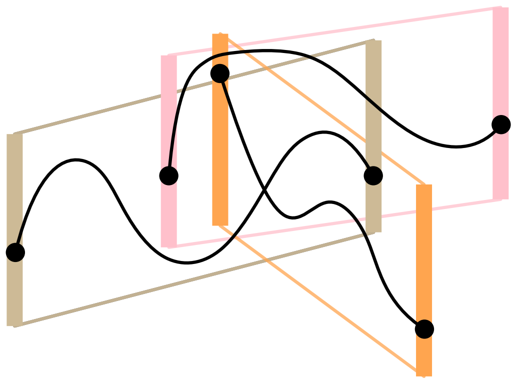
Non-crossing paths with geographic constraints

Rodrigo I. Silveira

UPC Barcelona

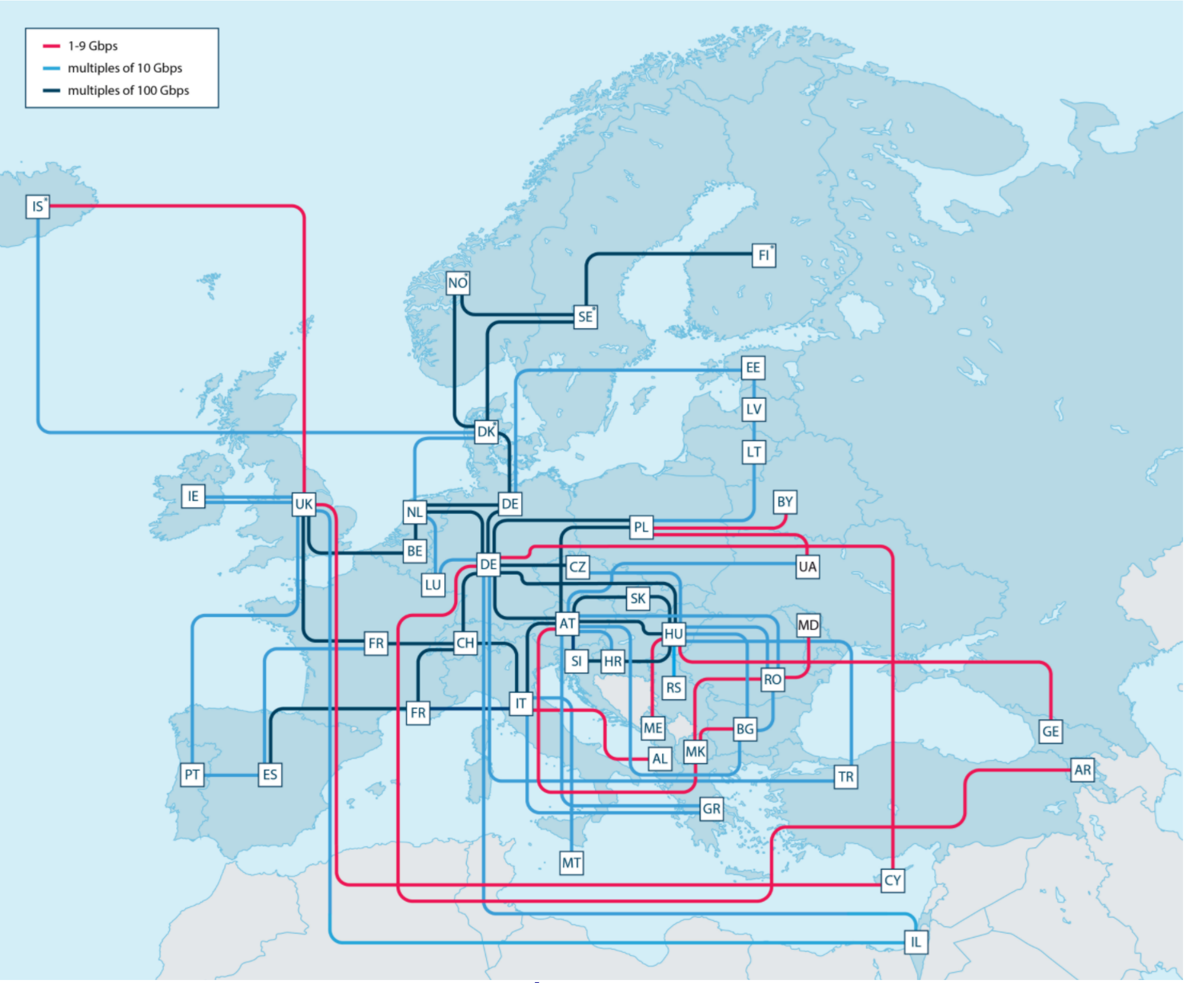


Bettina Speckmann

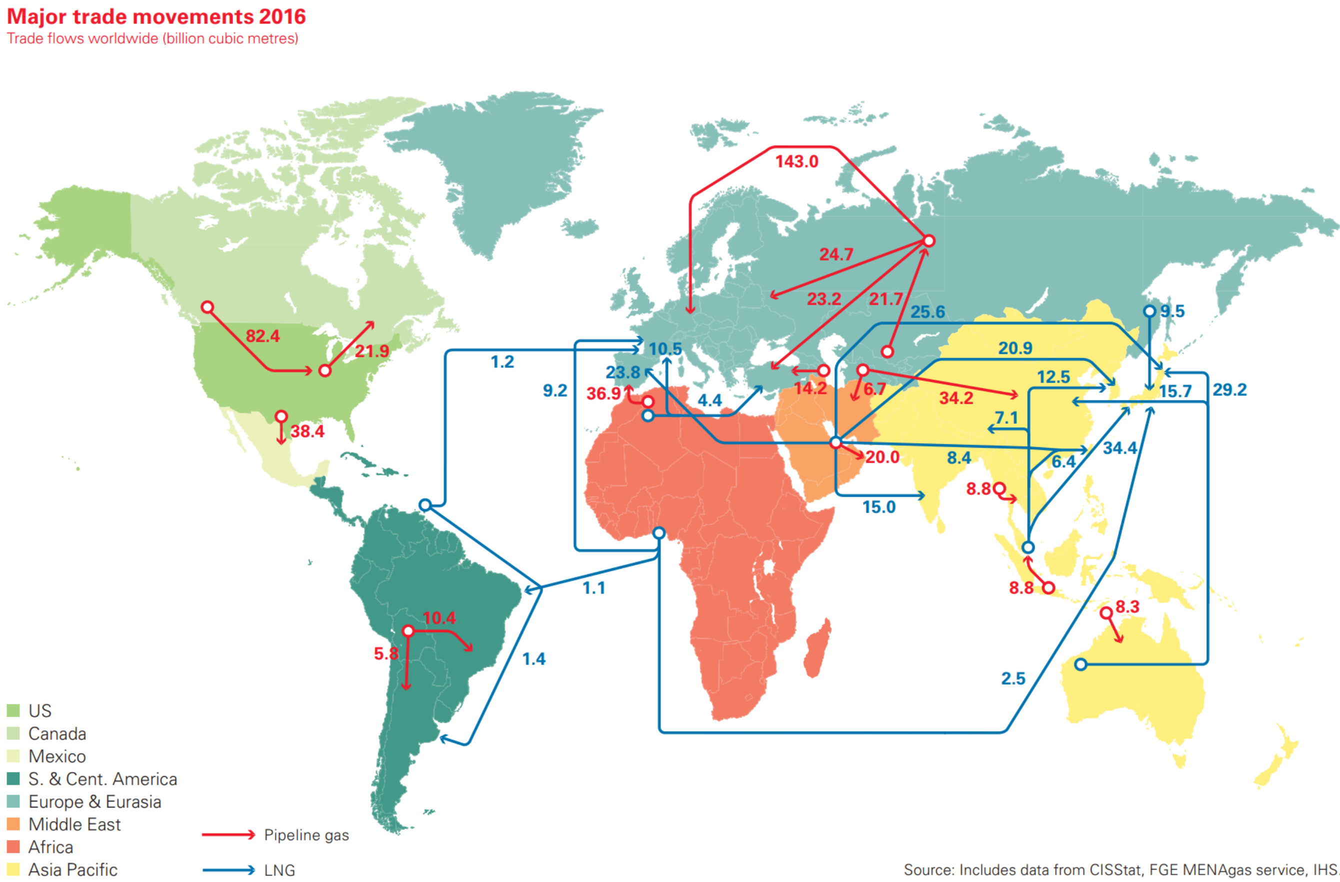
Kevin Verbeek

TU Eindhoven

Geographic networks



Topology map of GÉANT (pan-European research and education network) [geant.org]



Gas trade map [BP Statistical Review of World Energy '17]

Drawing geographic networks

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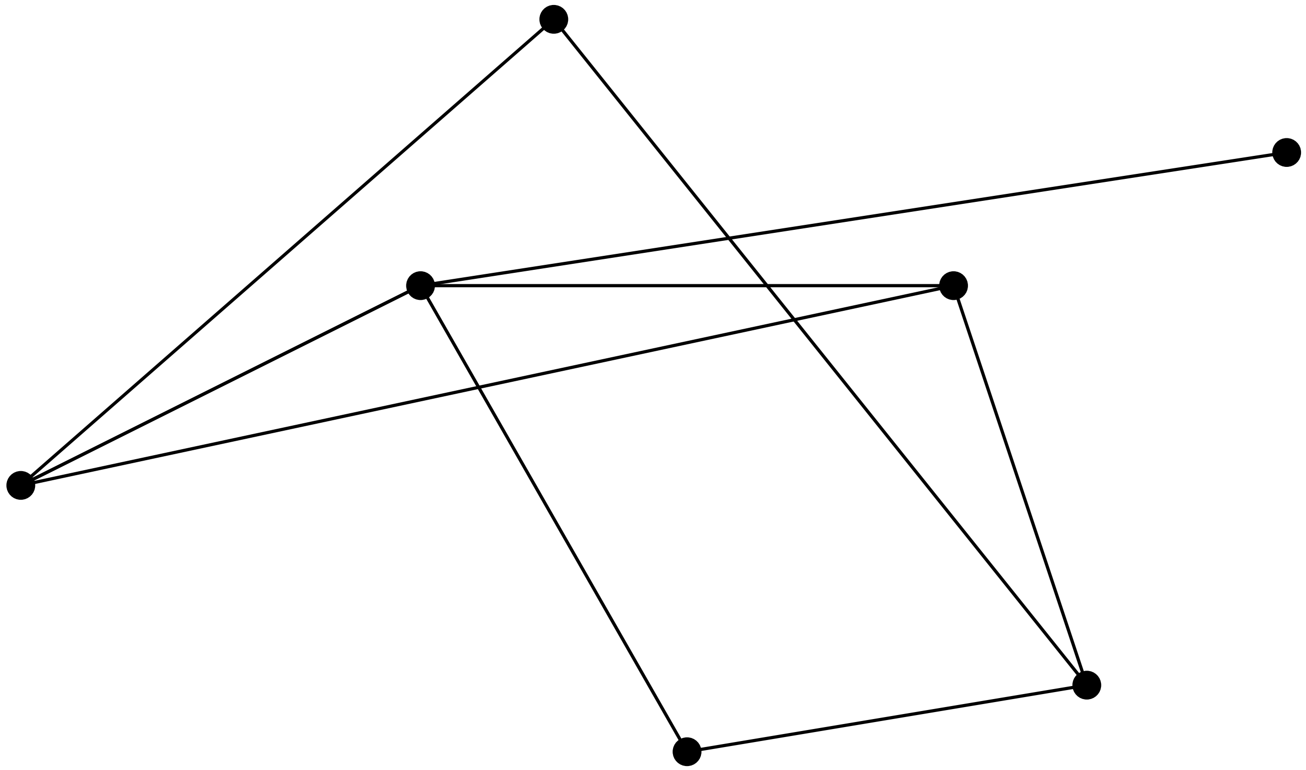

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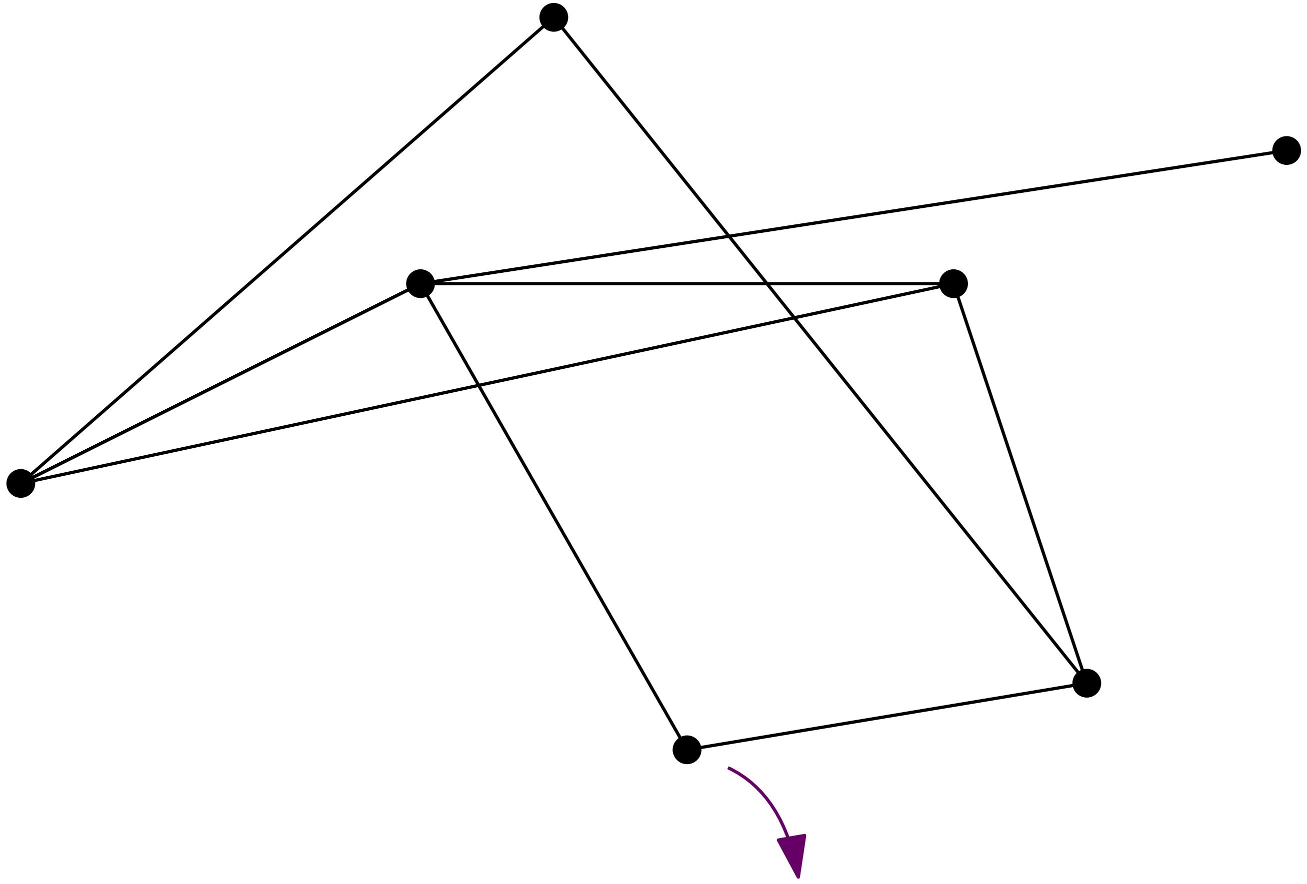
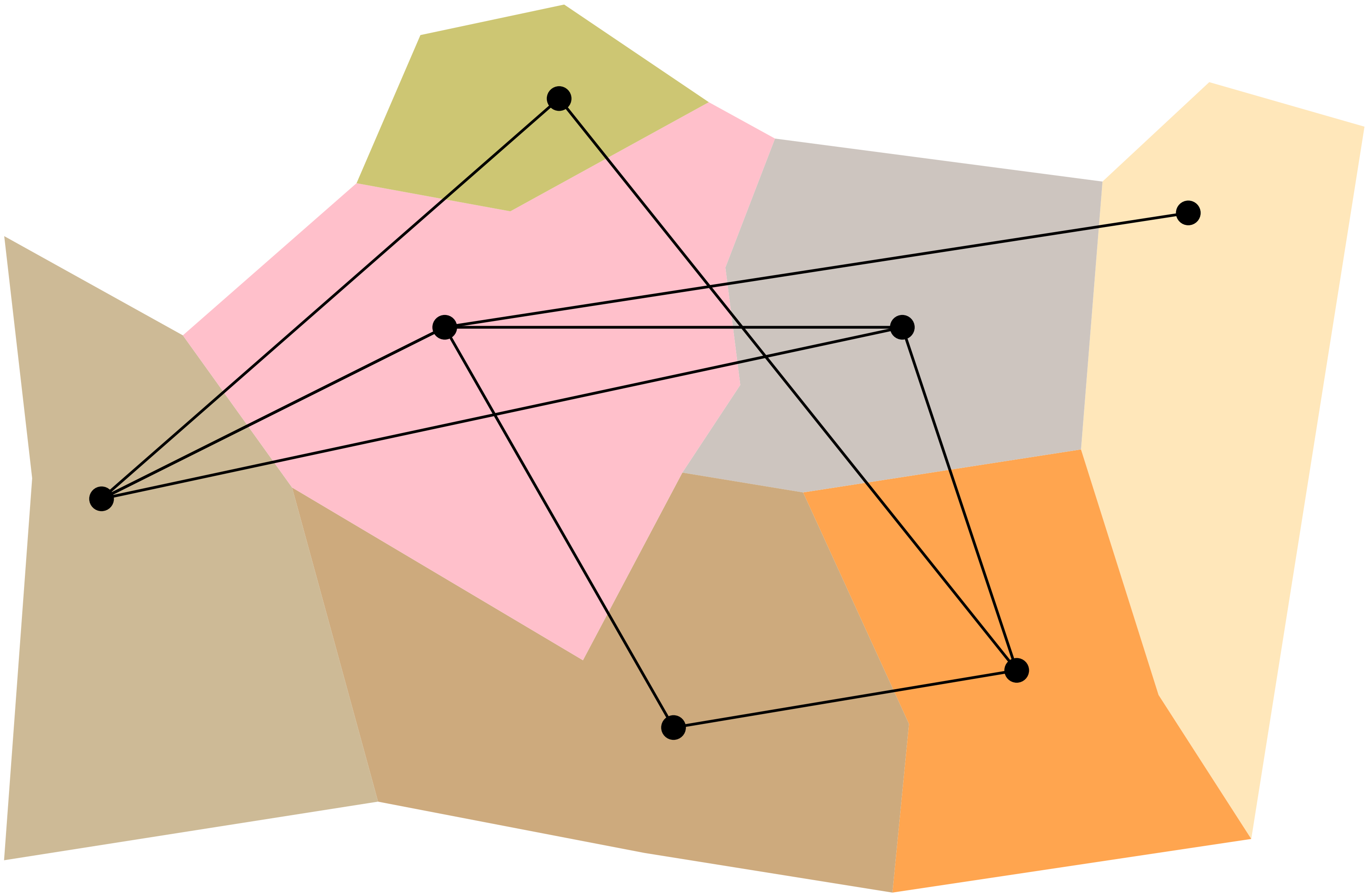


