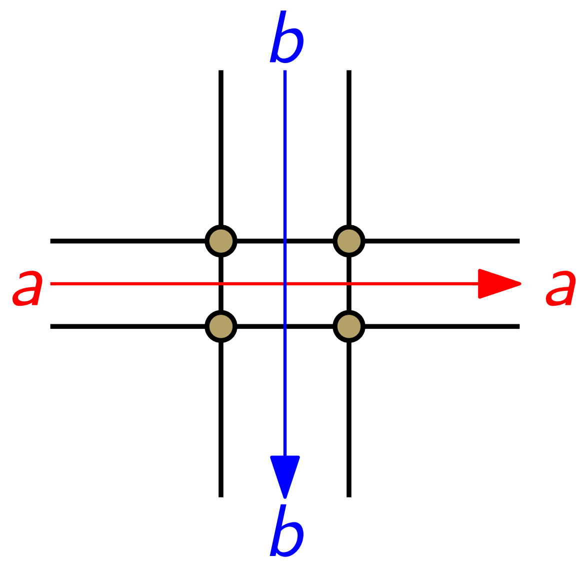

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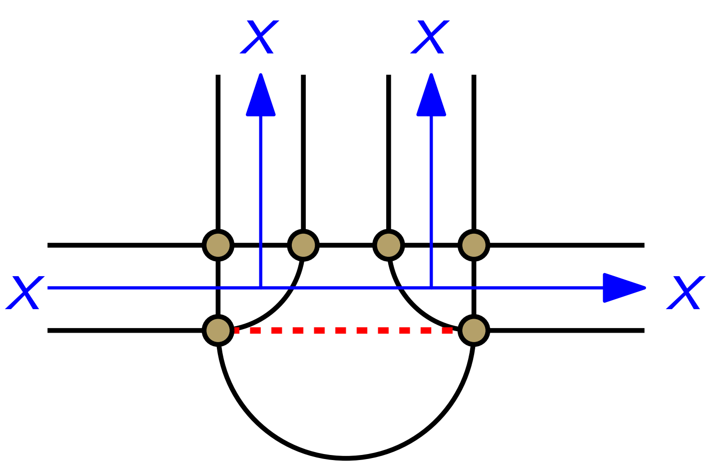

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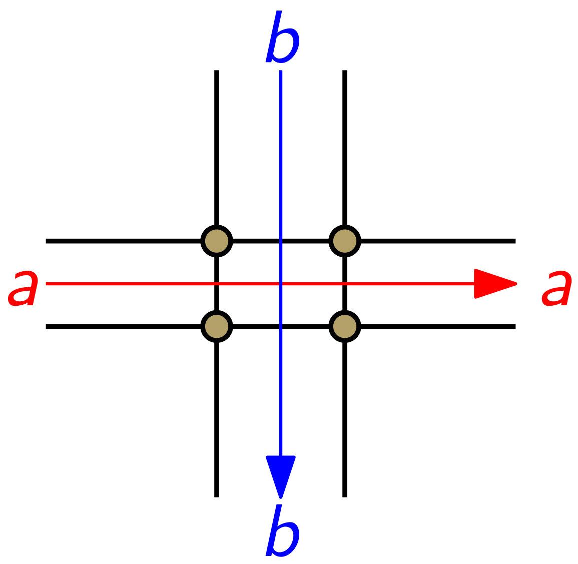
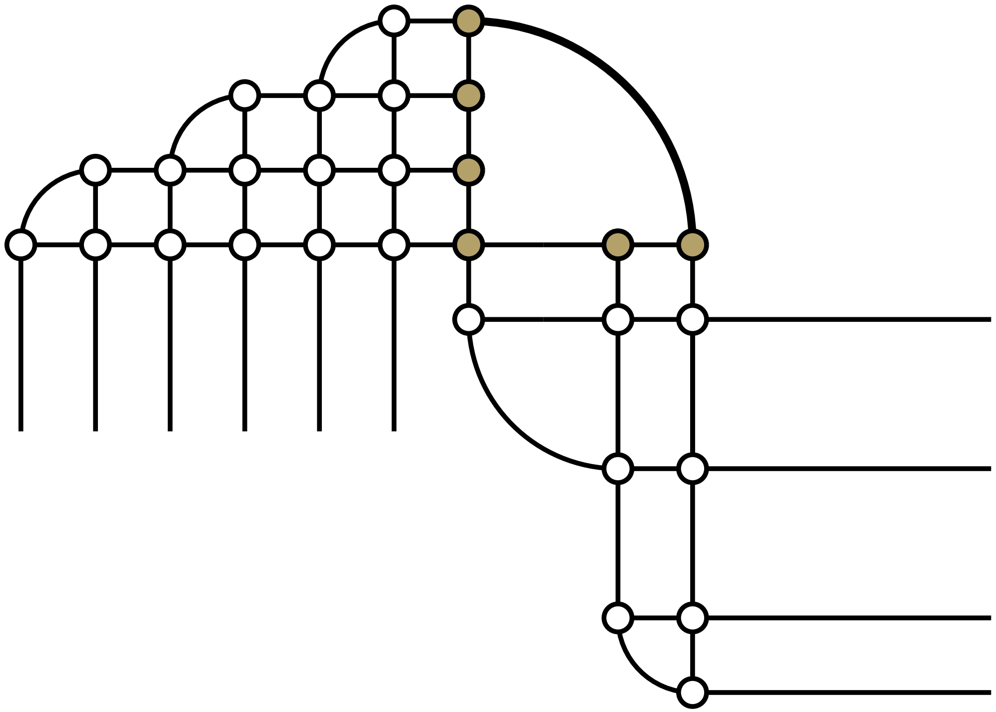


- We can connect literals and clauses properly

Variable Gadget



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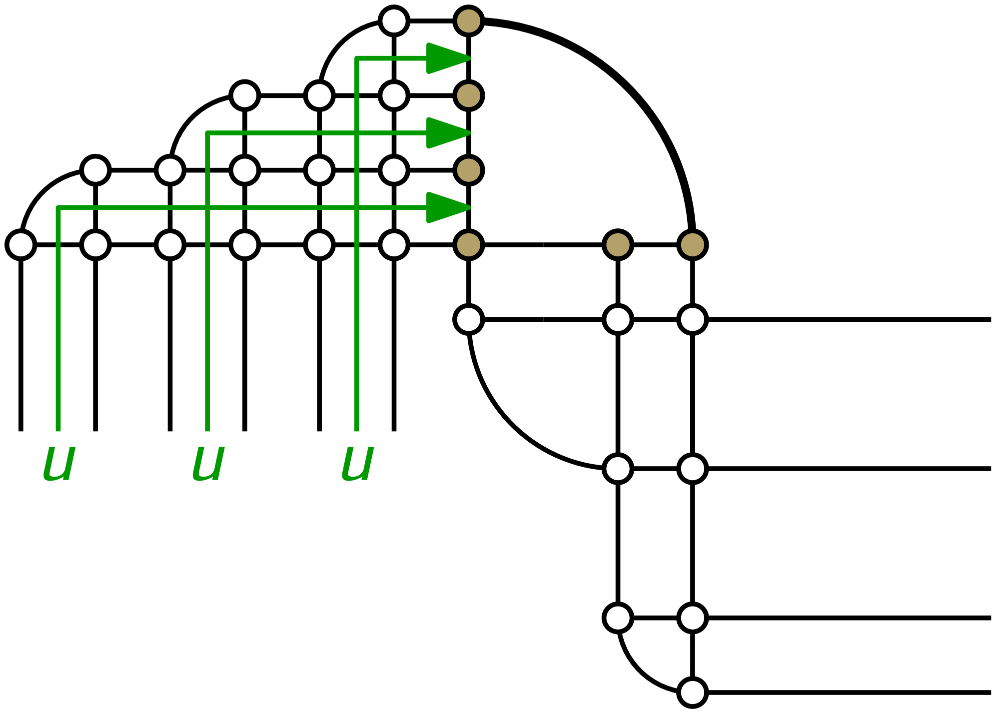

- Take 3 units of "flow" as input

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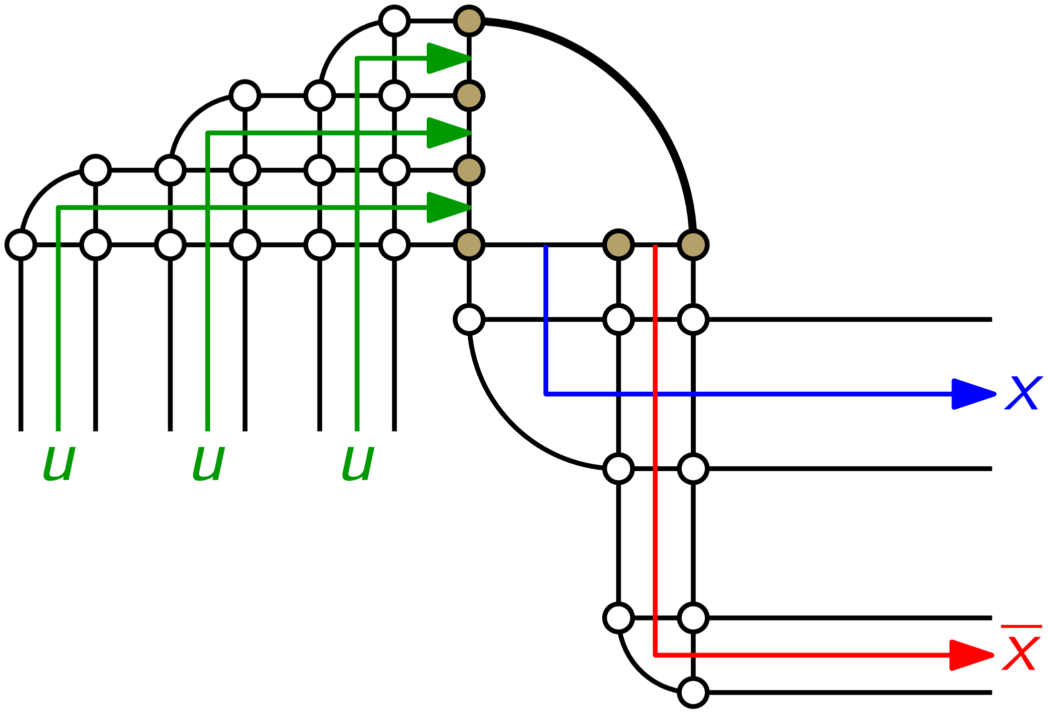