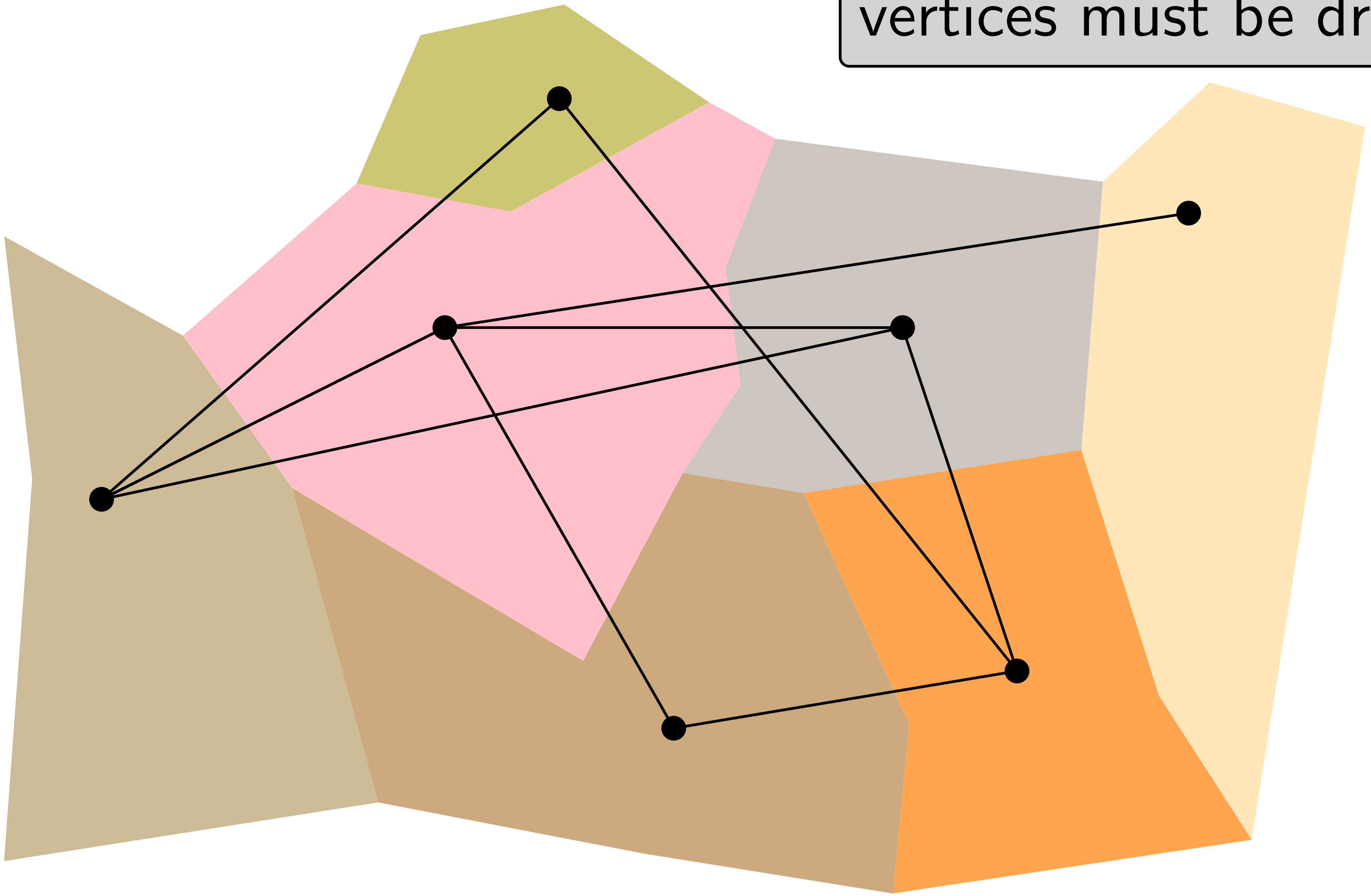
vertices: geographical regions

Drawing geographic networks



Drawing geographic networks

vertices must be drawn inside regions



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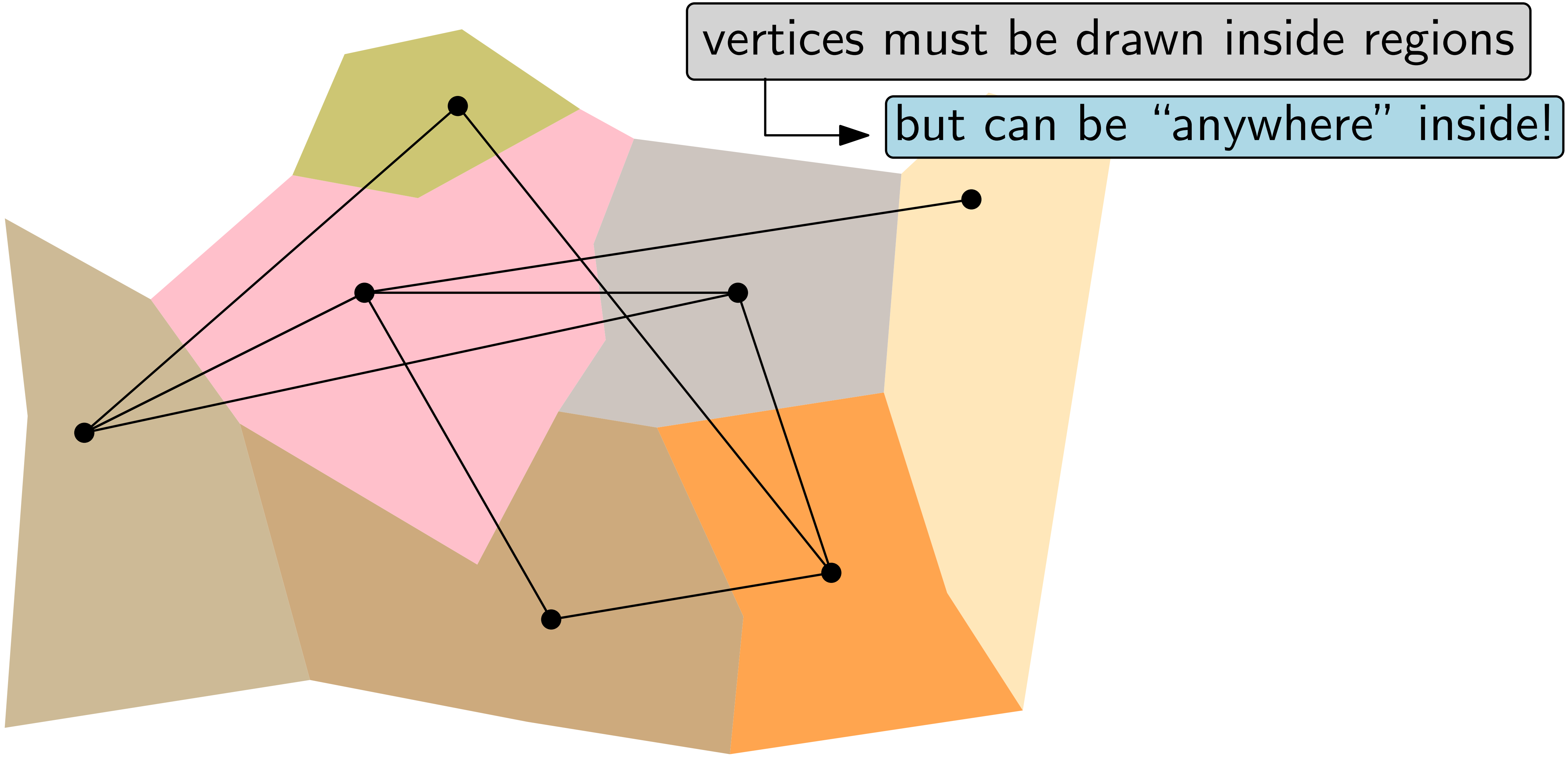
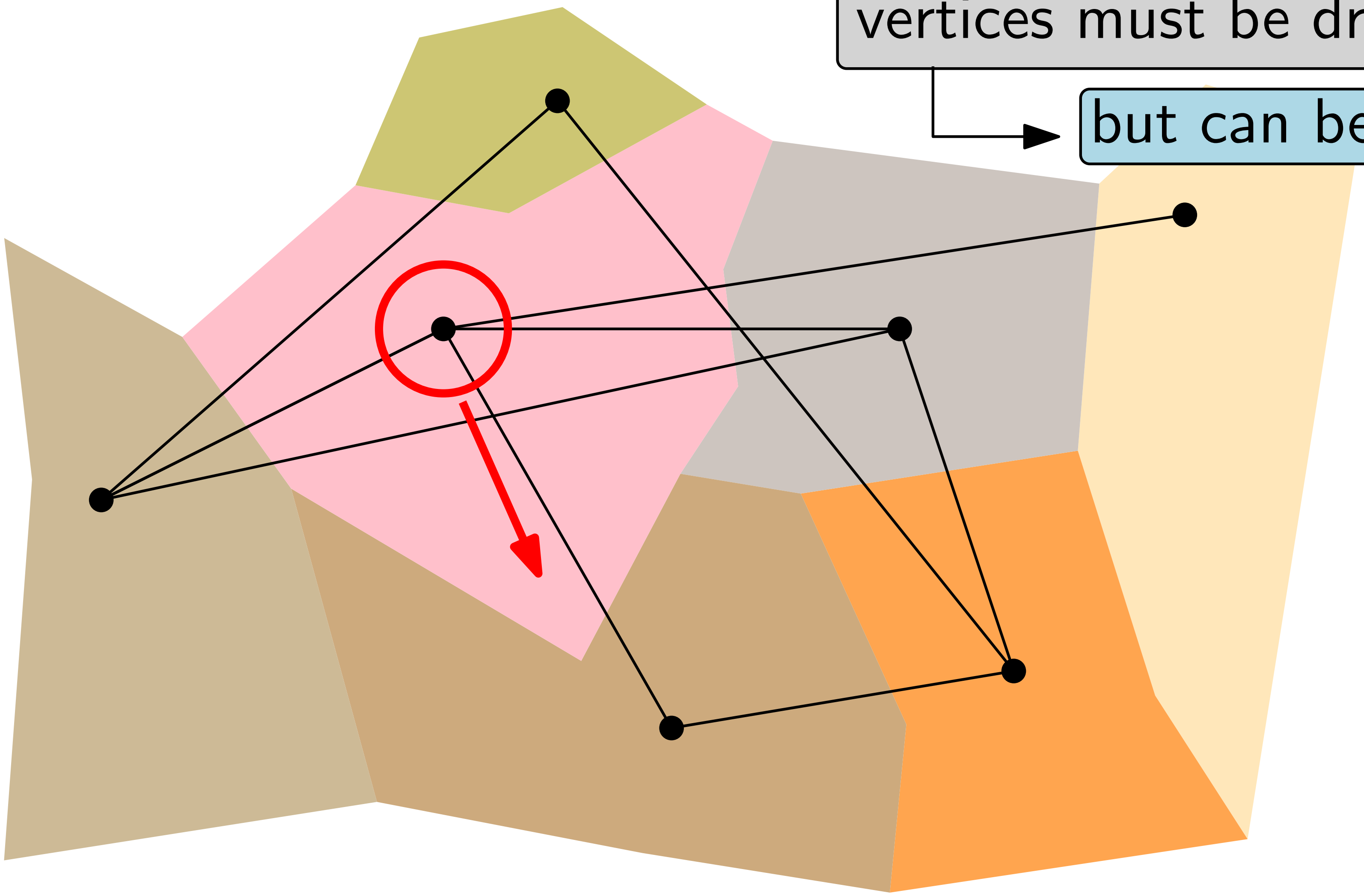


Drawing geographic networks

vertices must be drawn inside regions

but can be "anywhere" inside!



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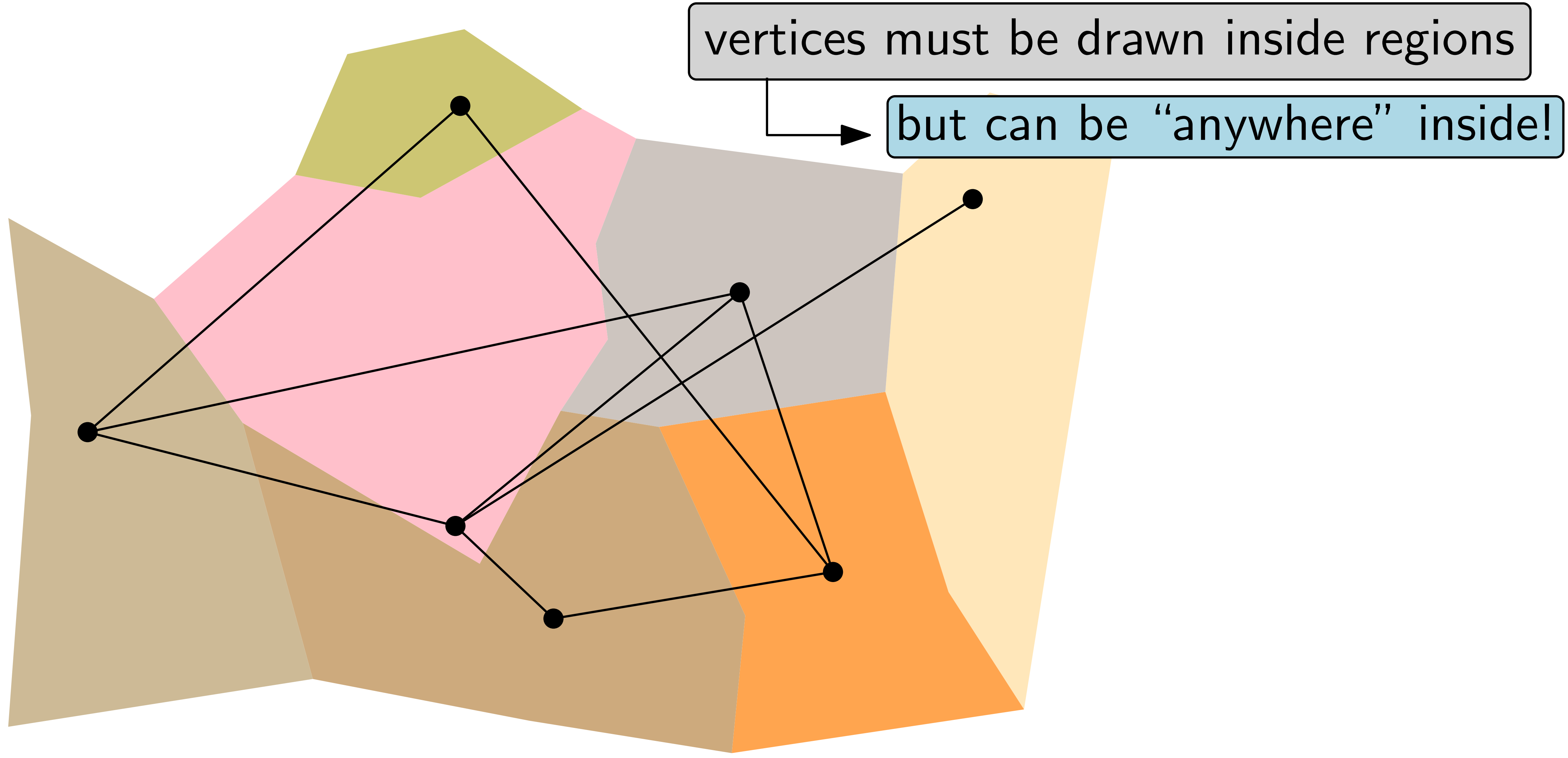

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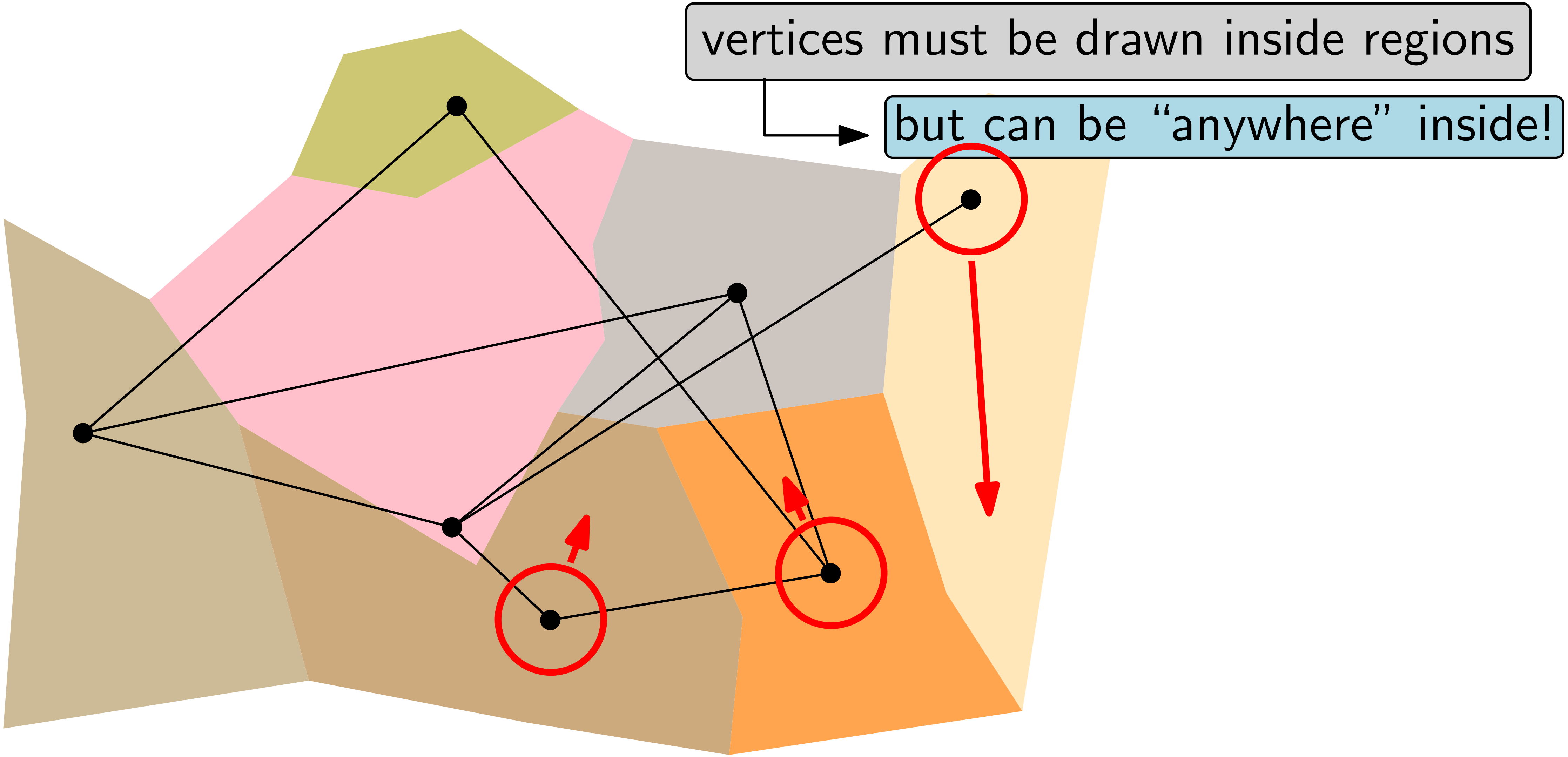

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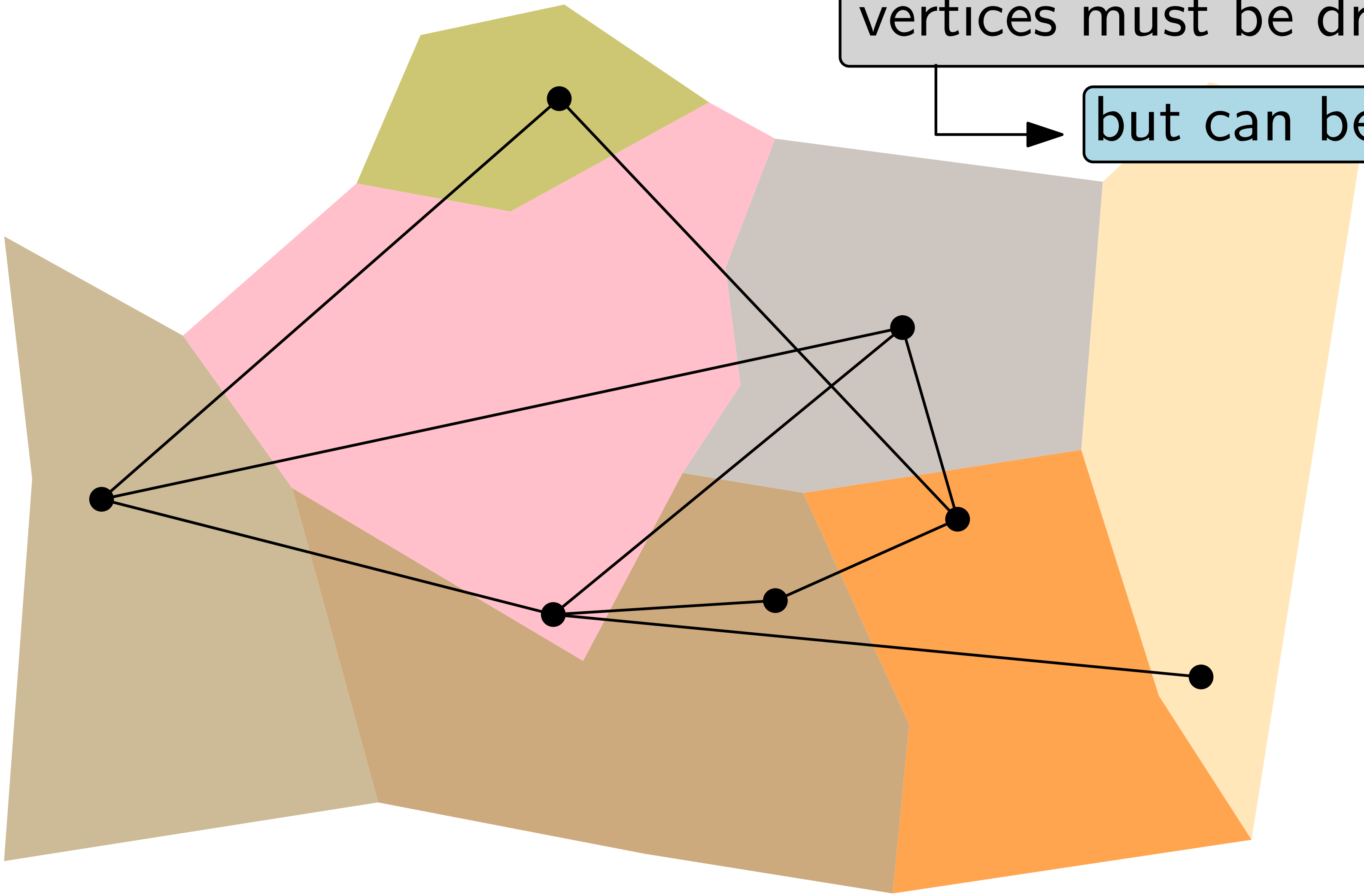

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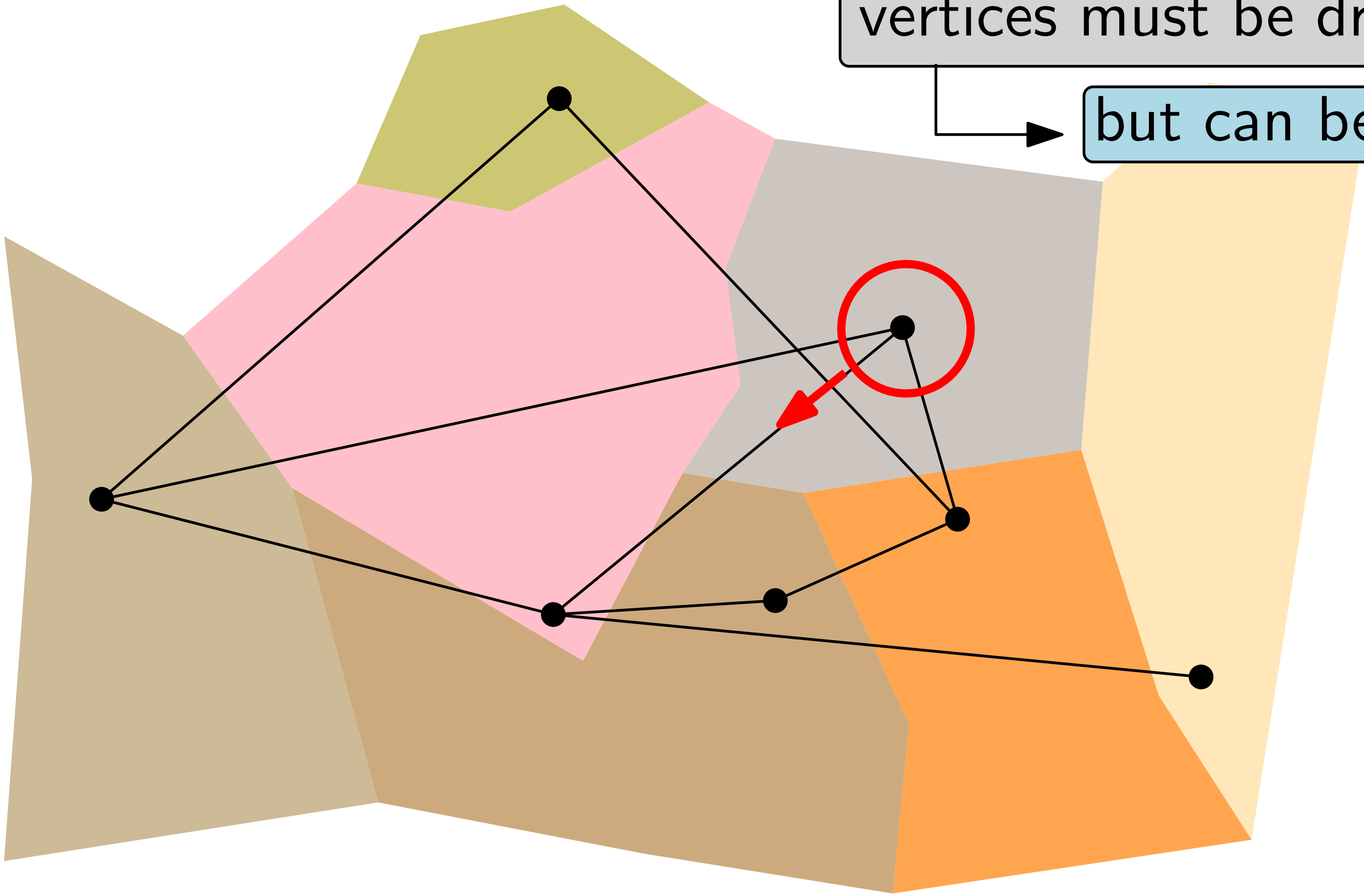

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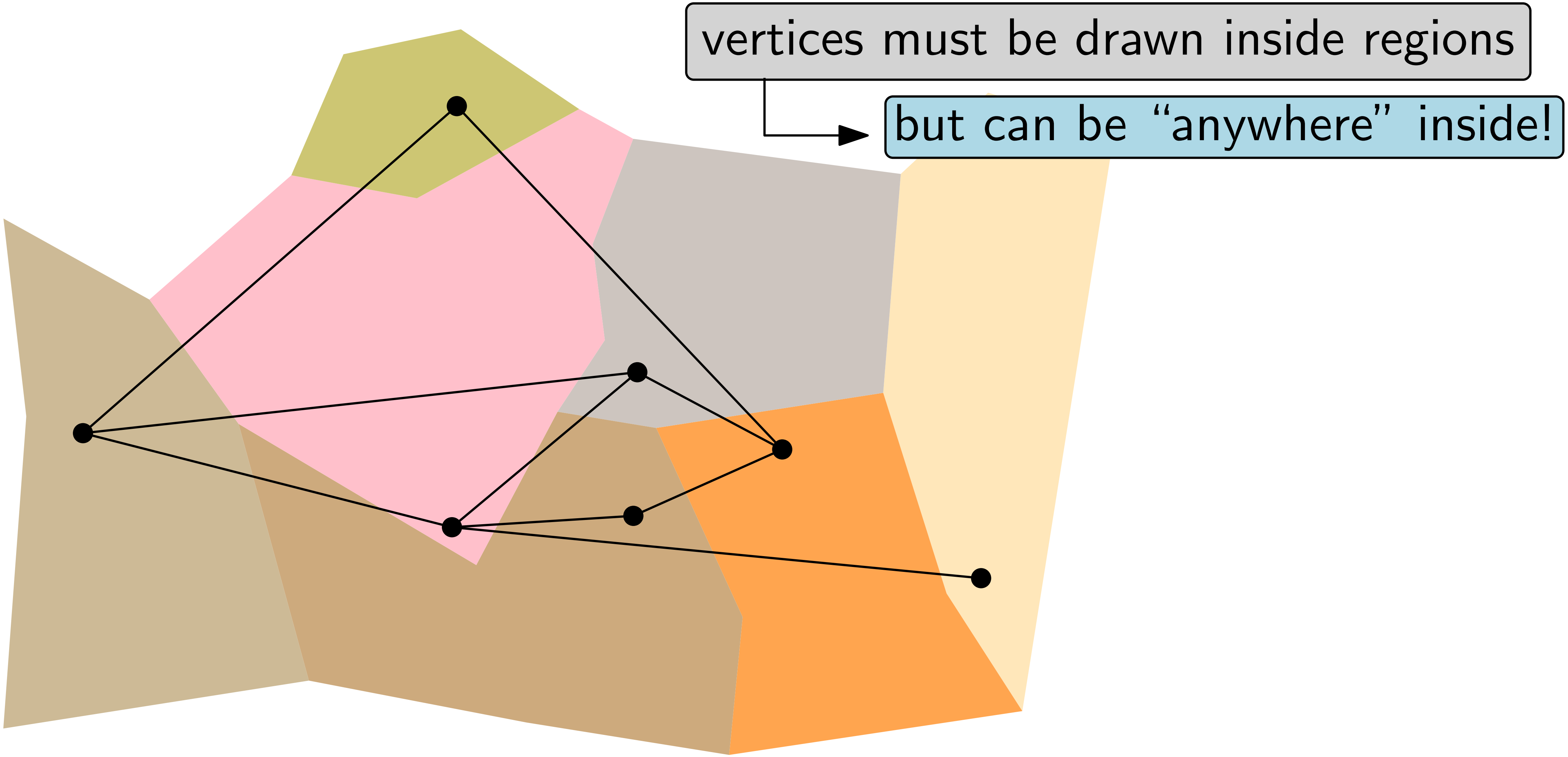

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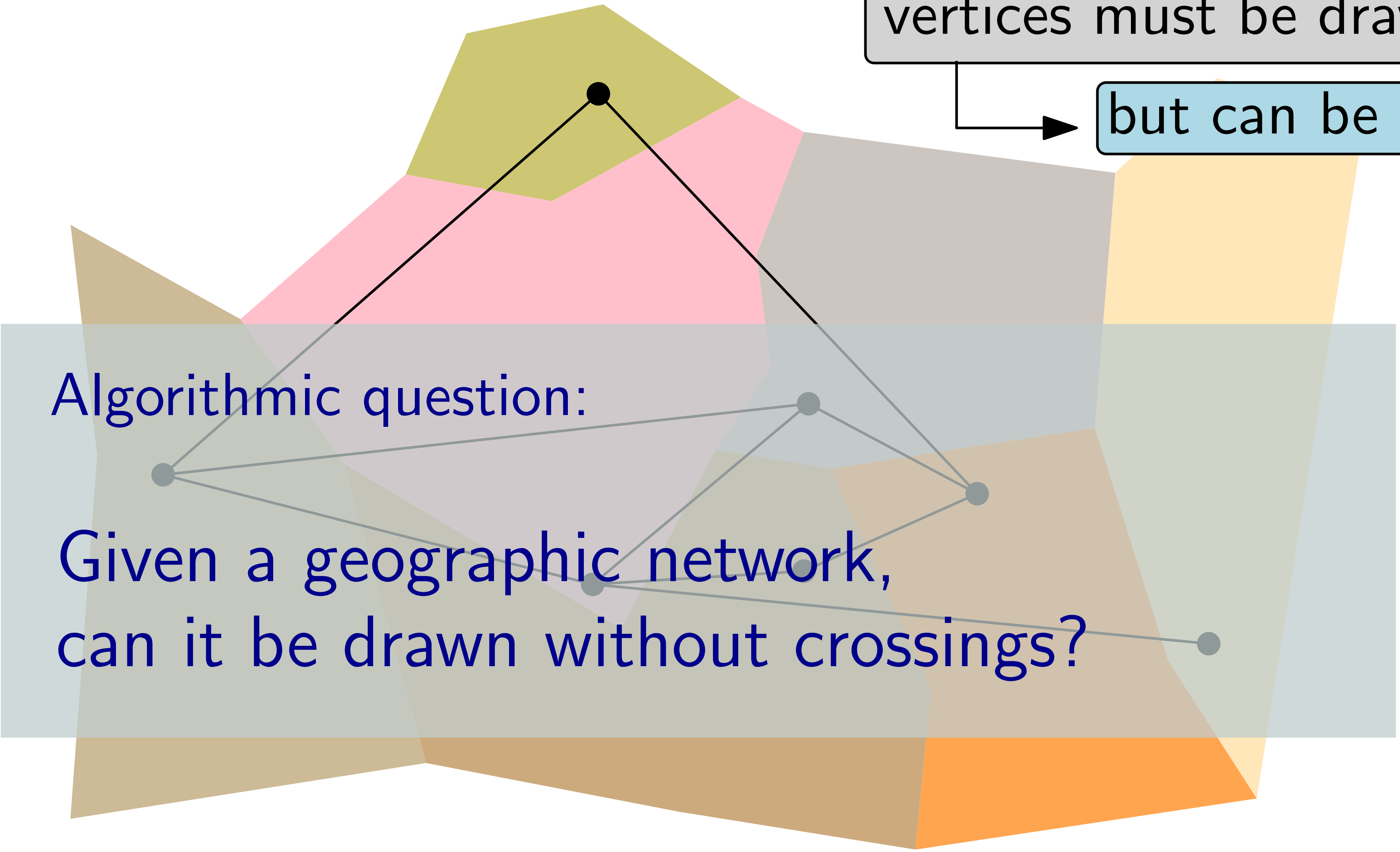

Algorithmic question:

Given a geographic network,
can it be drawn without crossings?