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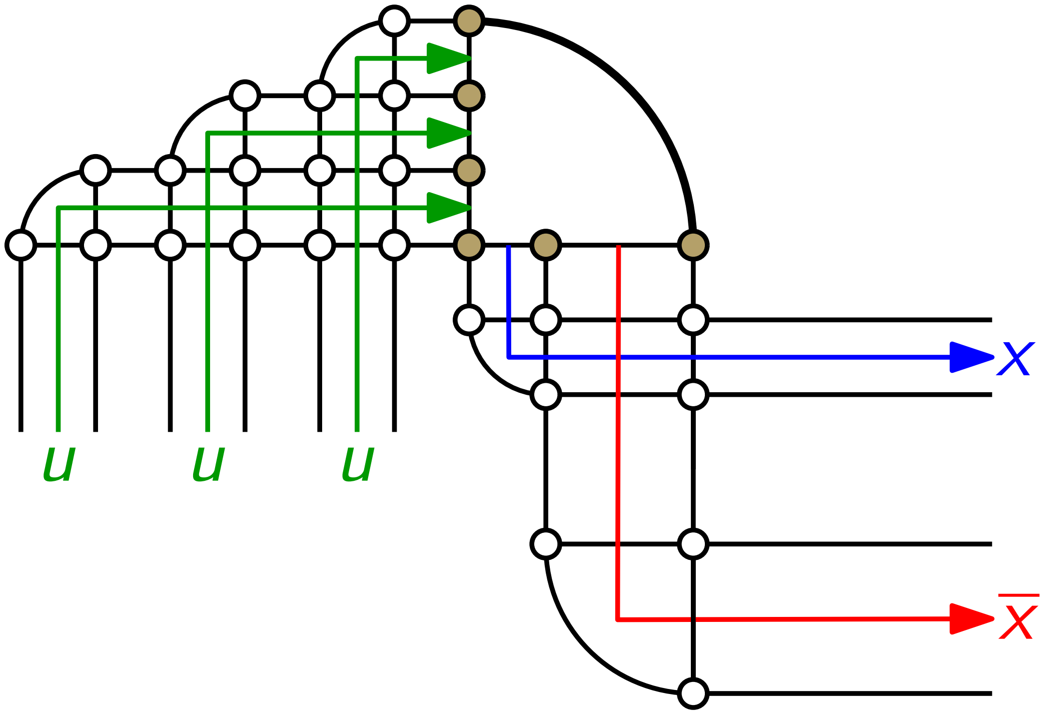

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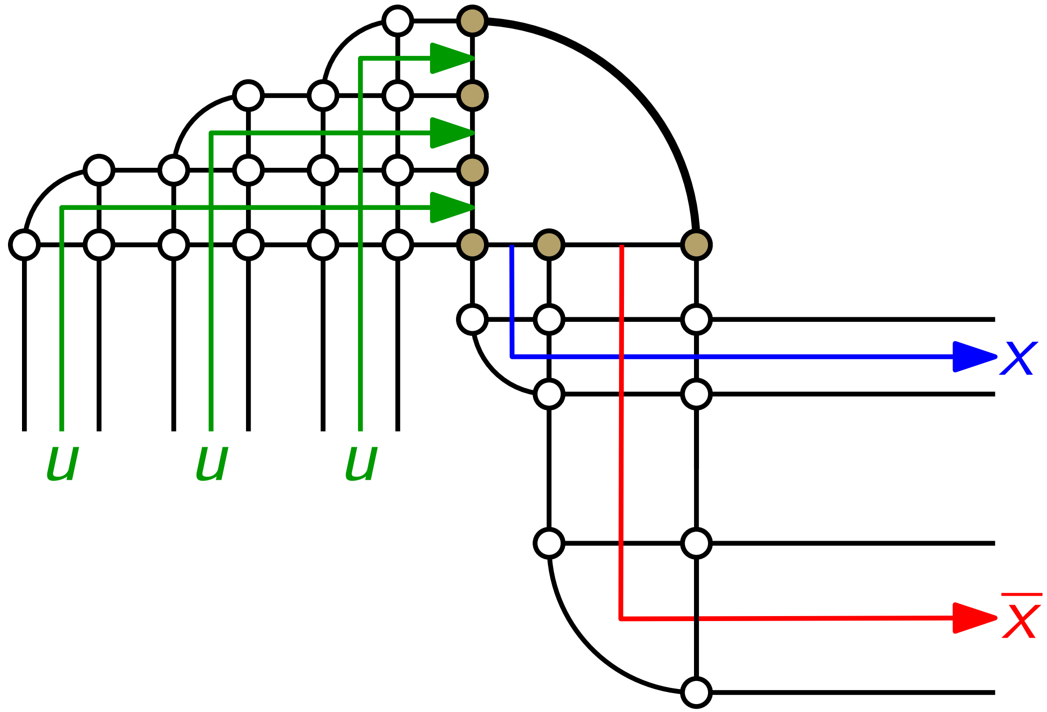

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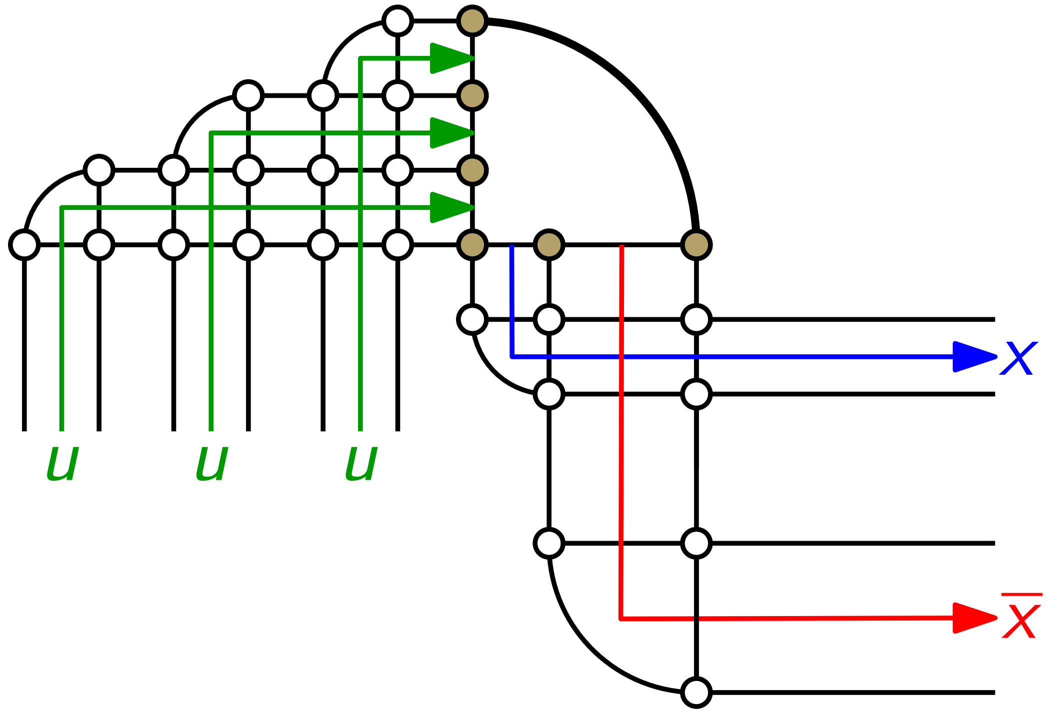

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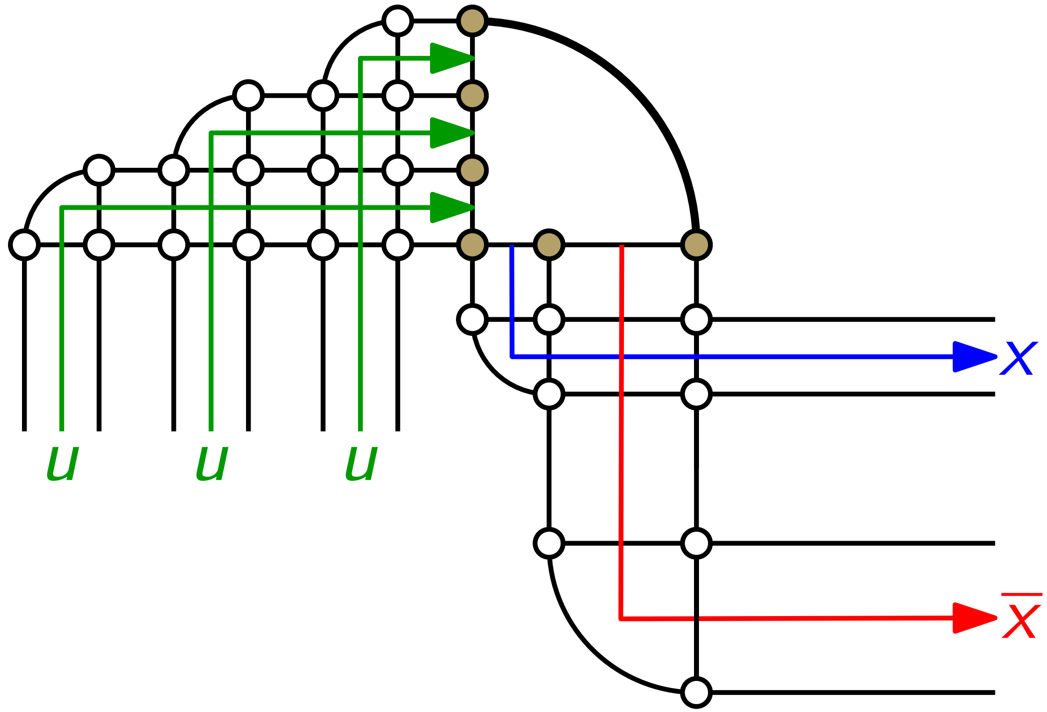

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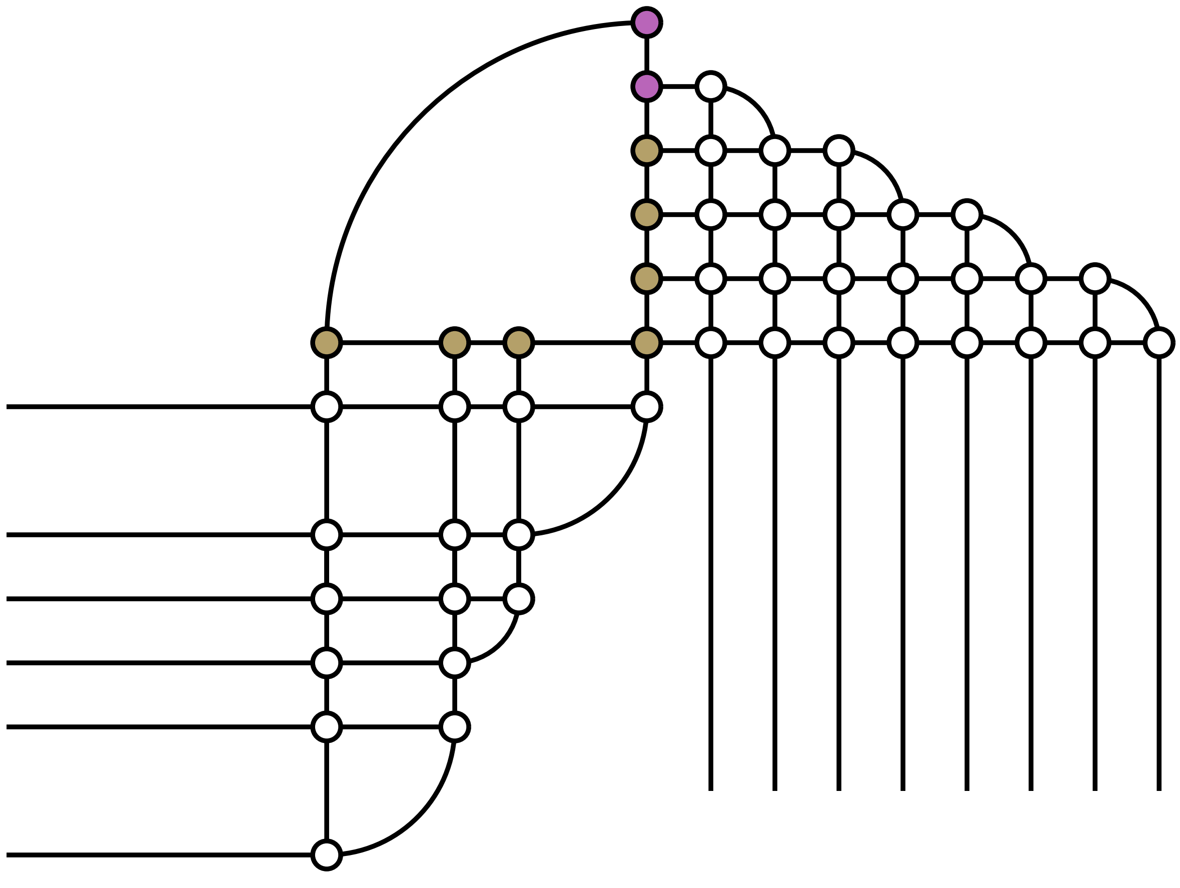
Clause Gadget