The big problem

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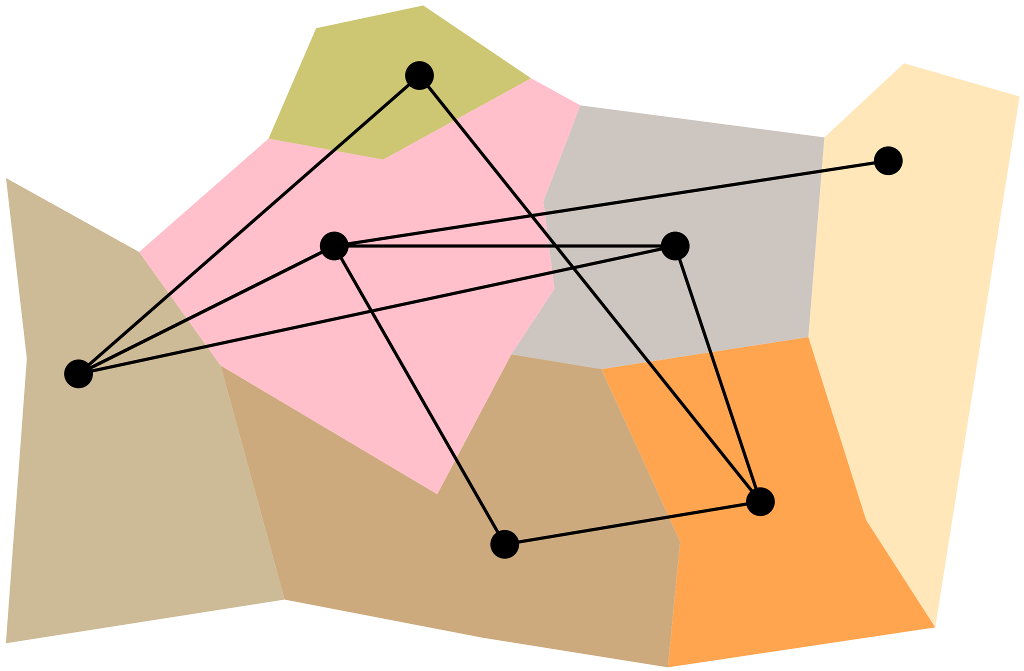

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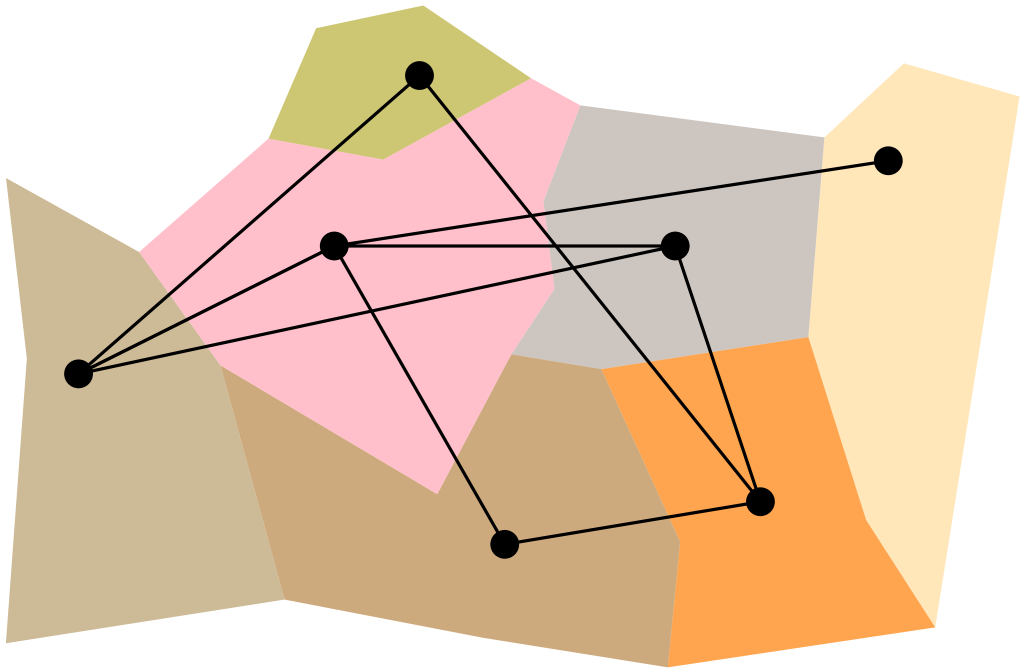

Need to precise:

- Type of graph
- Regions
- Curves to draw edges

Our problem

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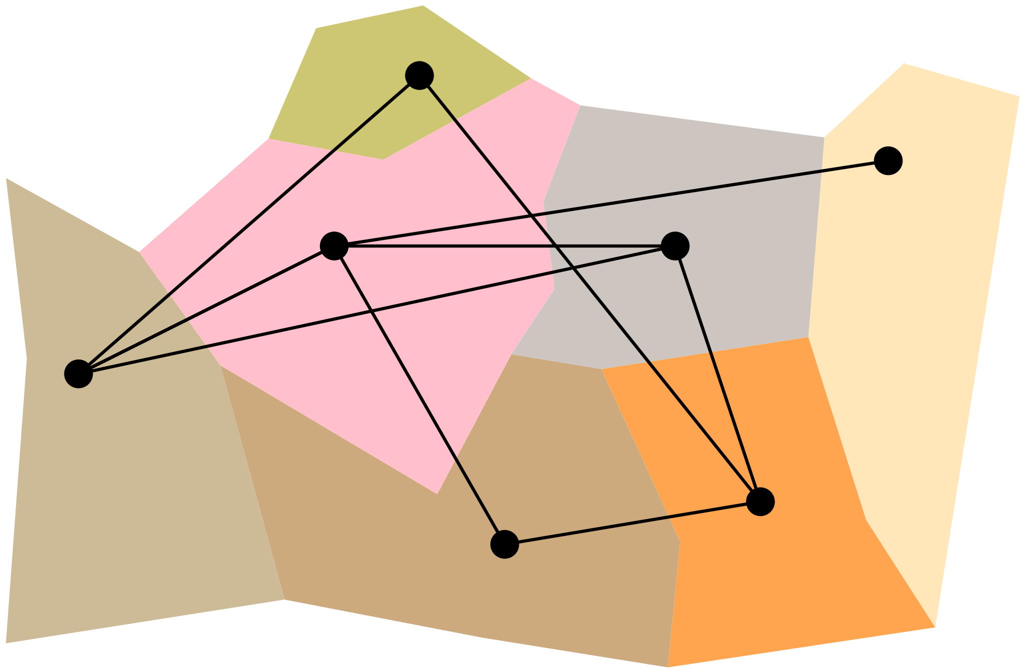

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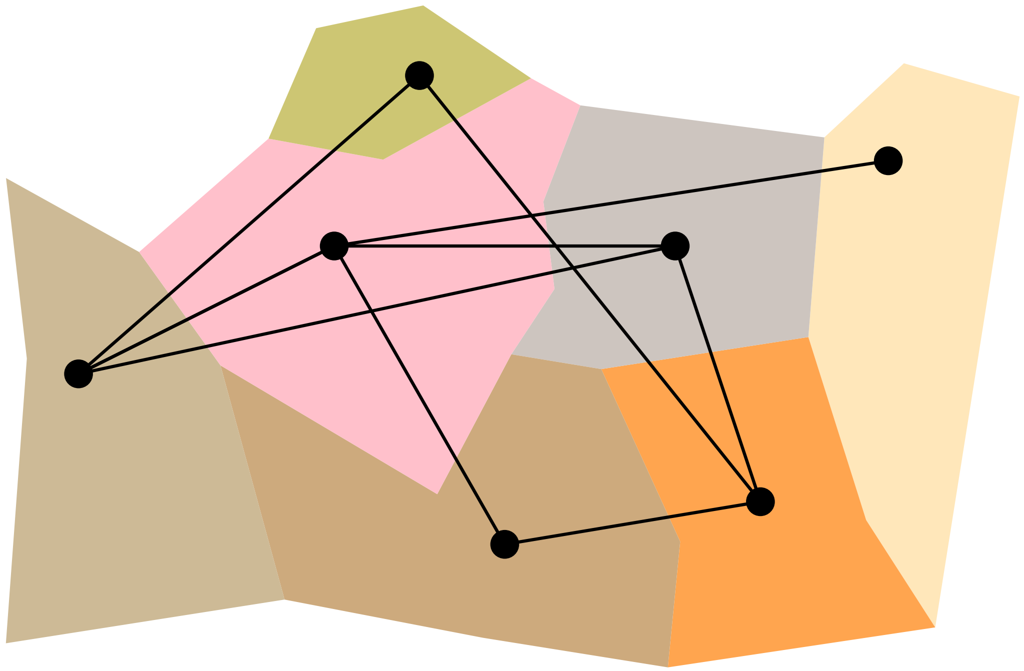

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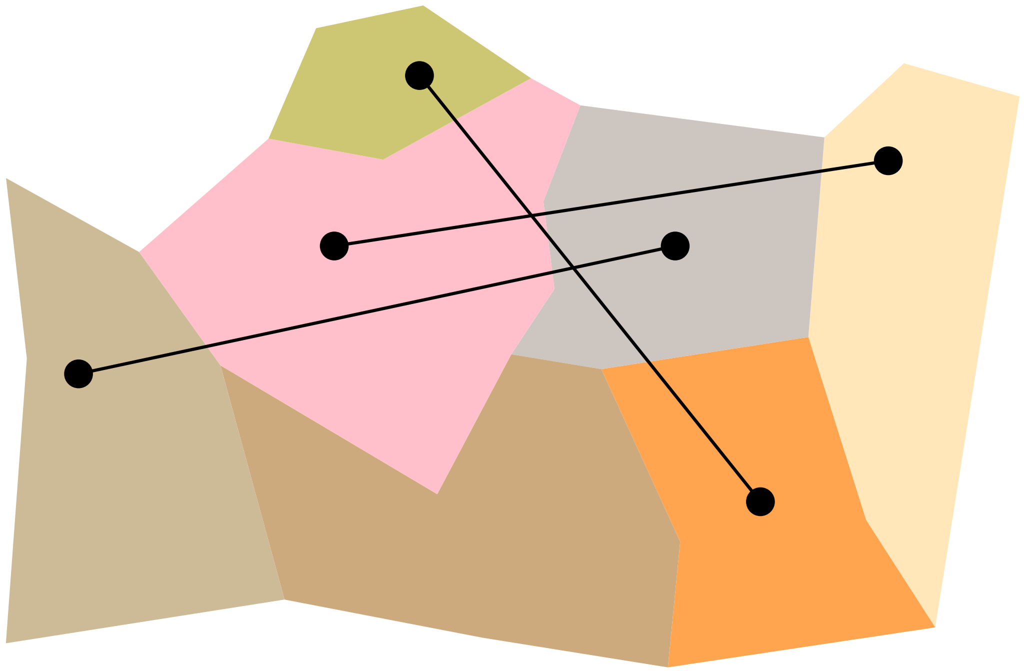

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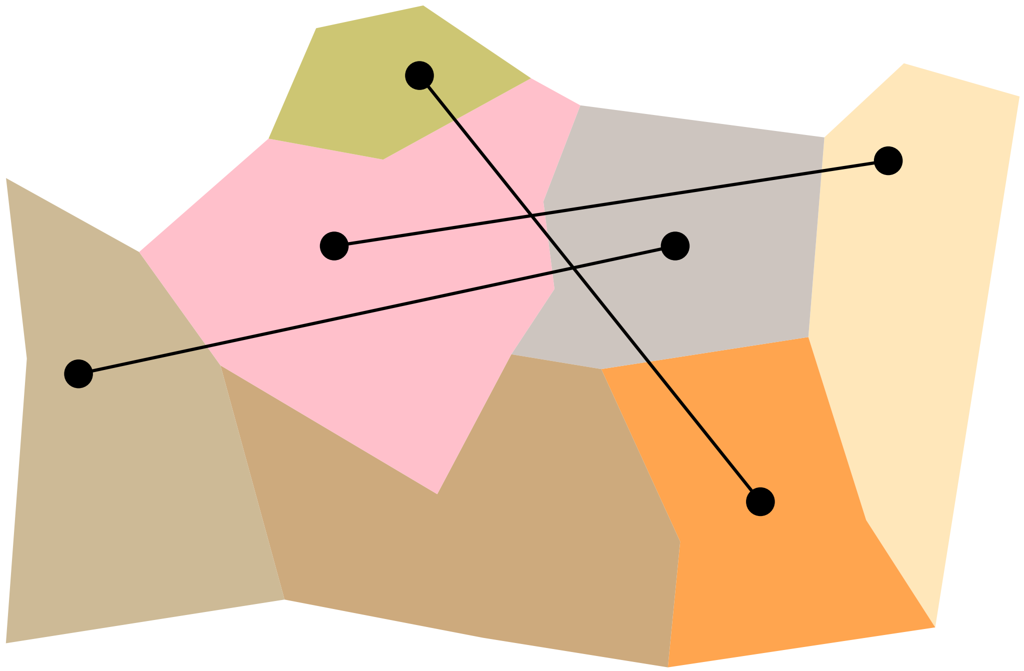

Need to precise:

- Type of graph $\leftarrow$ matching
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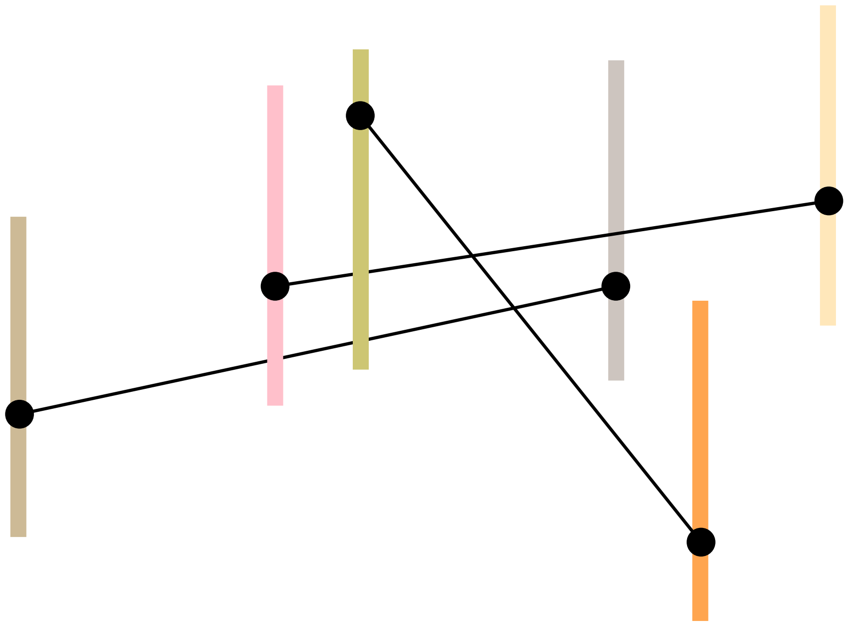

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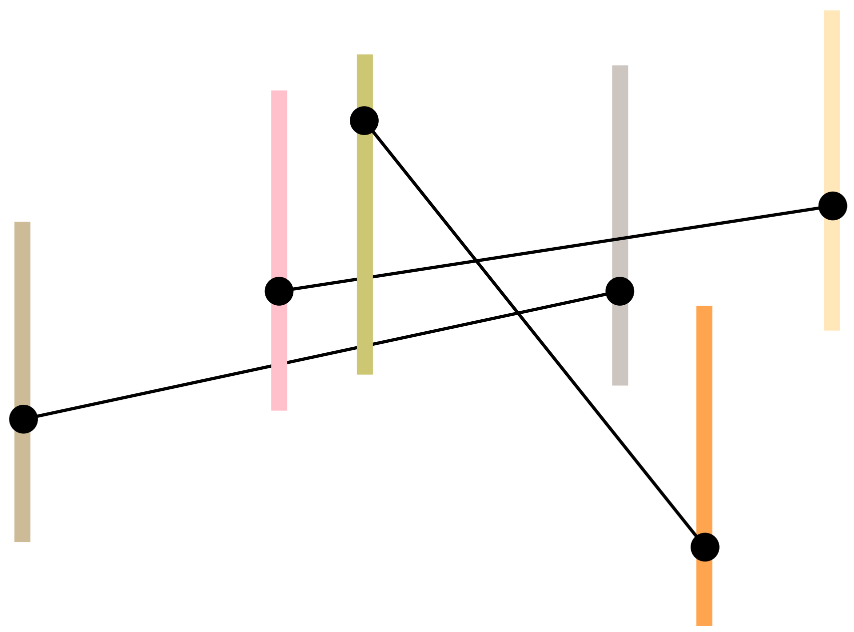