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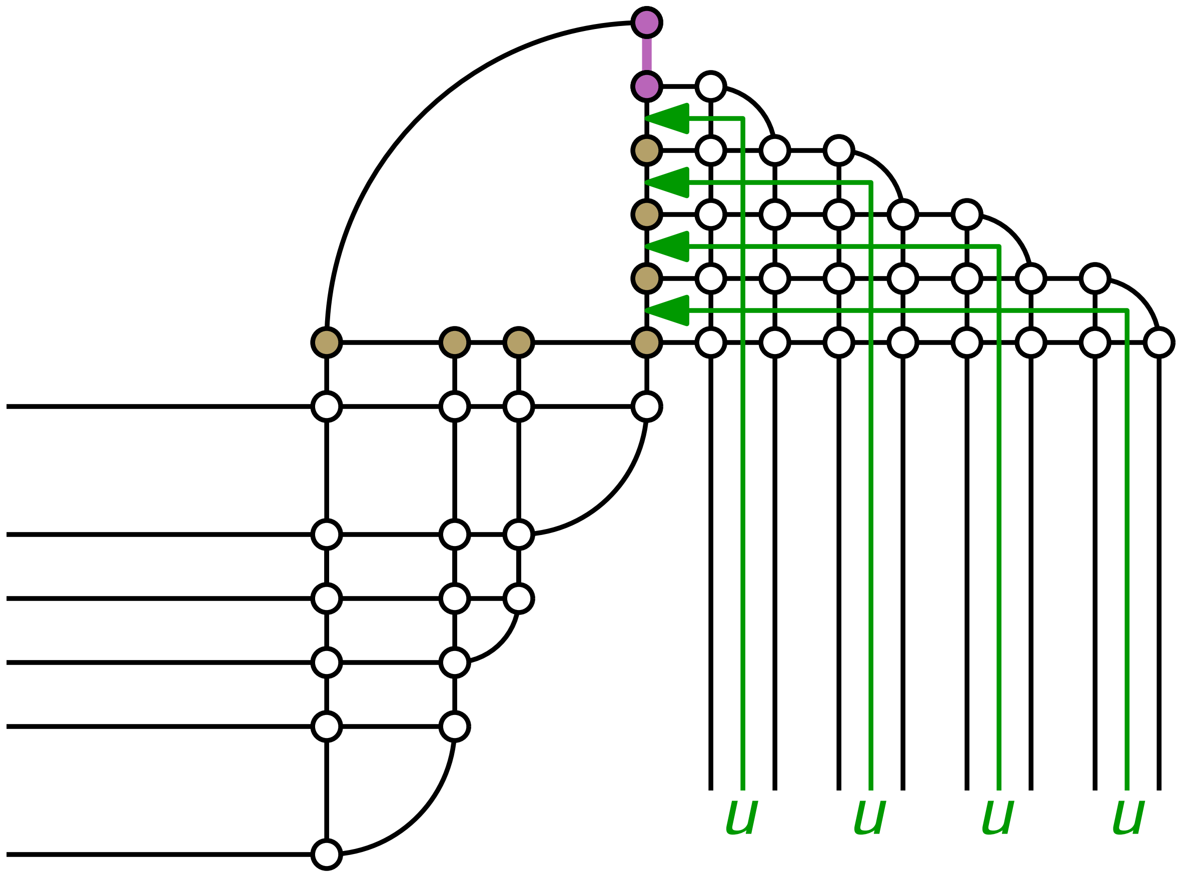

- One side of the arc is 4 units + a free edge's length long

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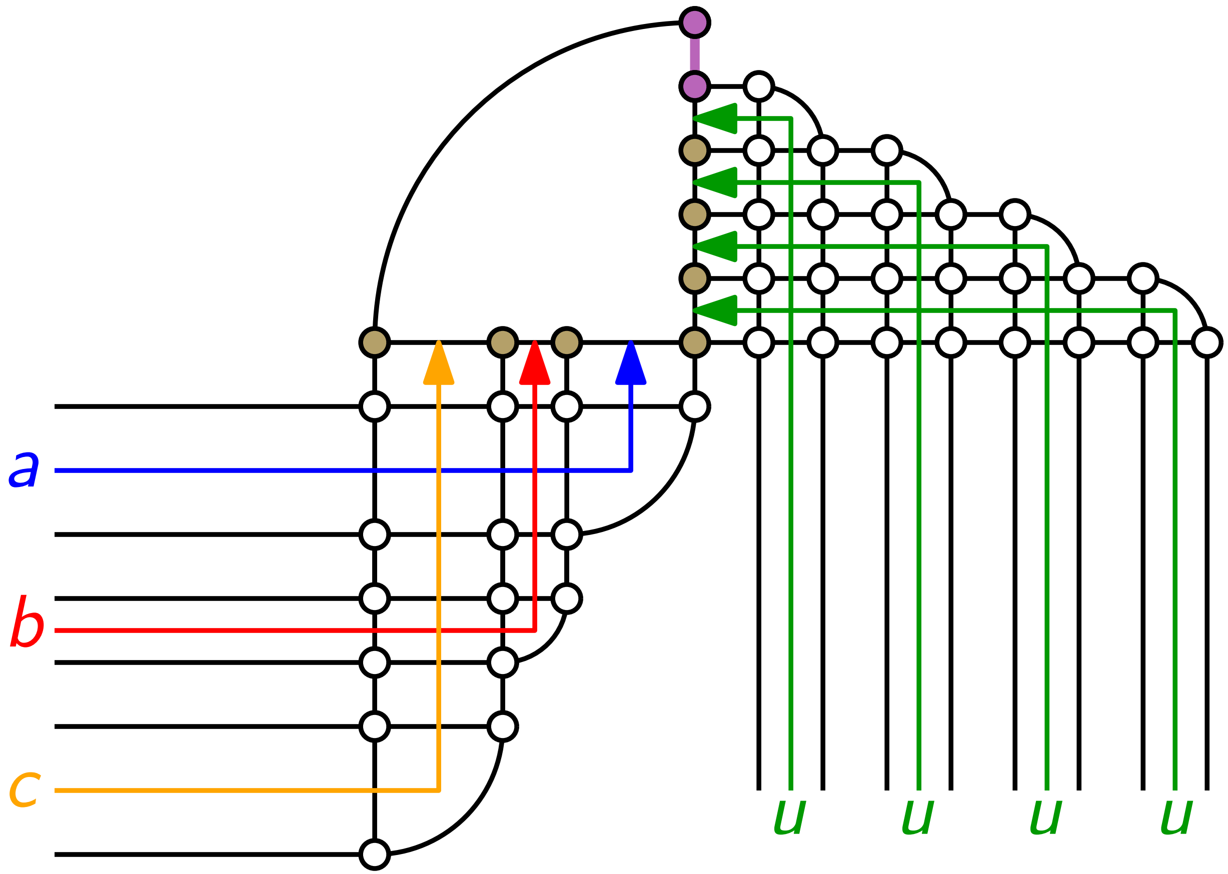

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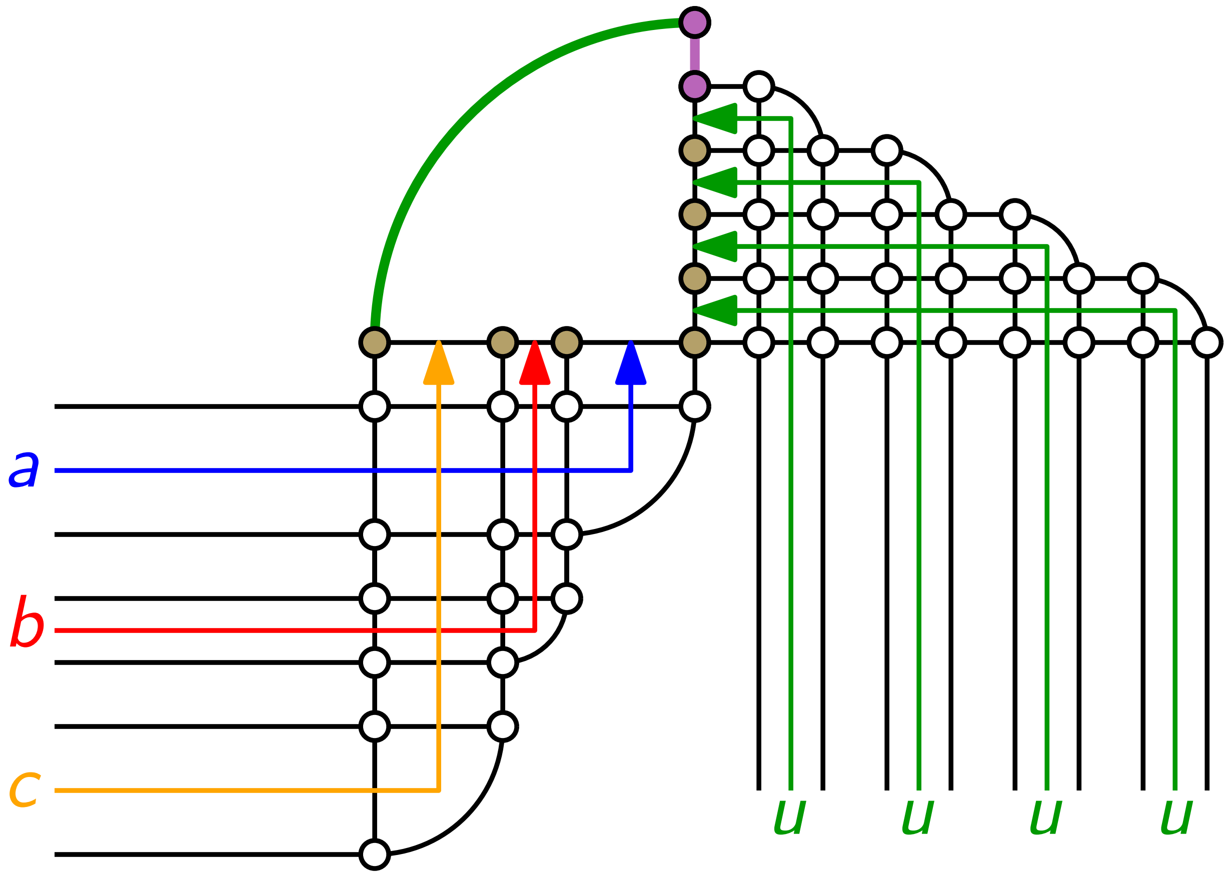