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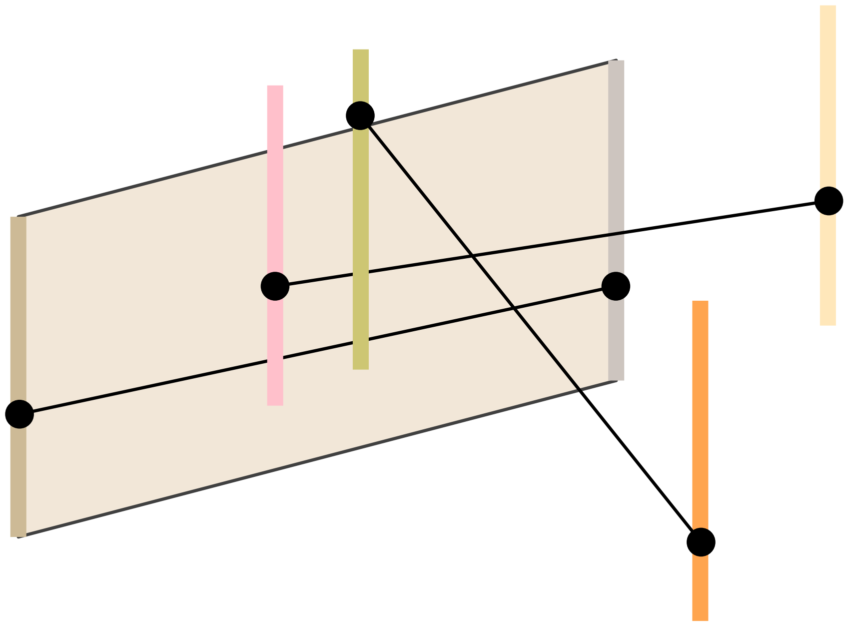

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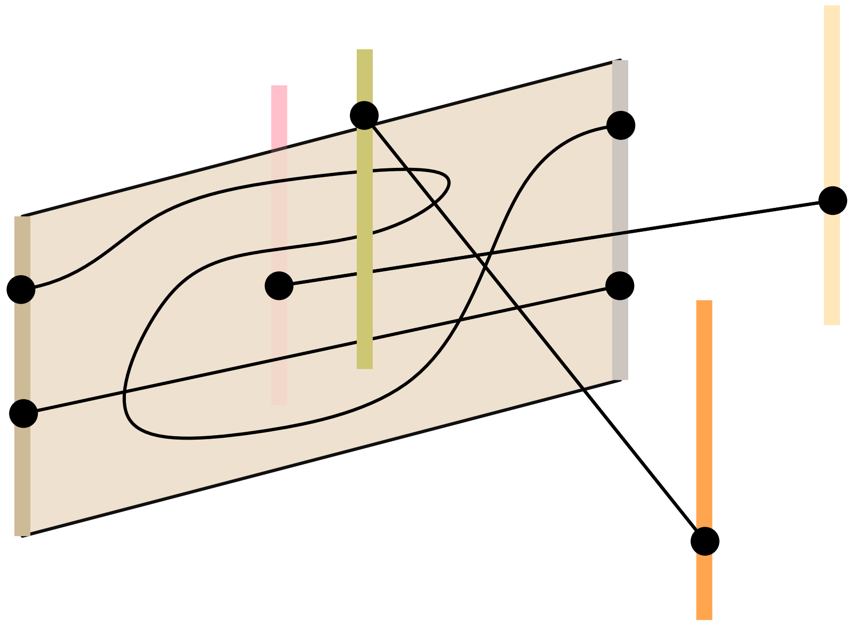

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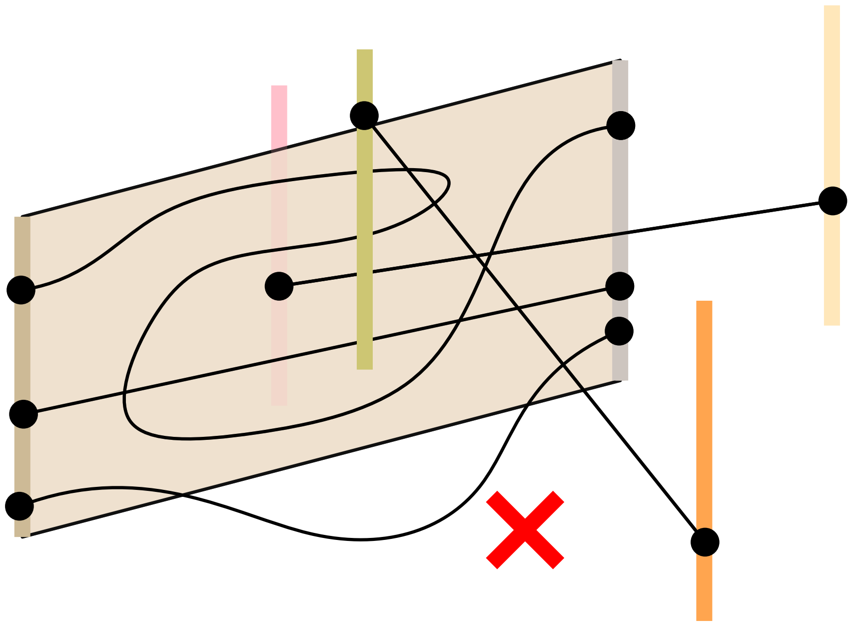

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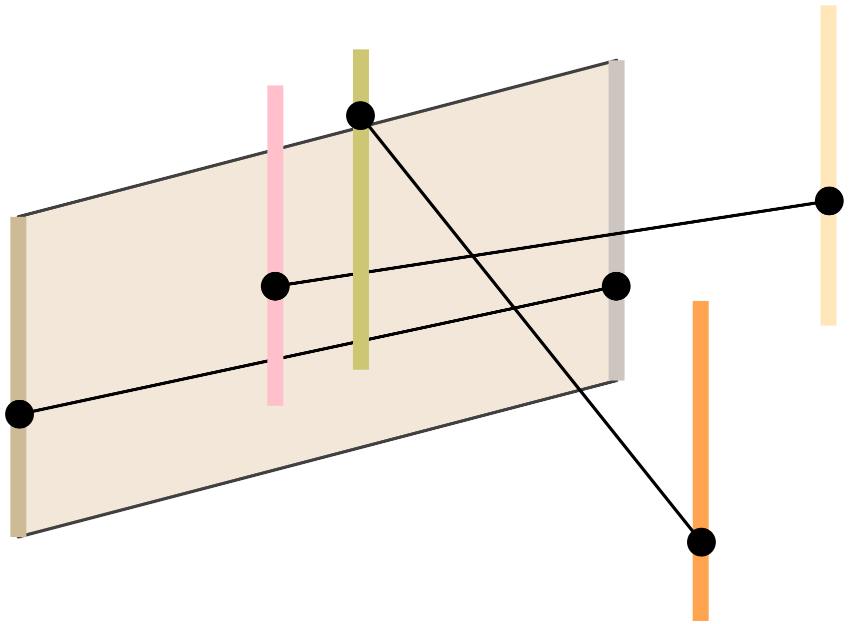

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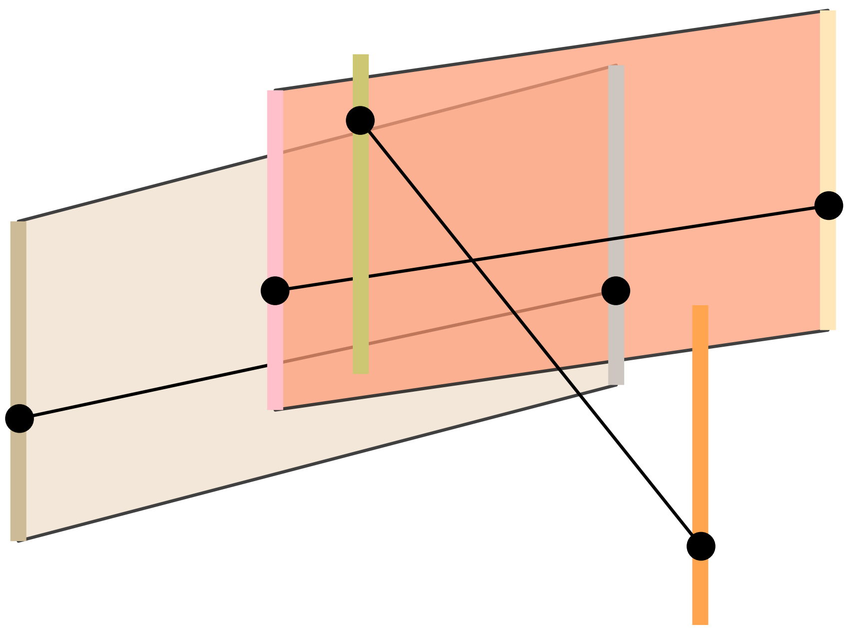

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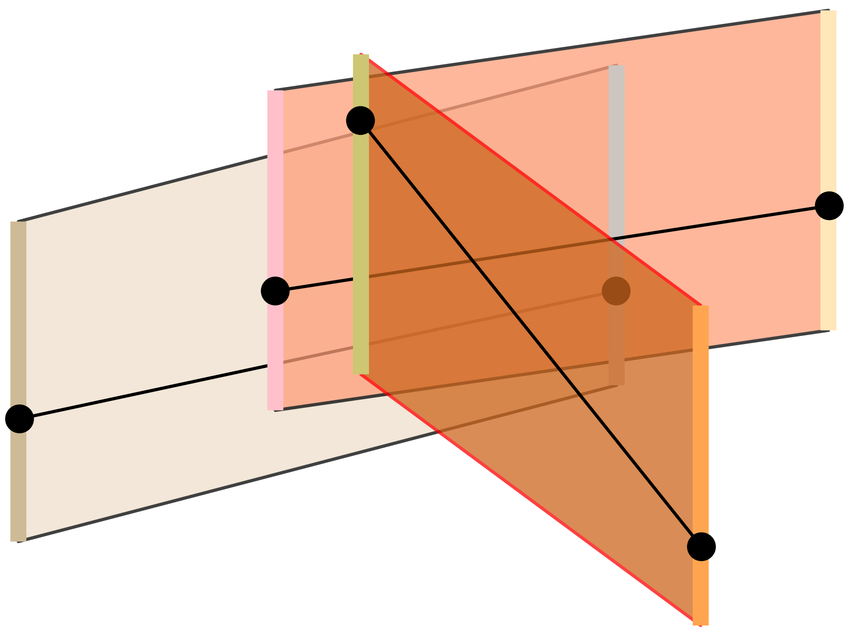

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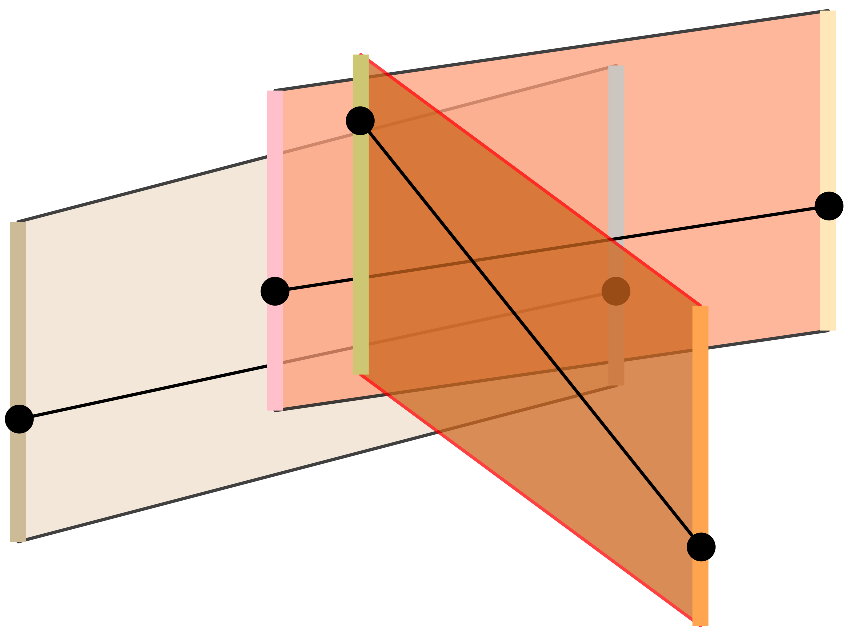

Need to precise:

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  ← three ways to draw edges

Our problem

Given a matching, where the endpoint of each edge must lie on a unit vertical segment, can it be drawn without crossings?

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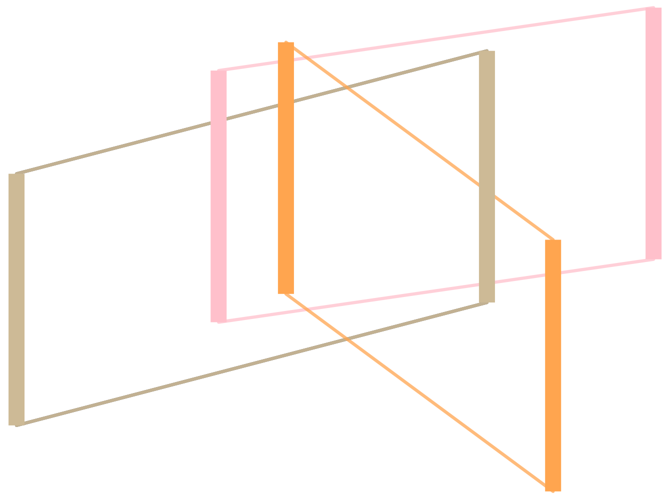
Three ways to draw edges inside tubes:



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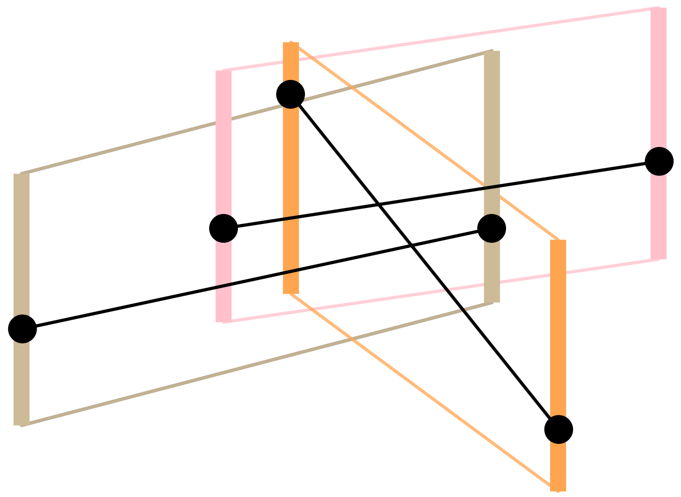
Three ways to draw edges inside tubes:

straight-line segment



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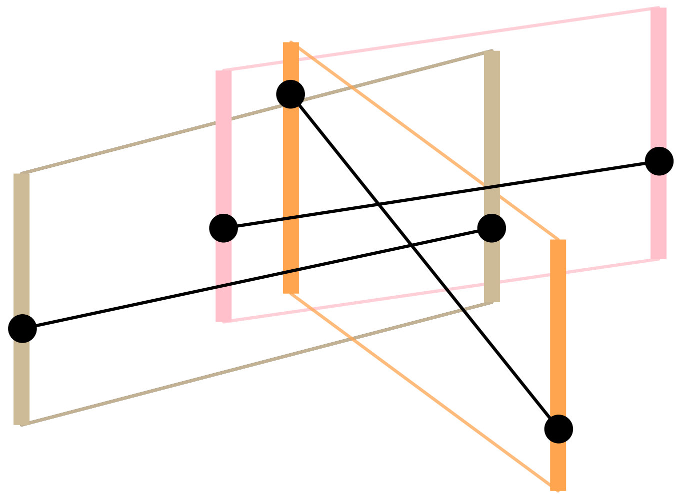
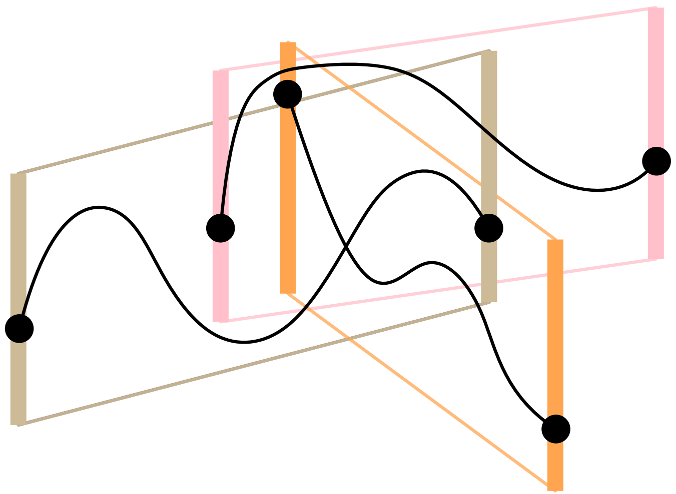


(x-)monotone paths



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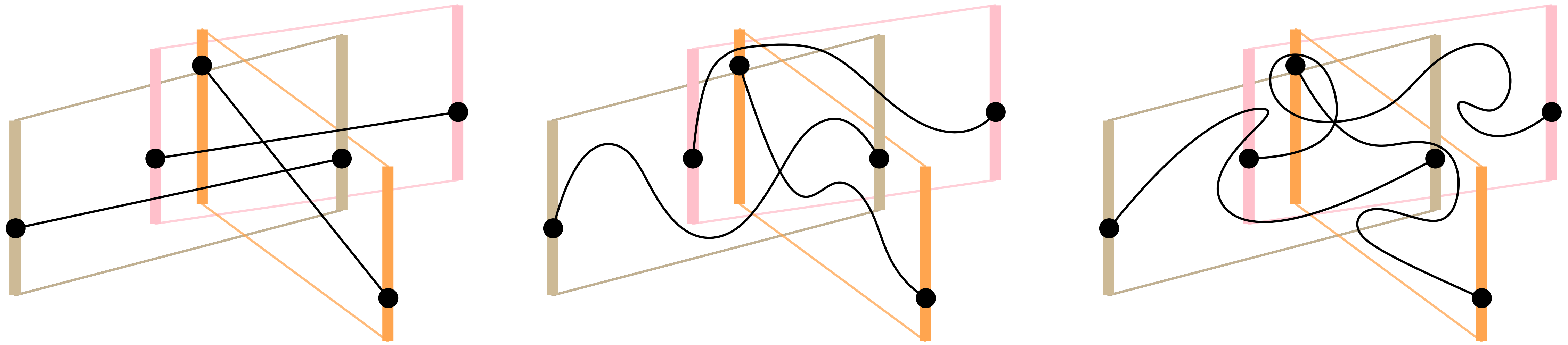
Given a matching, where the endpoint of each edge must lie on a unit vertical segment, can it be drawn without crossings?