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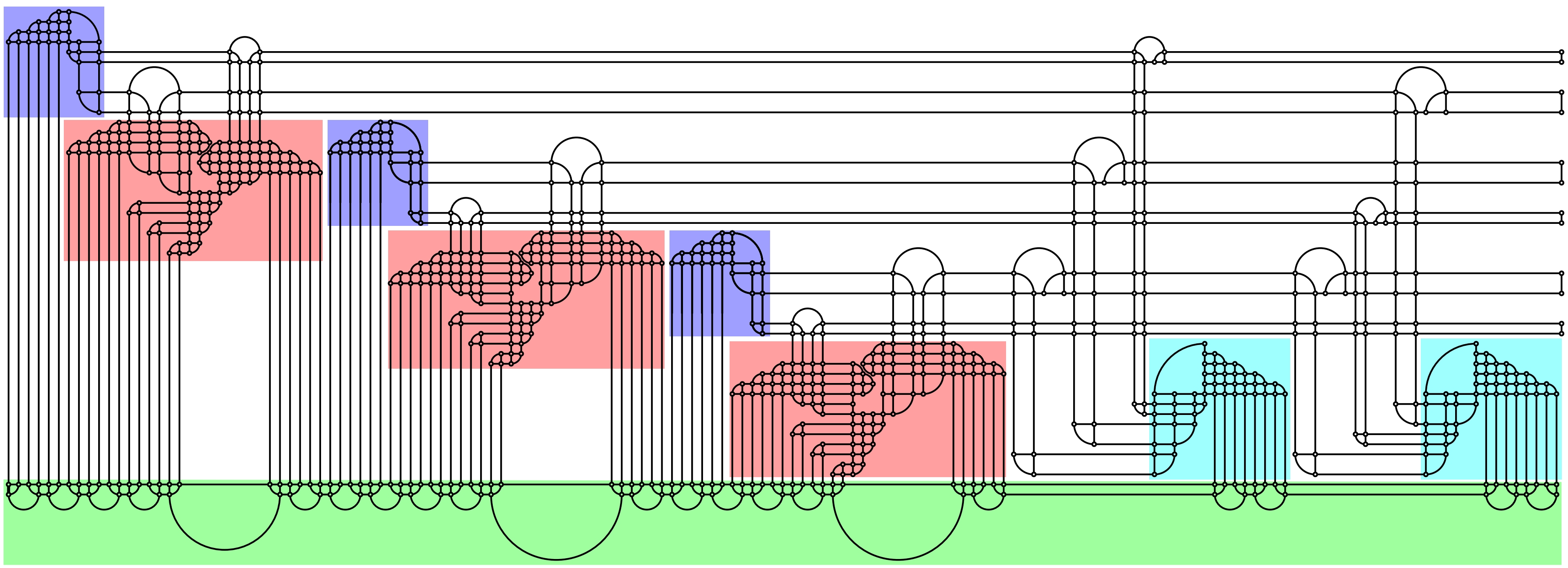
- $\ell(\texttt{true}) \gtrapprox 2\ell(u)$ and $\ell(\texttt{false}) \lessapprox \ell(u)$
  $\Rightarrow$ at least one literal must be true

A “Small” Example

$(a \vee b \vee c) \wedge (\overline{a} \vee \overline{b} \vee c)$ with $a =$ `false` and $b = c =$ `true`



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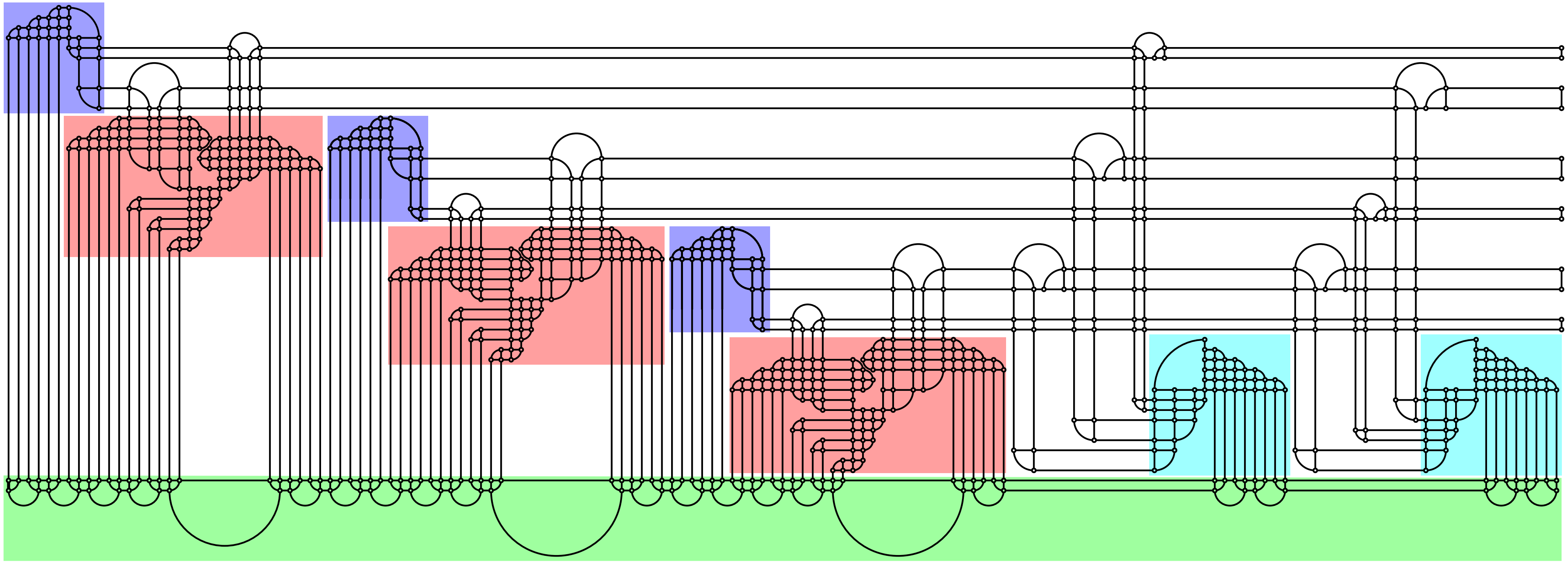

Copies of unit length

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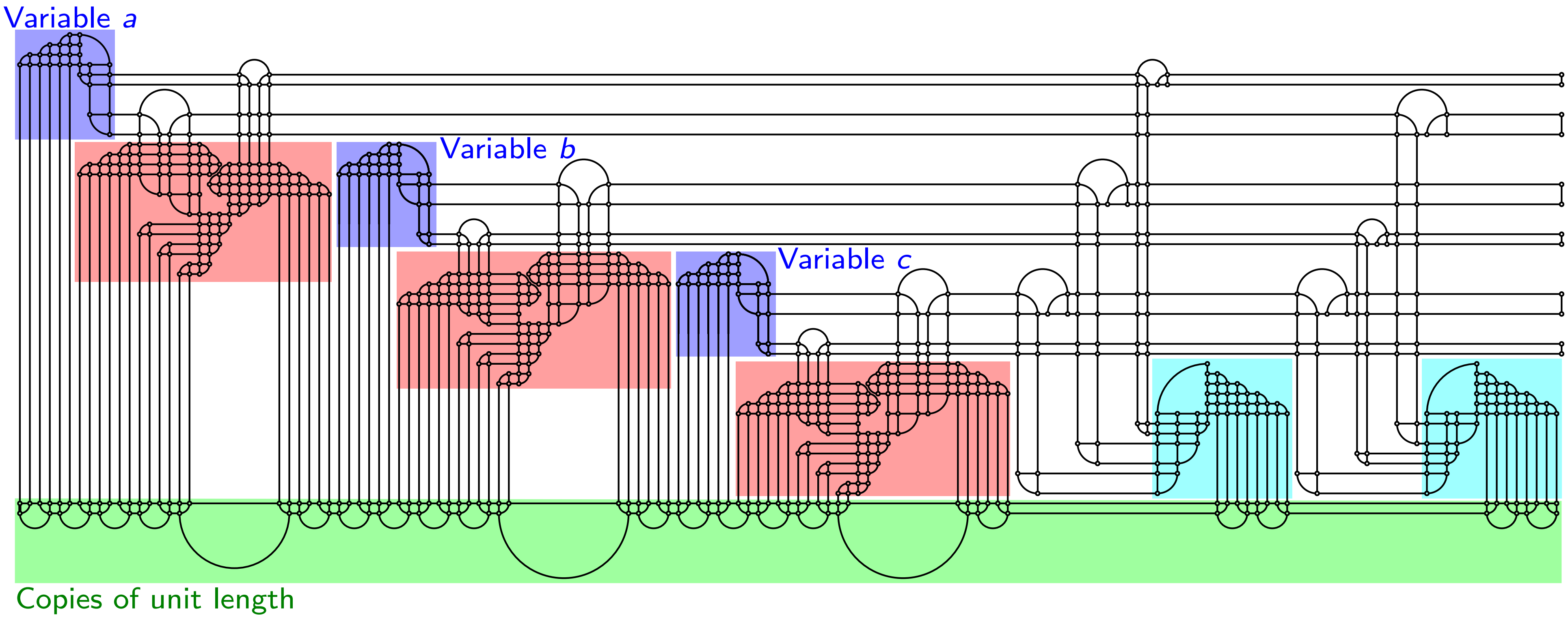

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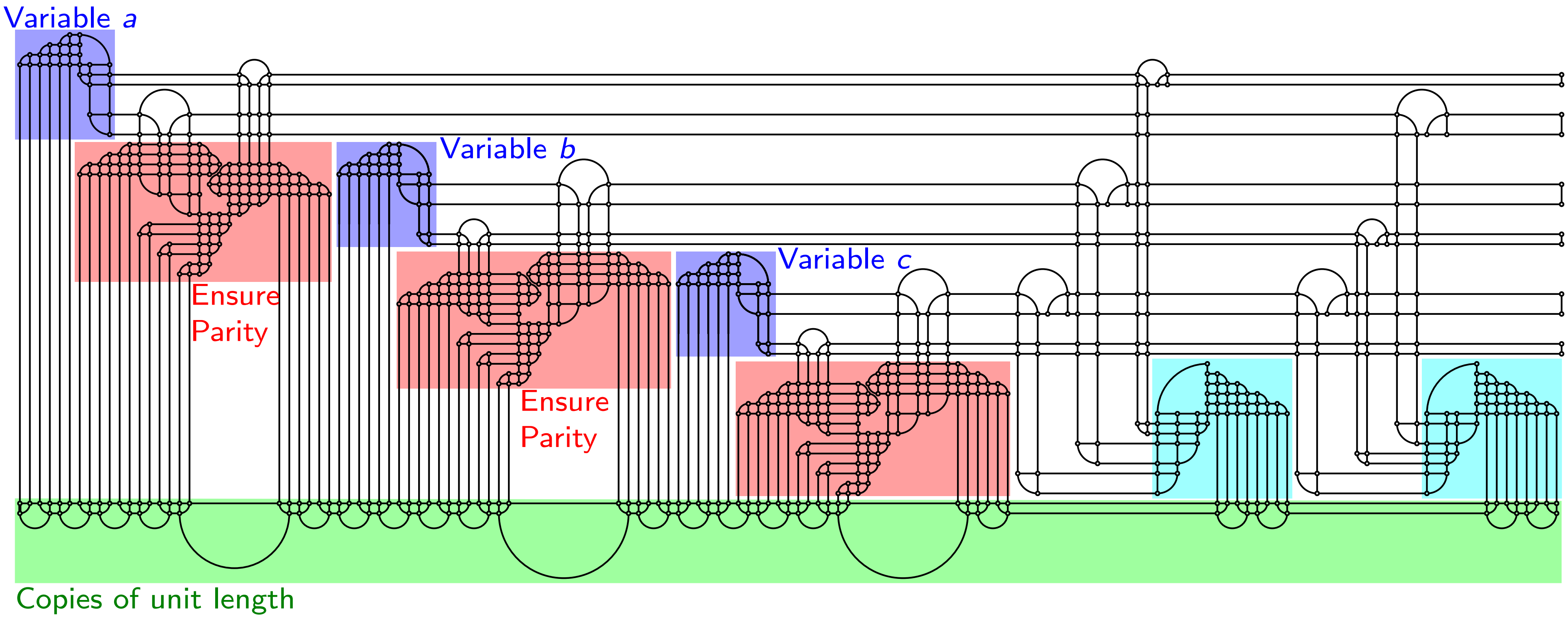

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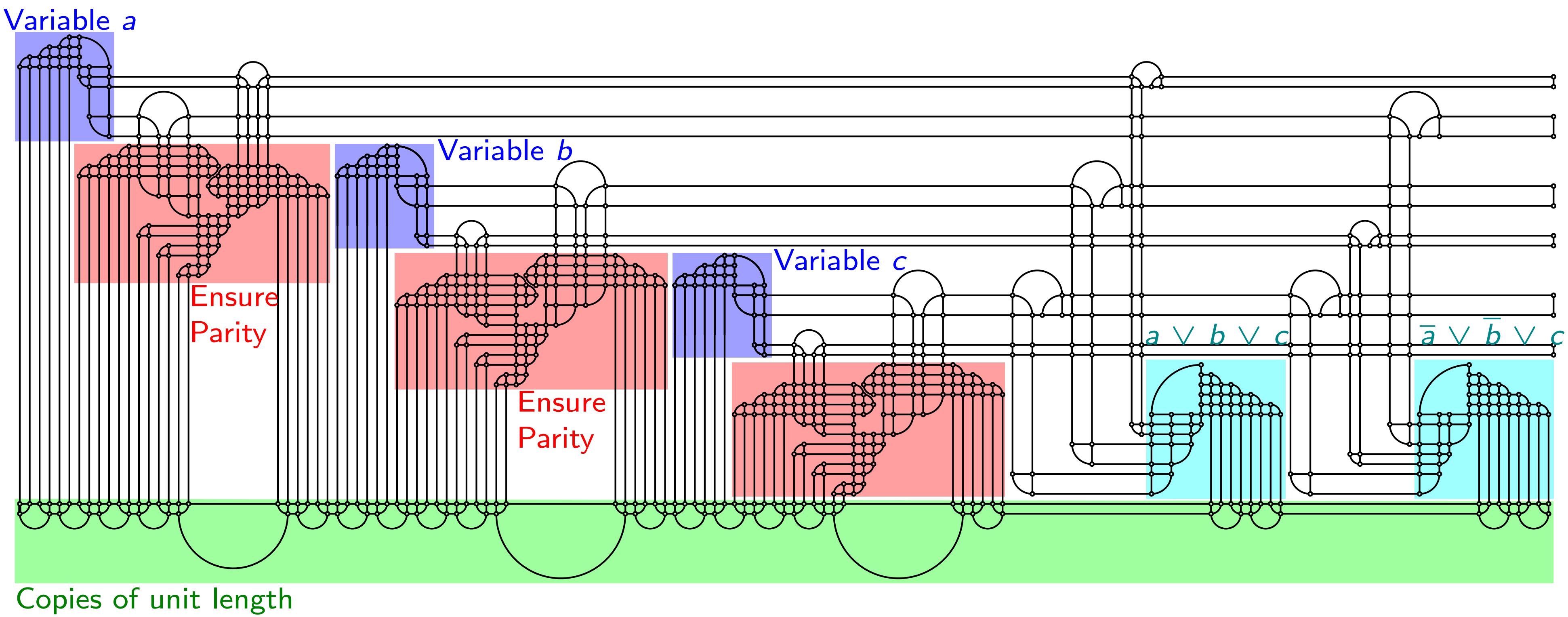

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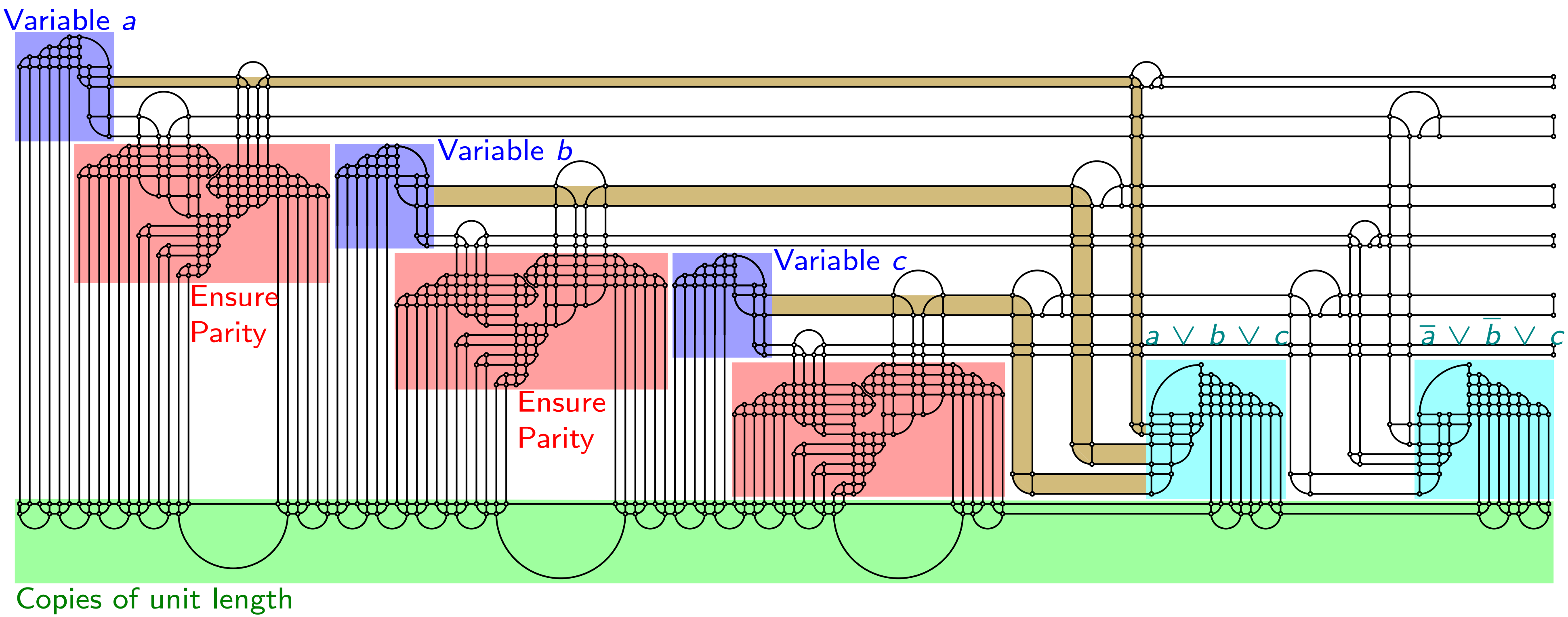

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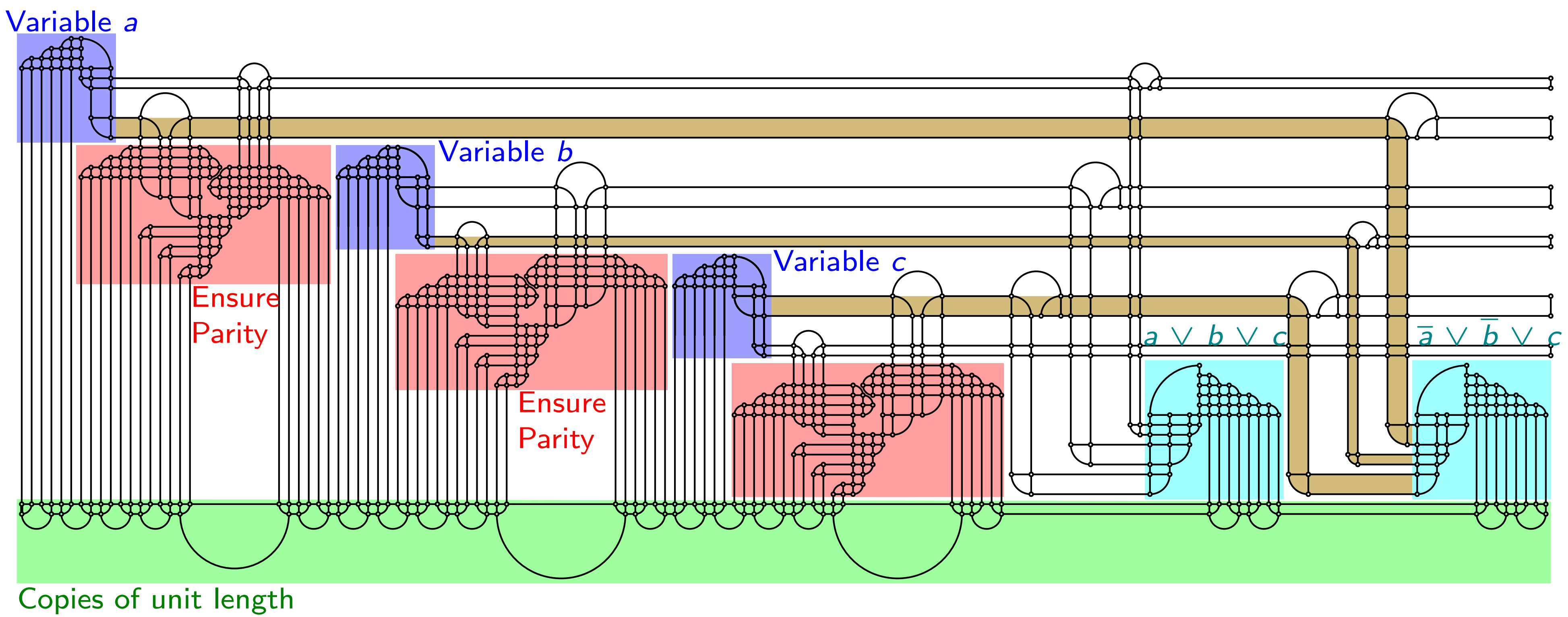