Three ways to draw edges inside tubes:

straight-line segment | (x-)monotone paths | arbitrary paths



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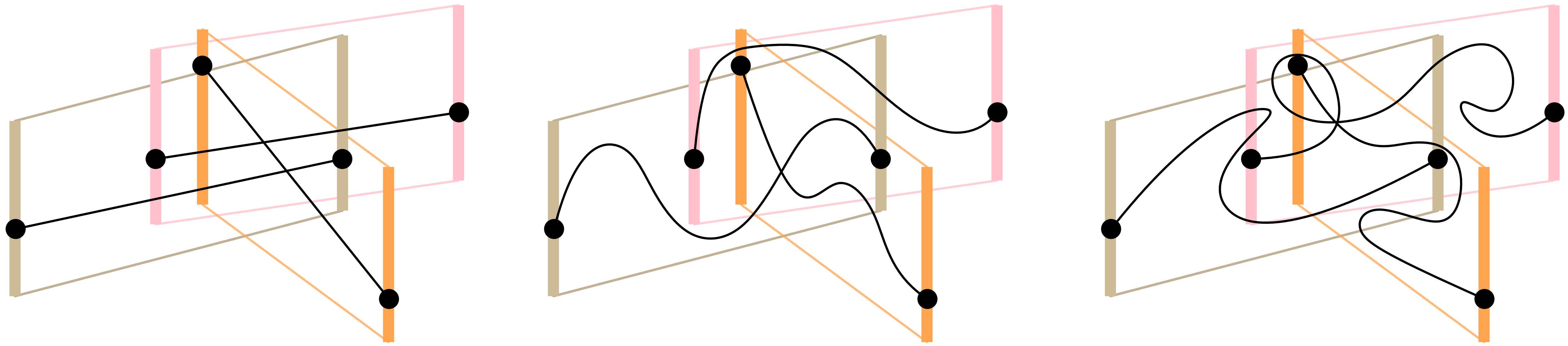
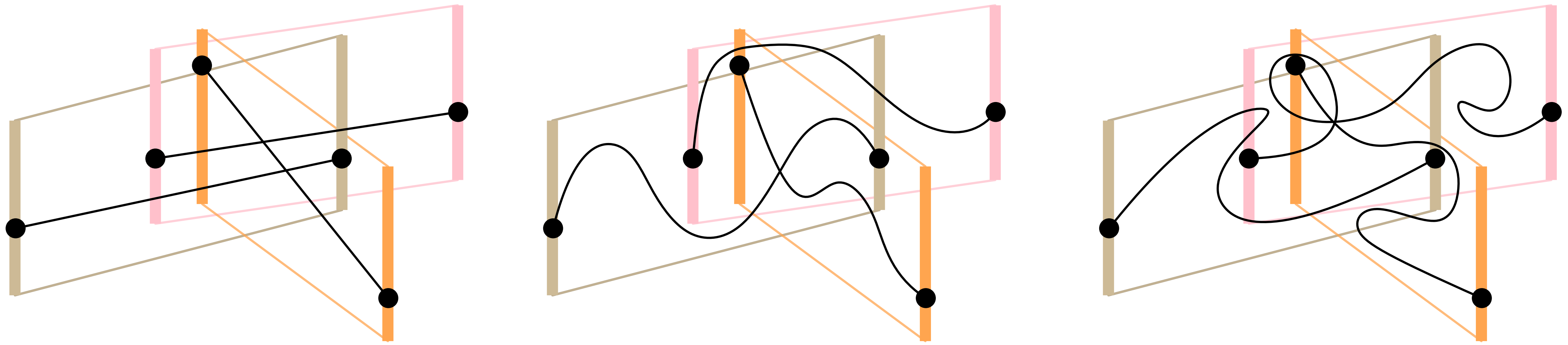


Recall: endpoints can be anywhere on the vertical line segments

Our results

Given a matching, where the endpoint of each edge must lie on a unit vertical segment, can it be drawn without crossings?

straight-line paths | (x-)monotone paths | arbitrary paths



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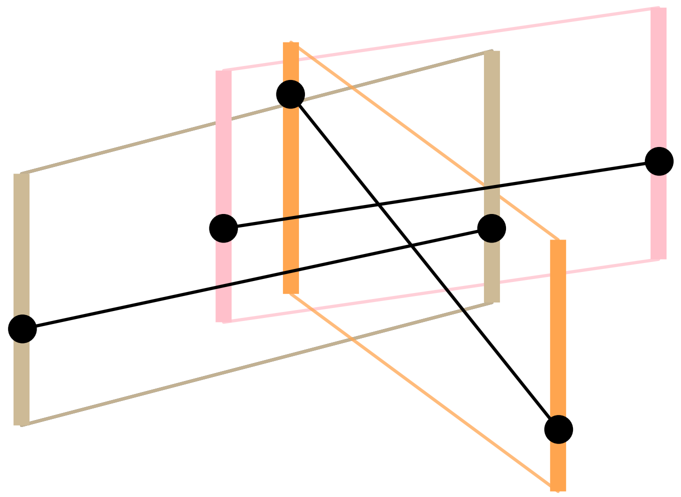

NP-complete

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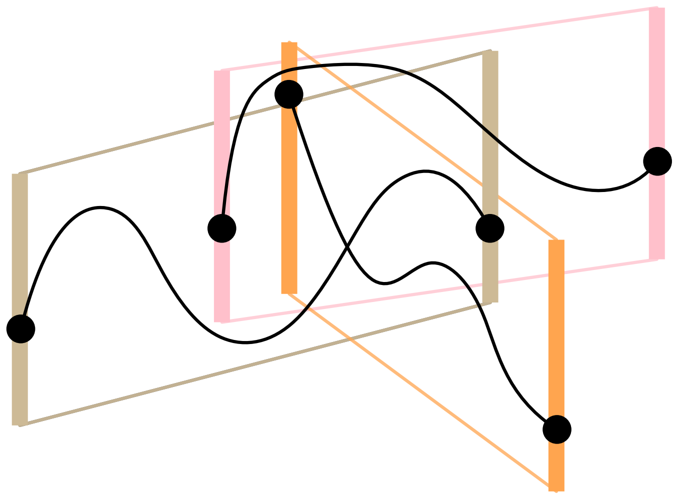
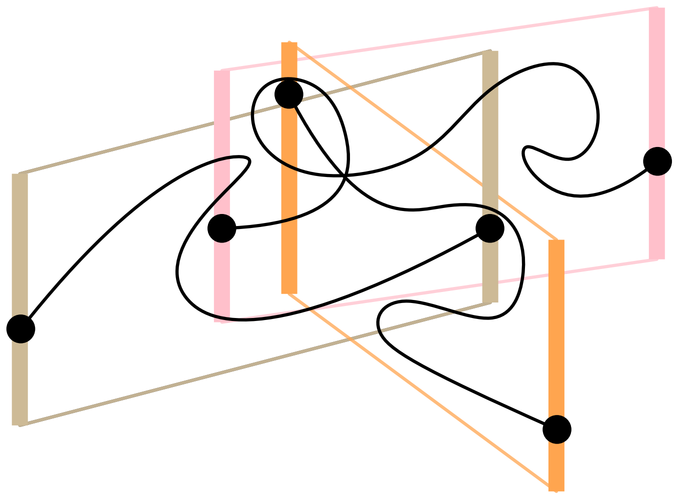


arbitrary paths



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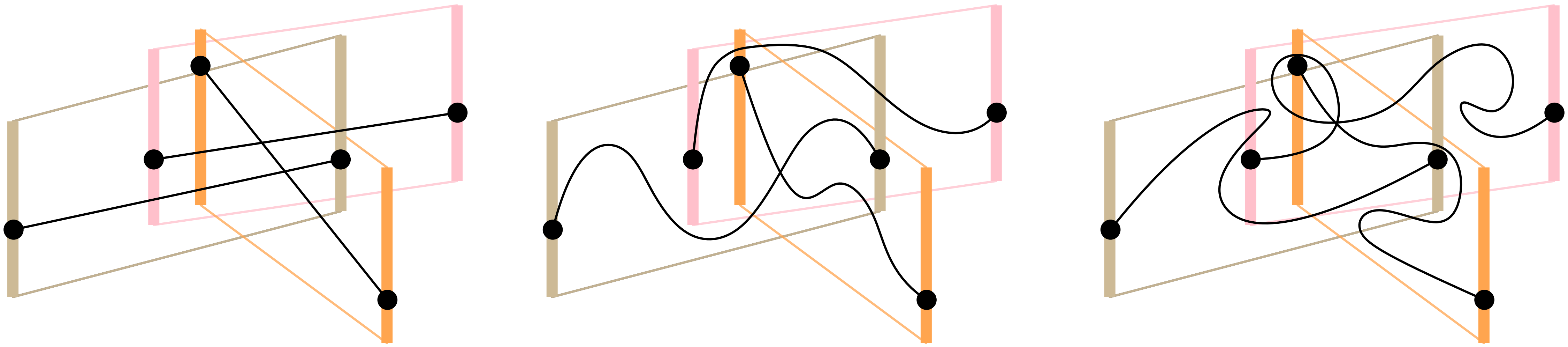

NP-complete | Polynomial

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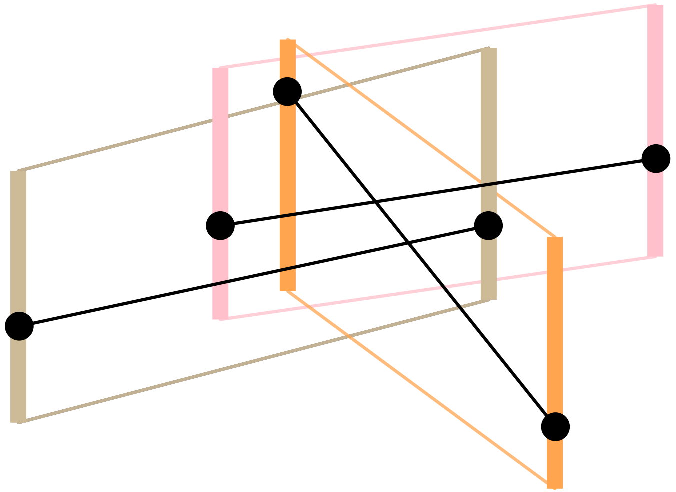

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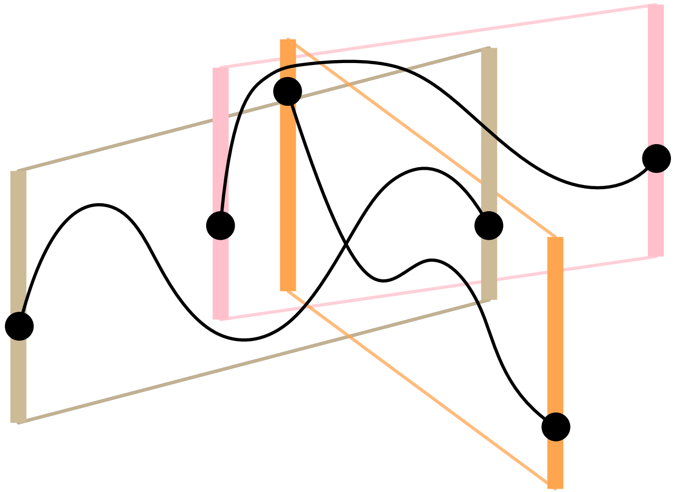
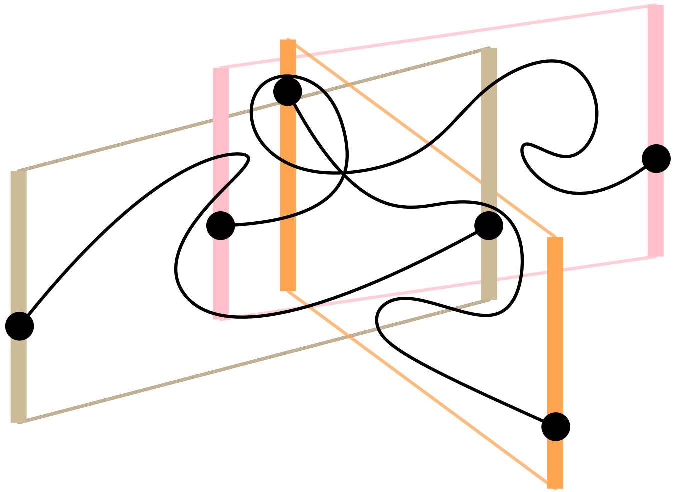


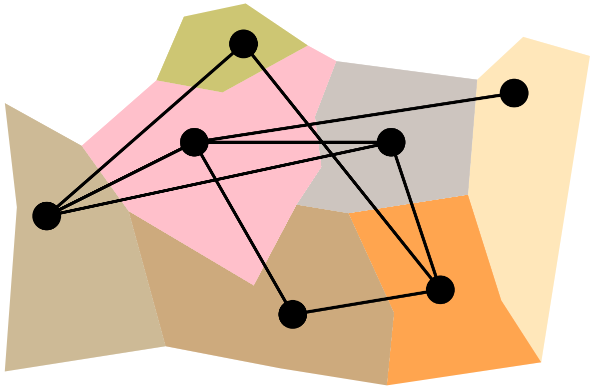
Polynomial

arbitrary paths



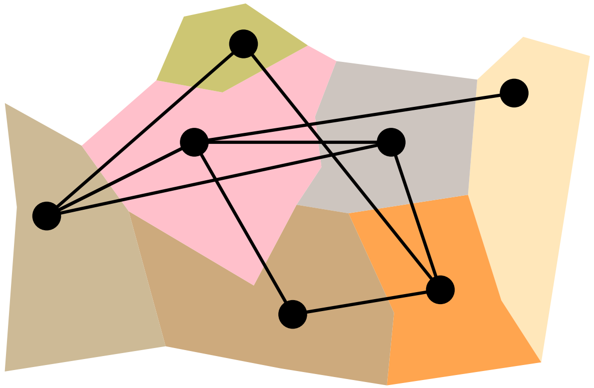
Polynomial under certain assumptions

Related work



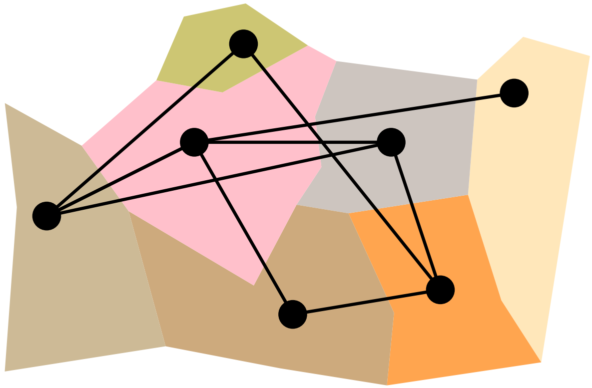
Related work

- Force-directed approach for general problem [Abellanas et al., 2005]



Related work

- Force-directed approach for general problem [Abellanas et al., 2005]
- Straight-line case shown NP-hard for:
  - Cycle graphs when regions are vertical segments [Löffler, 2011]
  - Matchings when regions are vertical segments [Aloupis et al., 2015; Verbeek, 2008]
  - General graphs when regions are unit squares [Angelini et al., 2014]



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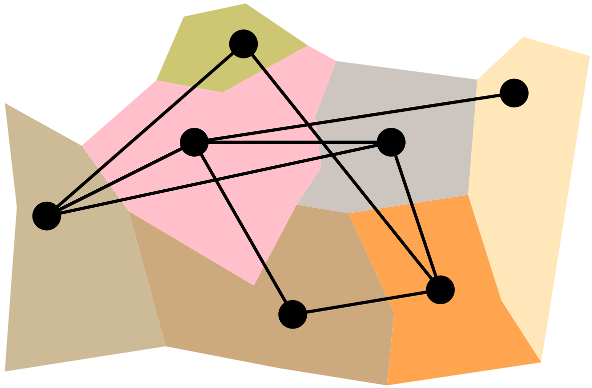
- Several other related problems:
  - Fitting planar graphs to planar maps [Alam et al., 2015]
  - c-planarity / ordered-level planarity [Feng et al., 1995, Klemz and Rote, 2017]
  - Manhattan geodesic planarity [Katz et al., 2009]
  - Non-crossing connectors in the plane [Kratochvíl and Ueckerdt, 2013]



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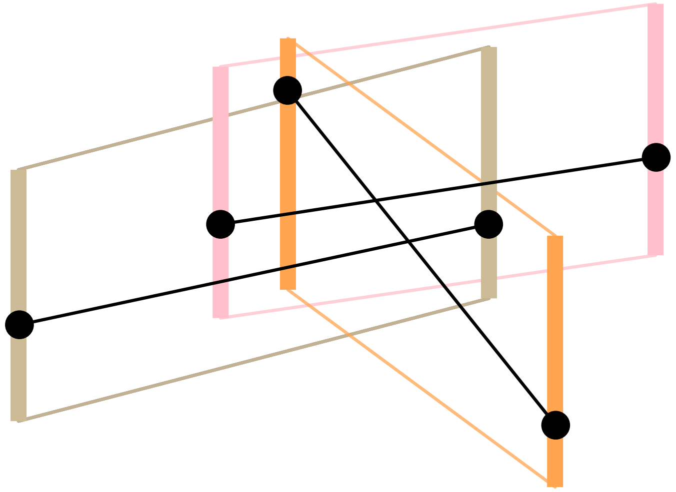

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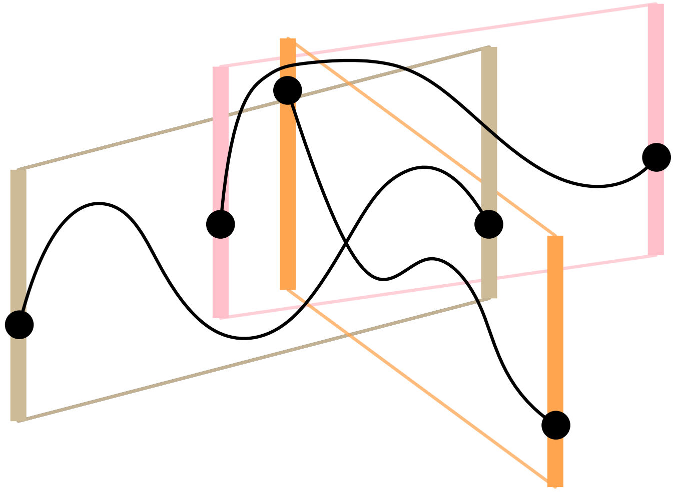

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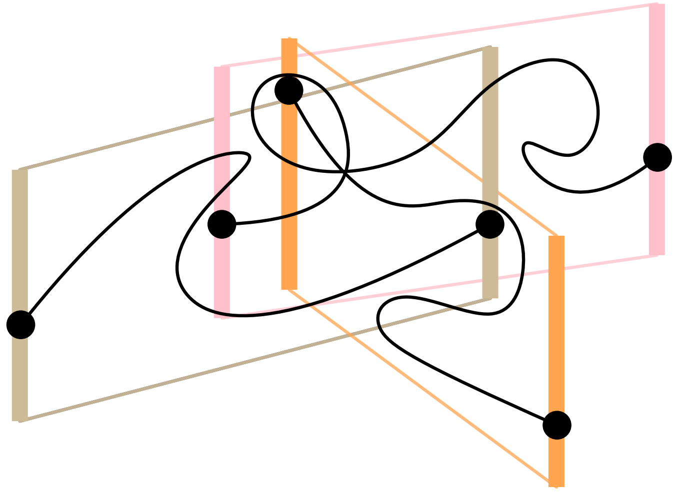

Polynomial under certain assumptions

How can two tubes intersect?

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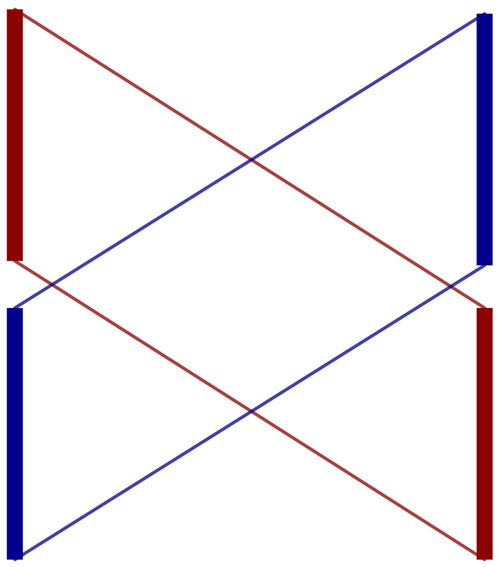
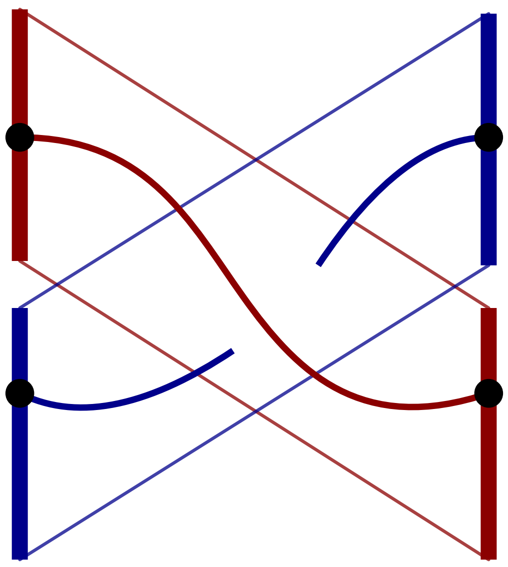


full crossing

(no solution)

How can two tubes intersect?



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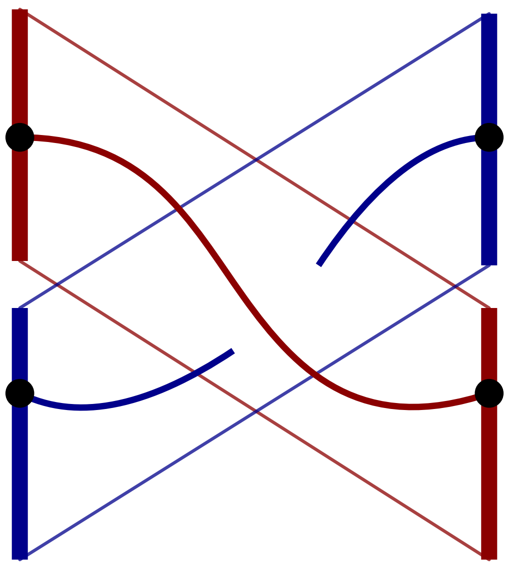

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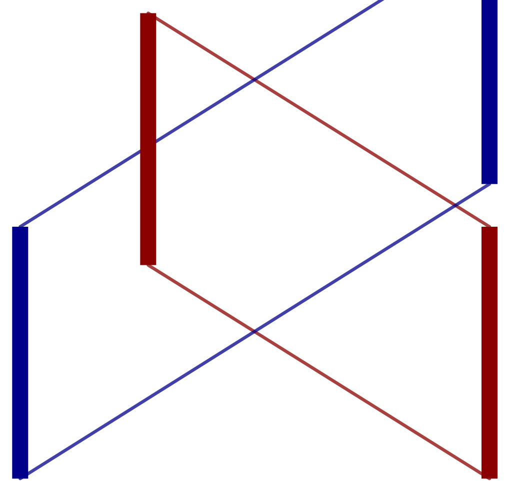

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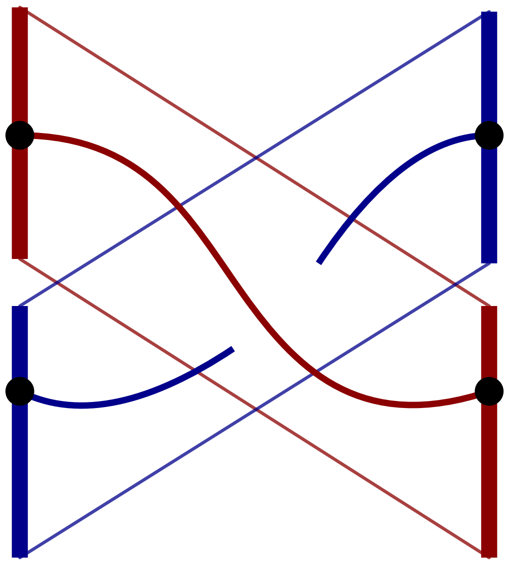

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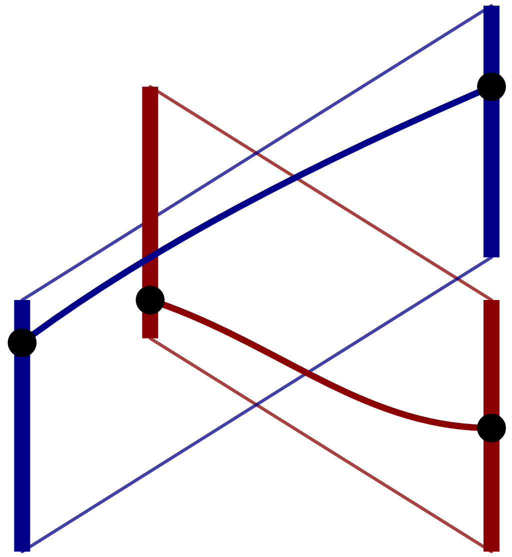

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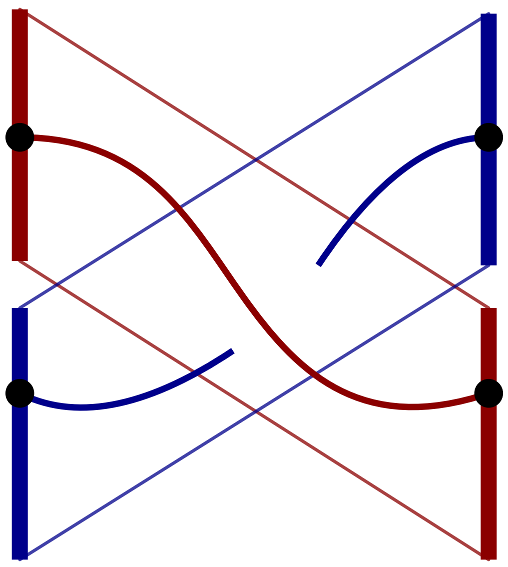

full crossing

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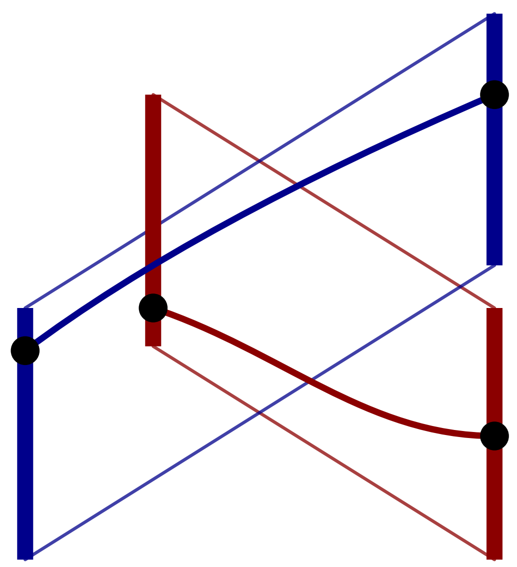
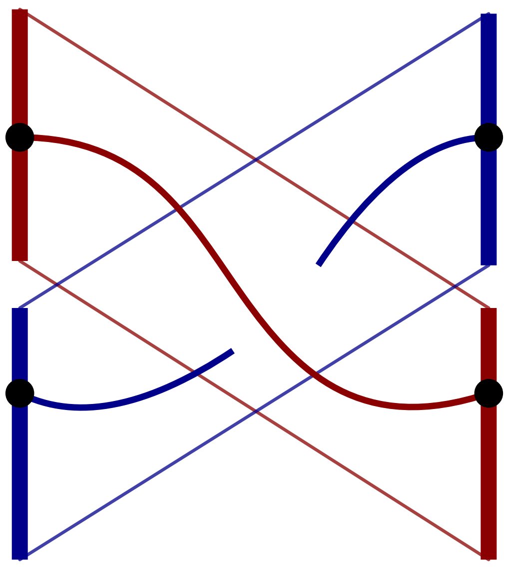


single intersection

induces vertical order between paths

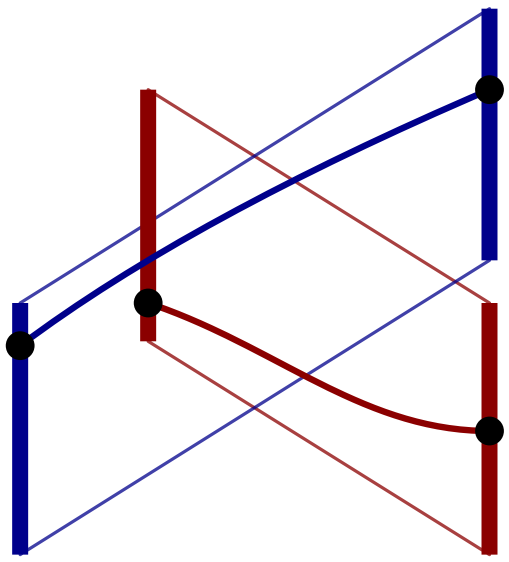
e.g., blue path is "above" red

How can two tubes intersect?



full crossing

(no solution)



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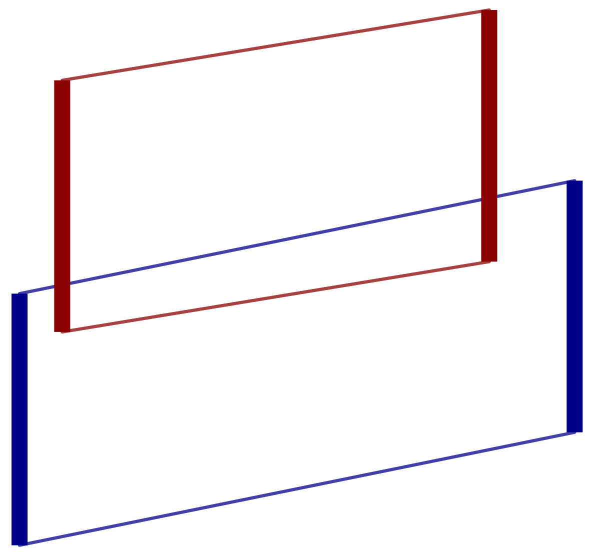

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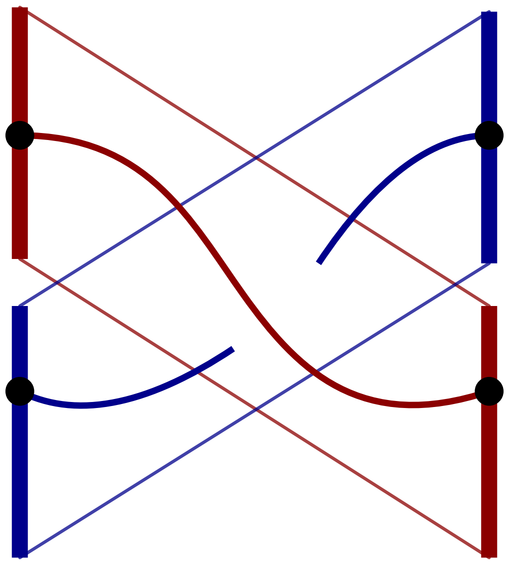

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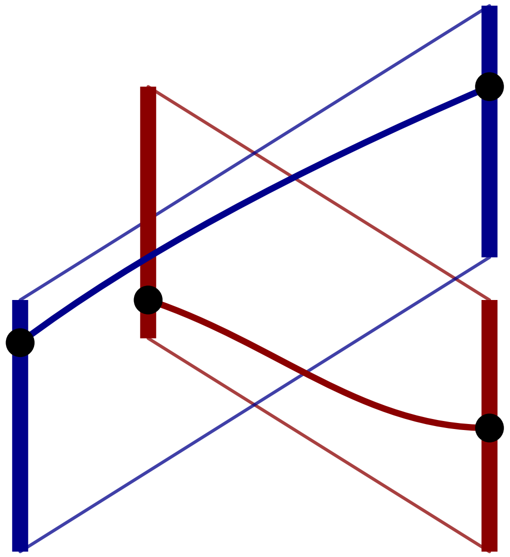

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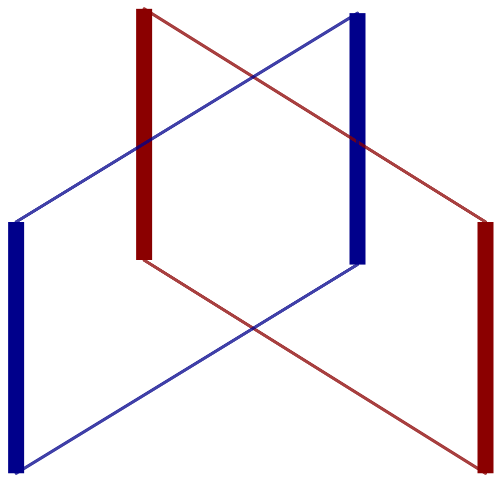

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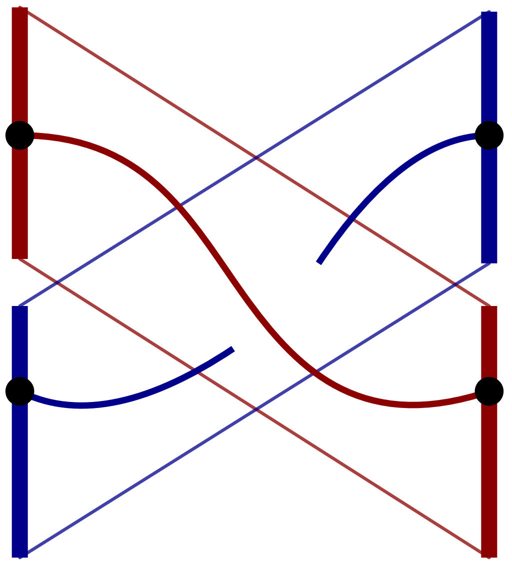

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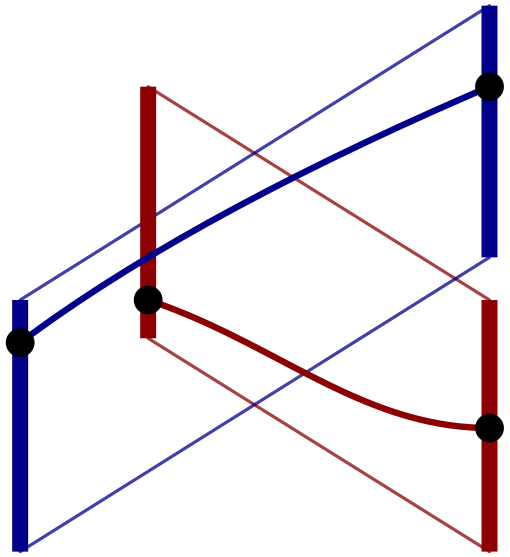