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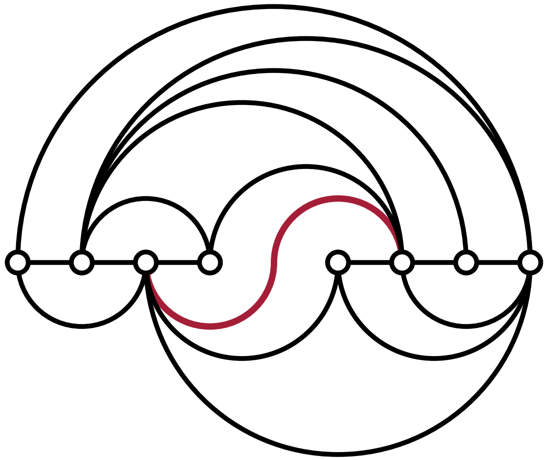
  - Same reduction scheme, most gagdets easy to transform
- TSM approach not suitable for smooth orthogonal and octilinear drawings

Kandinsky Drawings

- Kandinsky model in smooth orthogonal setting so far: Book Embedding Inspired [Bekos et al. 2013]



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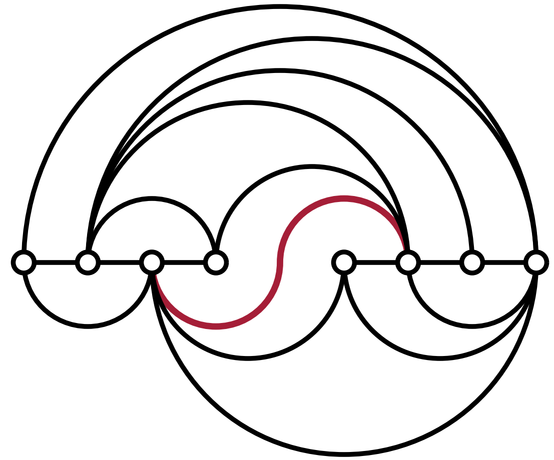

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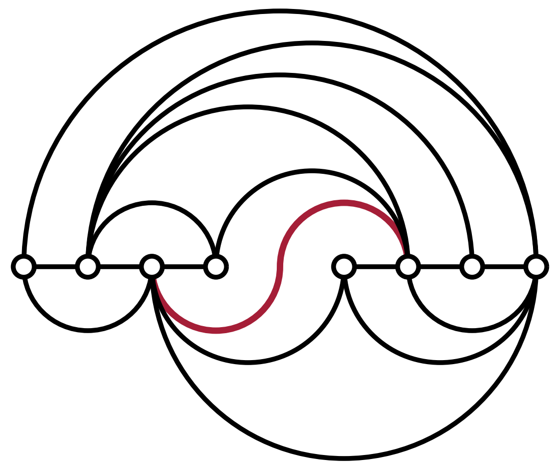

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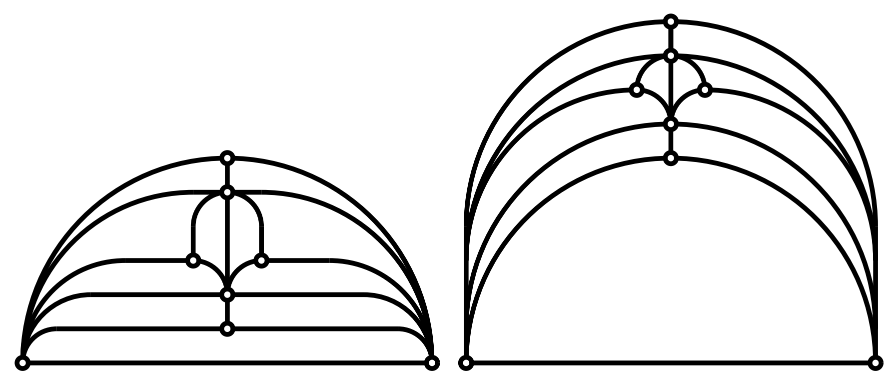
- Possible improvements:
  - Distribute vertices more evenly
  - Draw edges x, y-monotone



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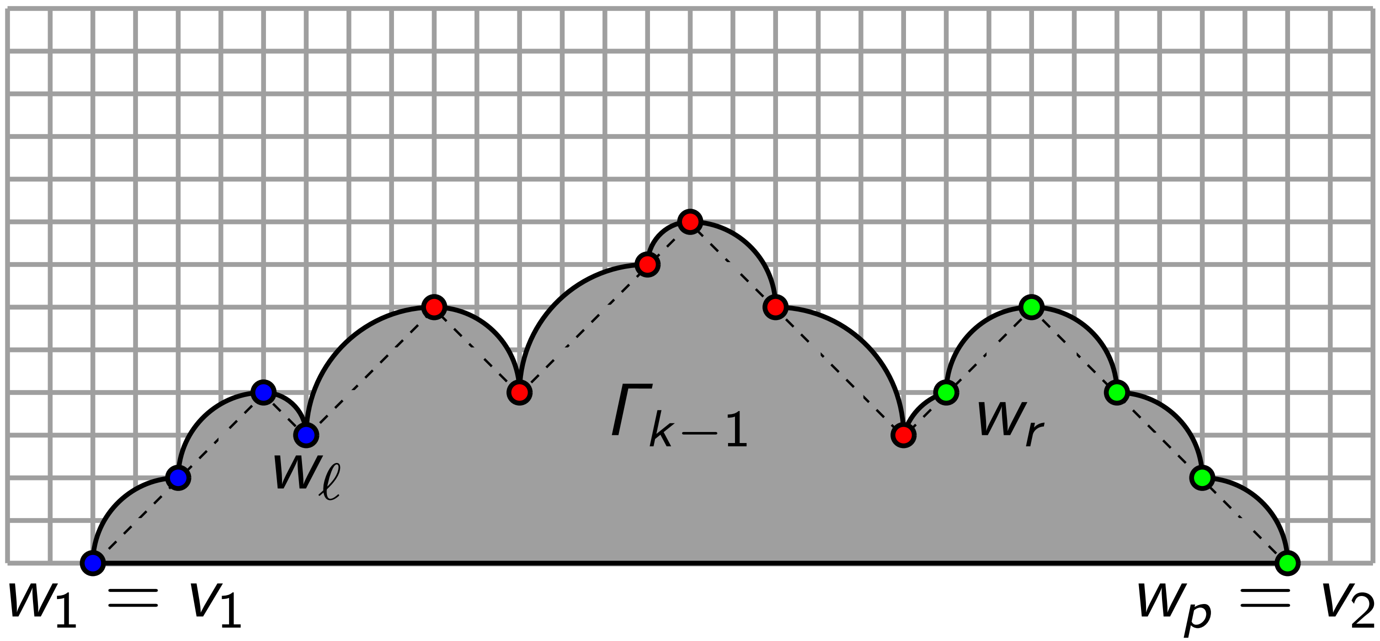

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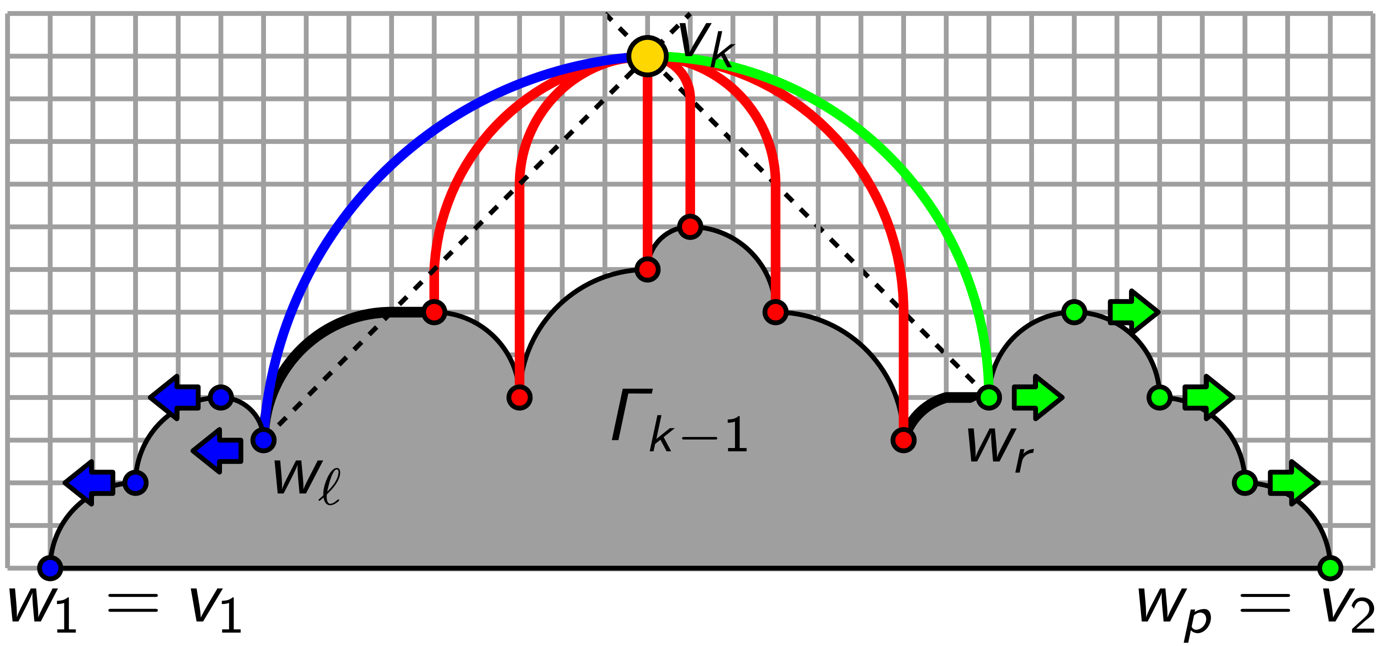

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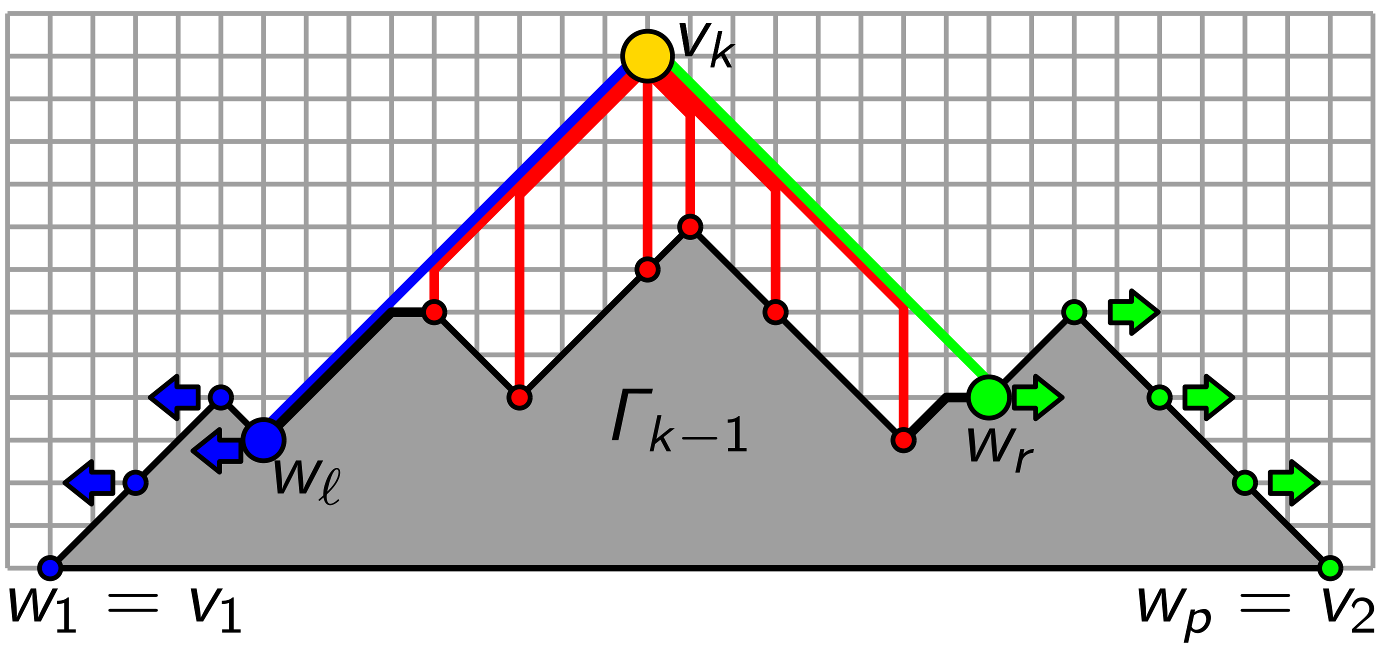

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  - We can use this approach for octilinear Kandinsky drawings too!

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Thanks to the anonymous reviewers!

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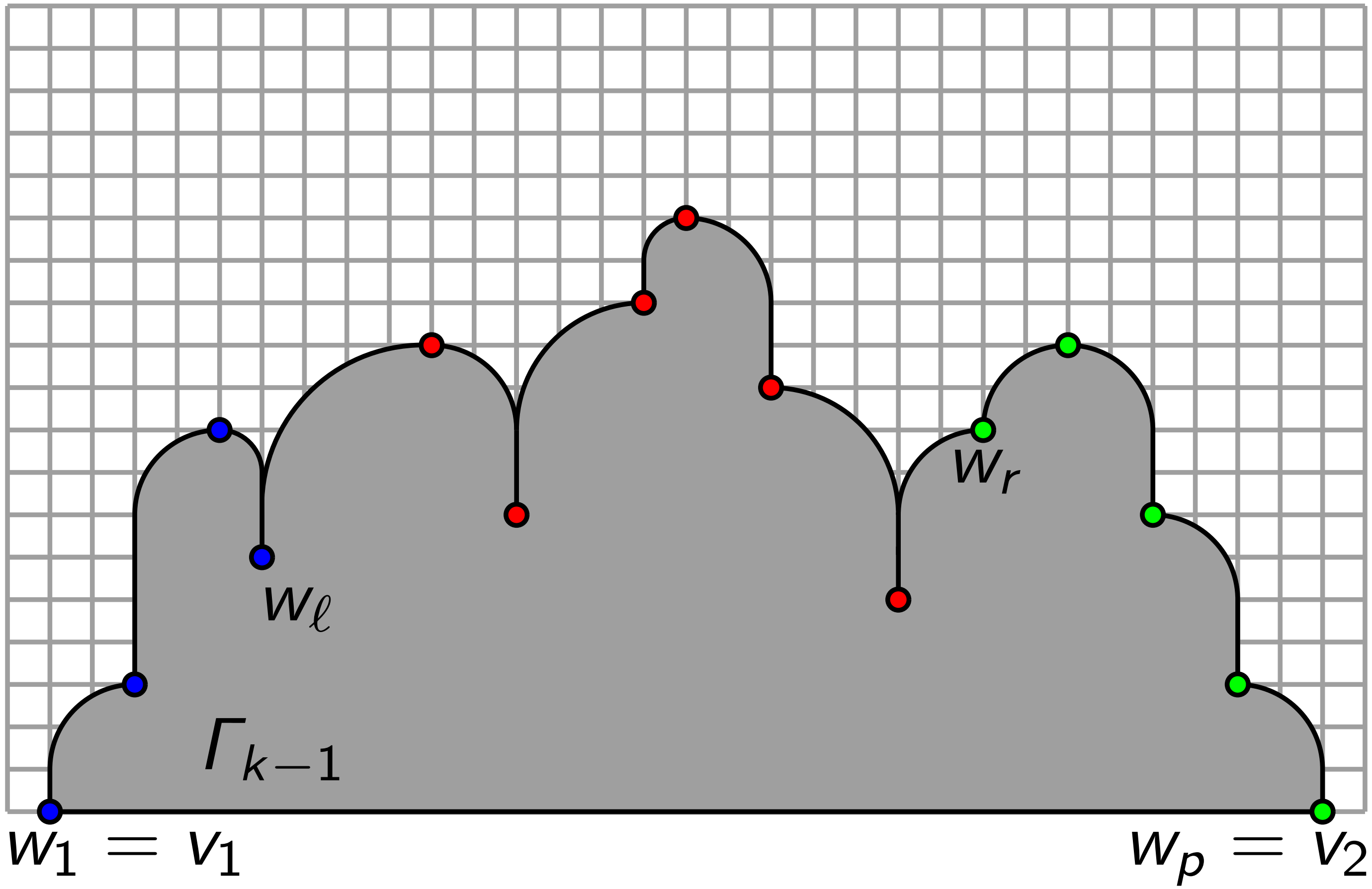
- Now: steep mountains as contour



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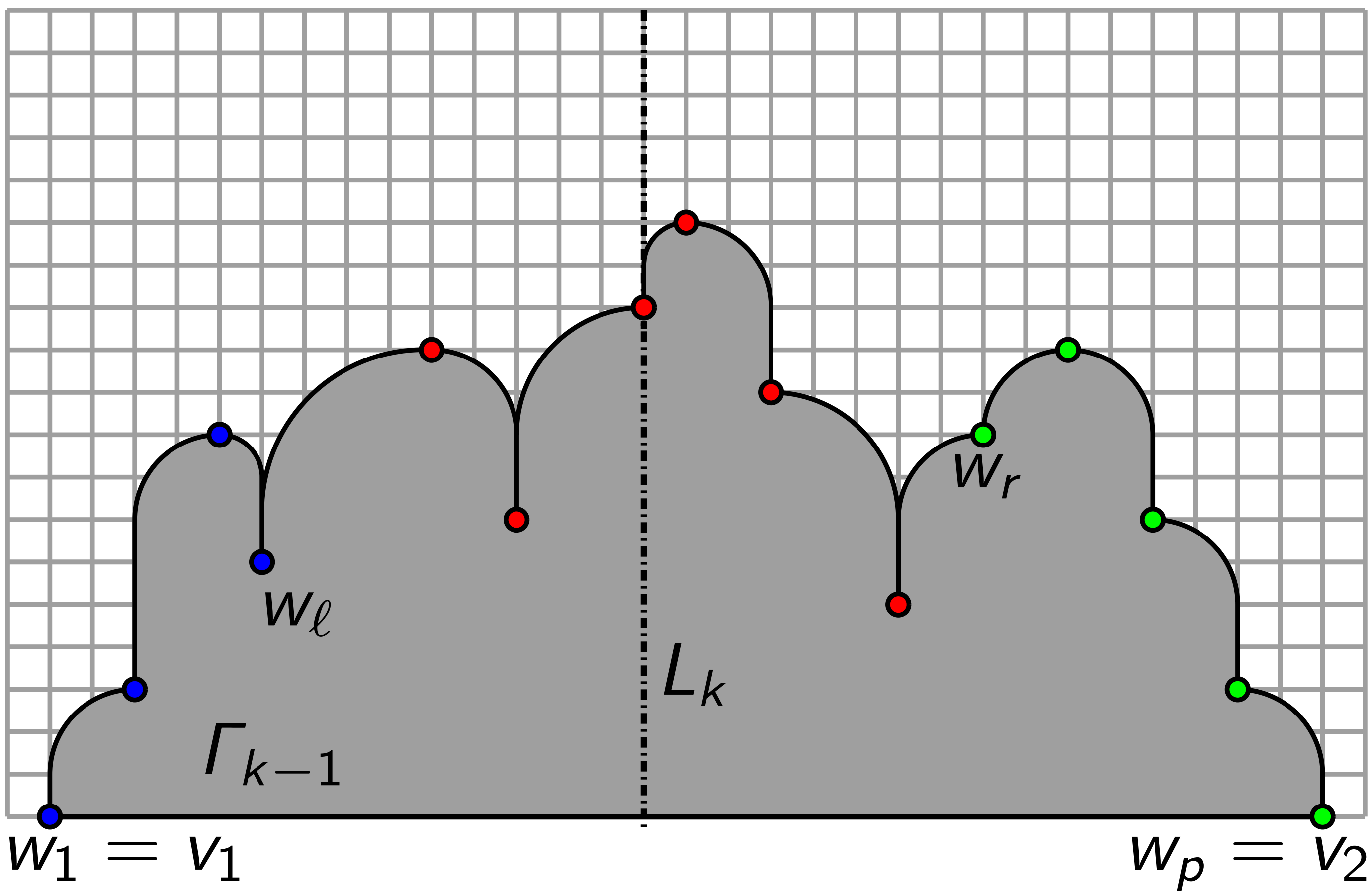
- $x(v_k)$ is fixed



Drawings with Less Bends

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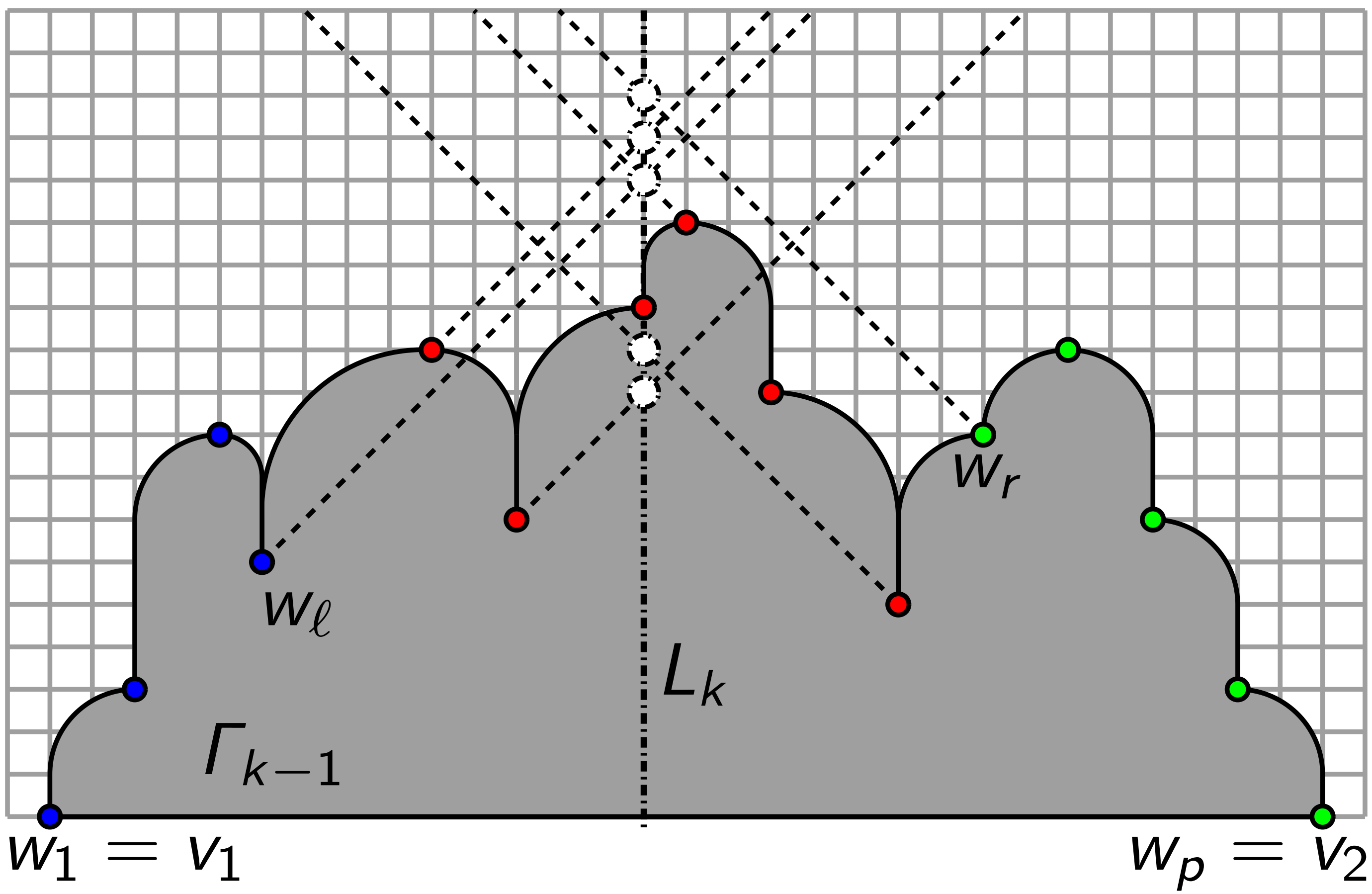

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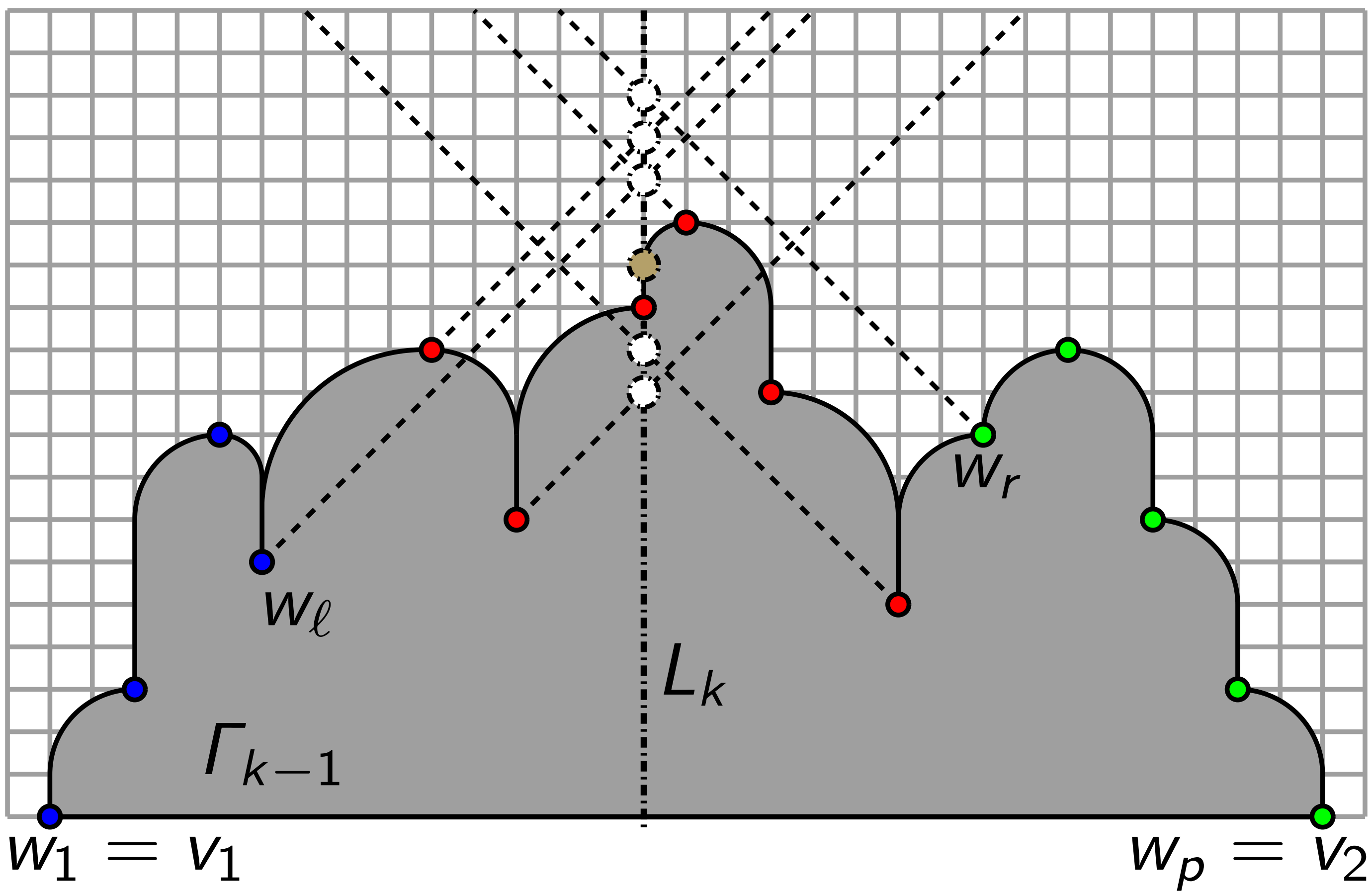

Drawings with Less Bends

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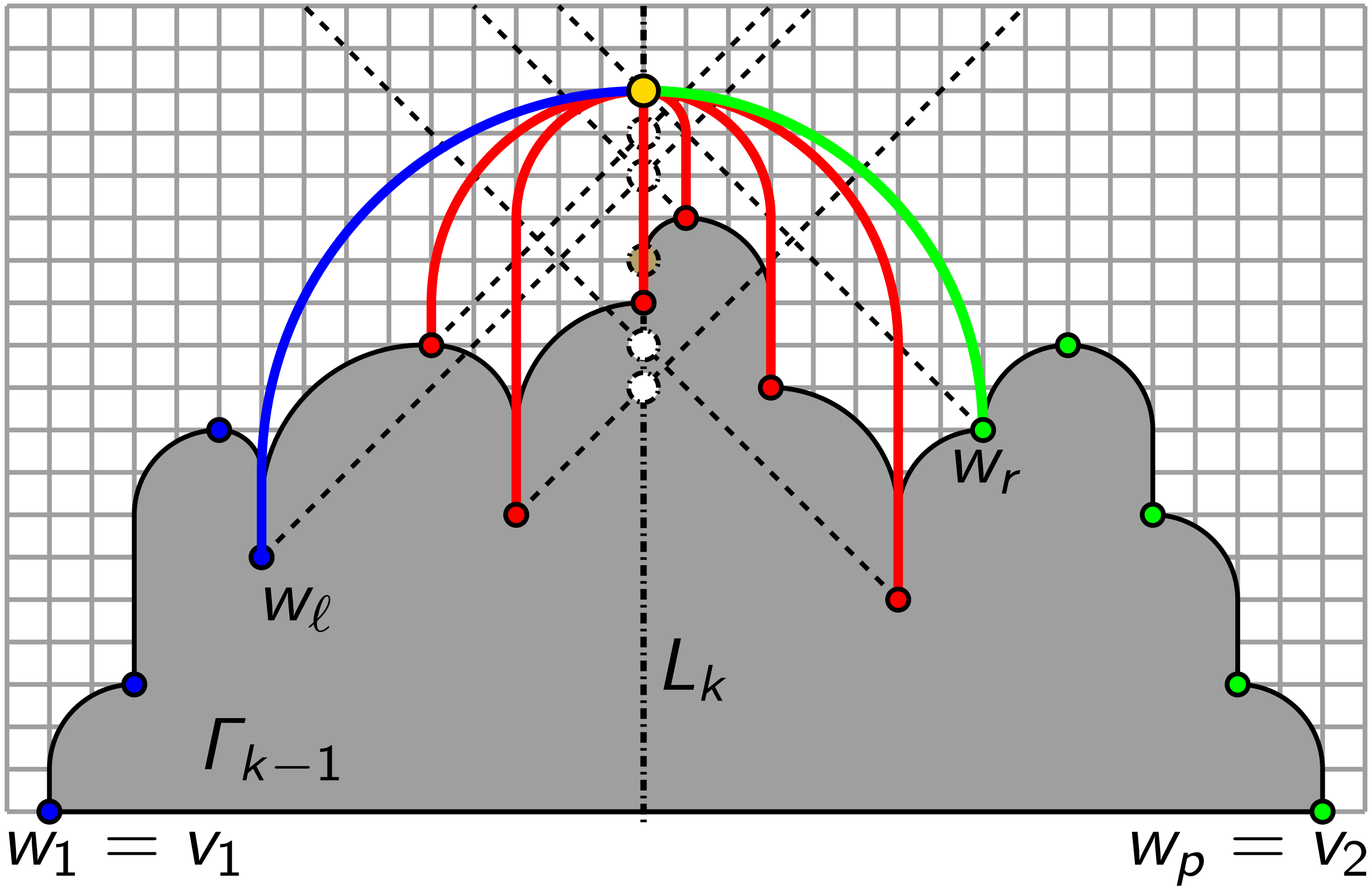
- Use highest candidate position to ensure planarity and contour condition



Open Problems

- **Relations**

- **Complexity**

- **Kandinsky Drawings**

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  - Higher degree: (smooth) d-linear drawings
  - Relations of smooth d-linear and $2d$-linear on max-degree d
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Thanks for your attention!