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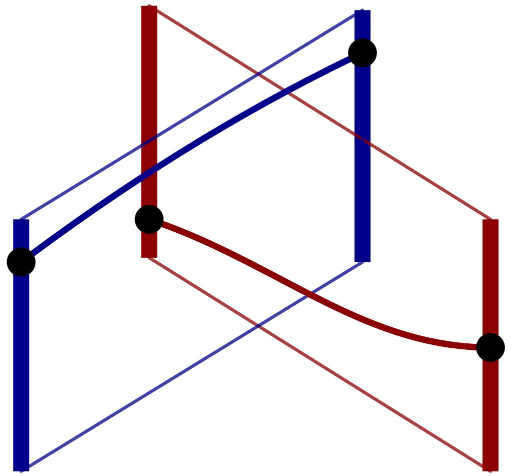

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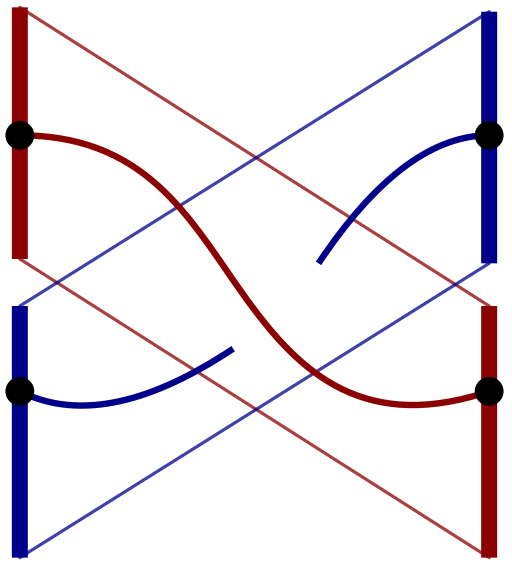

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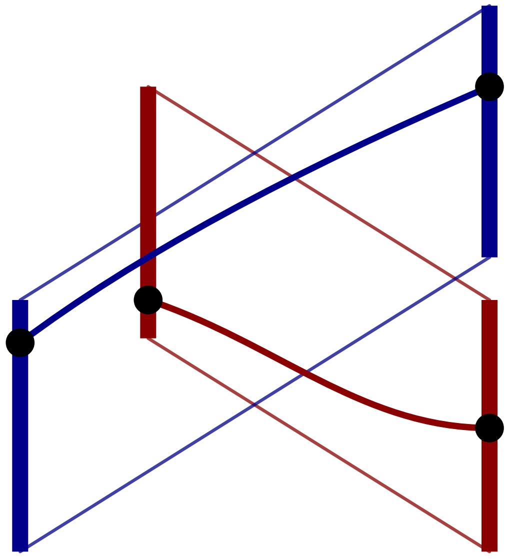

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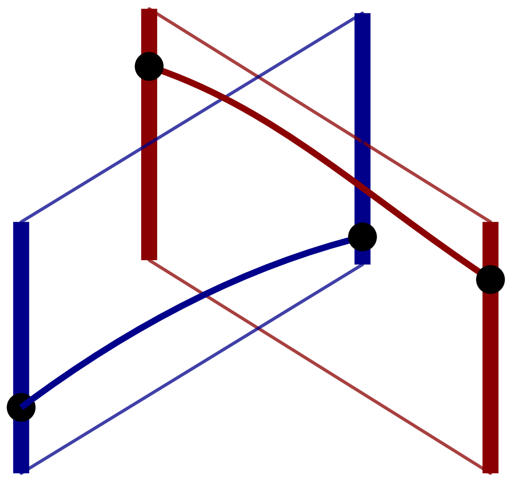

double intersection

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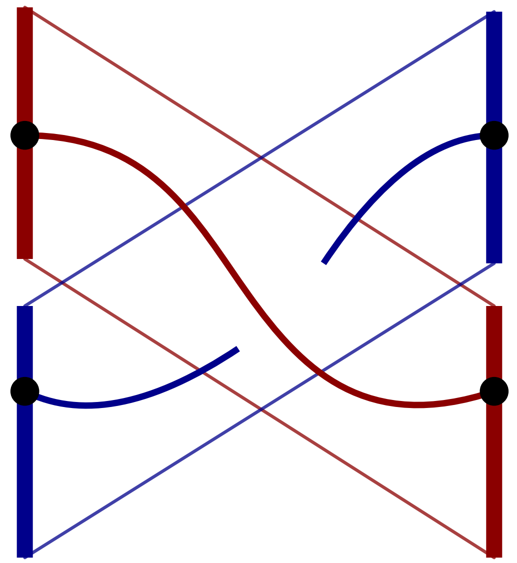

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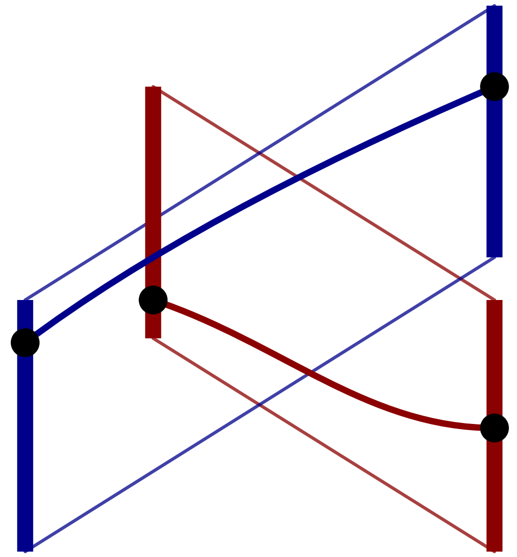

single intersection

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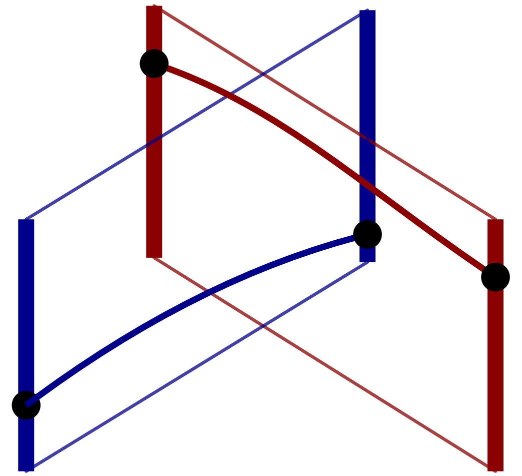
→ we can define an *order graph*

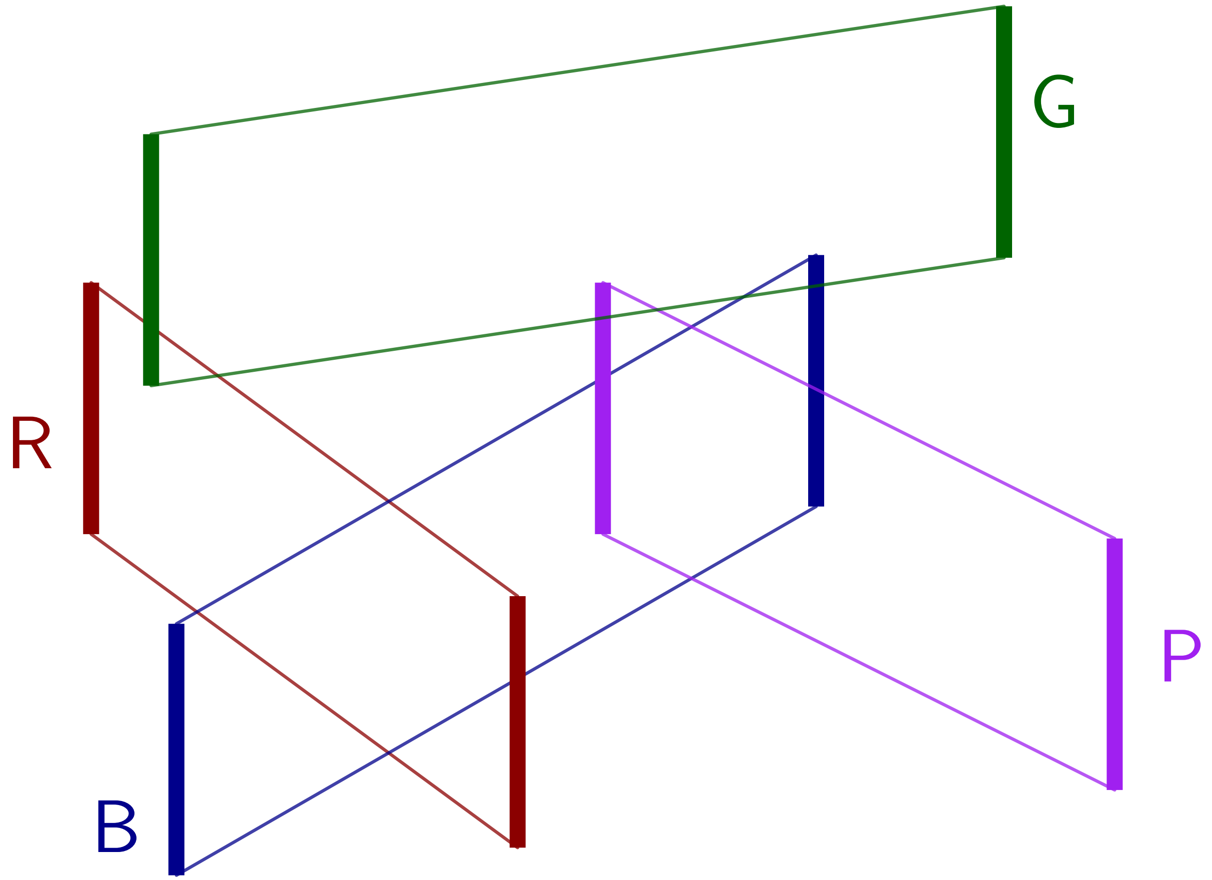


double intersection

Order graph

Order graph

Tubes



Order graph

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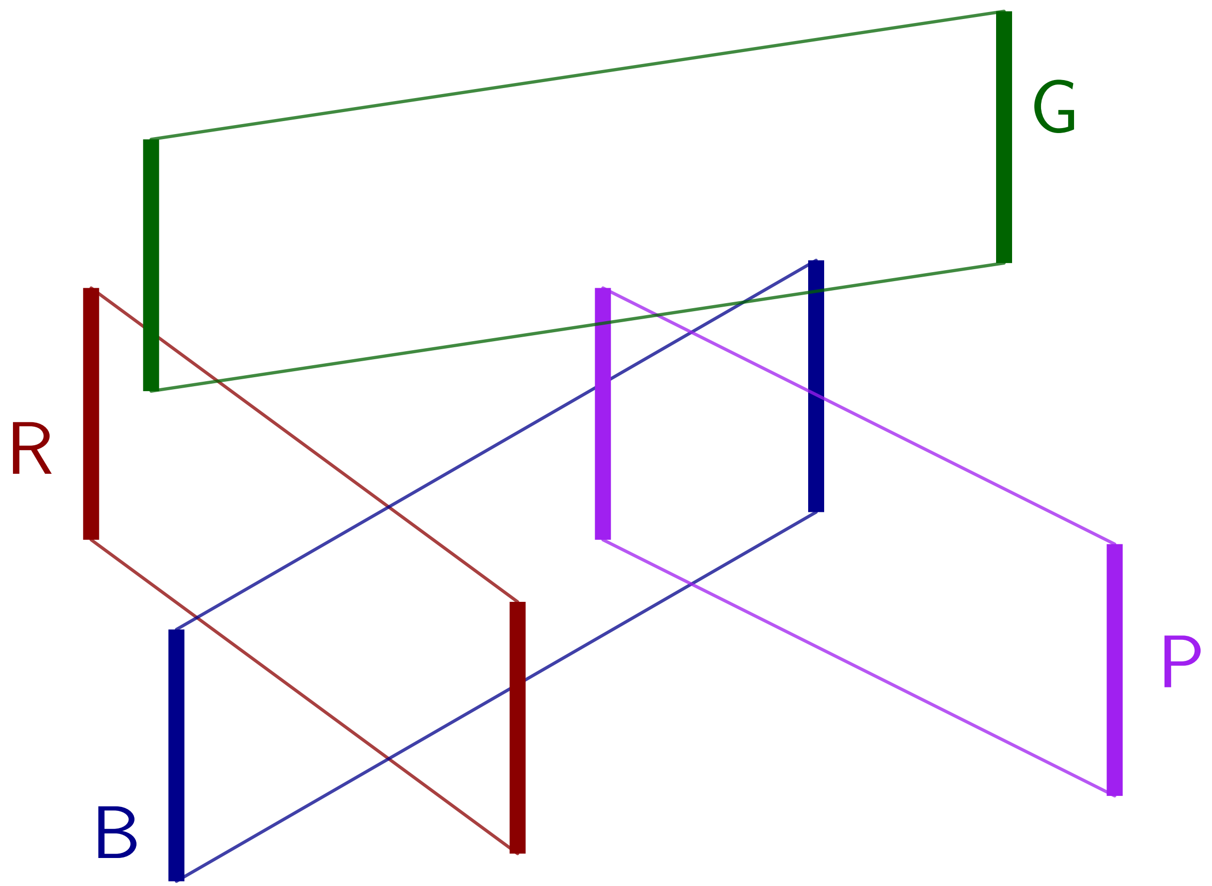
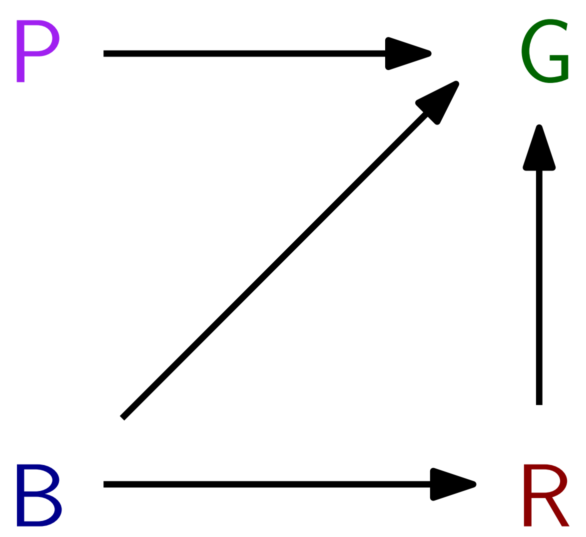
Tubes

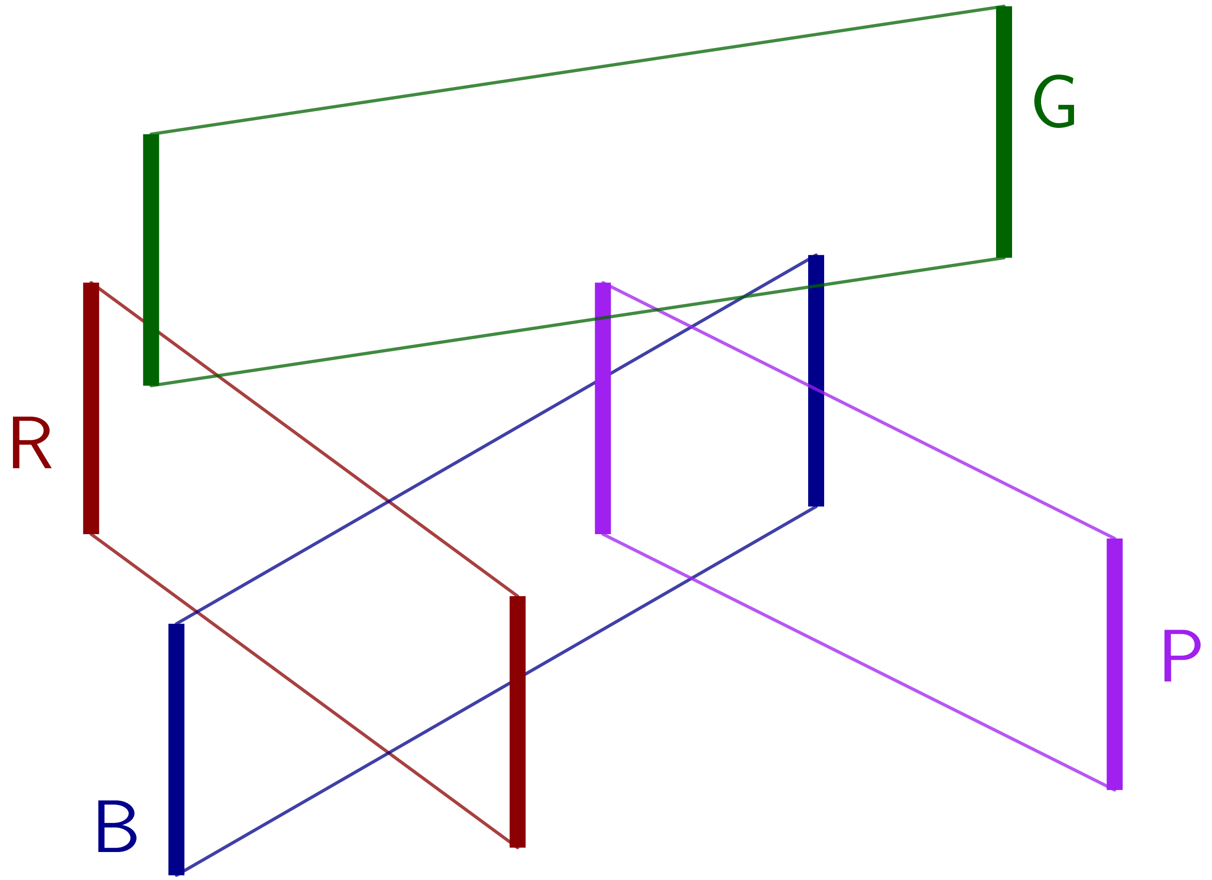


Order graph

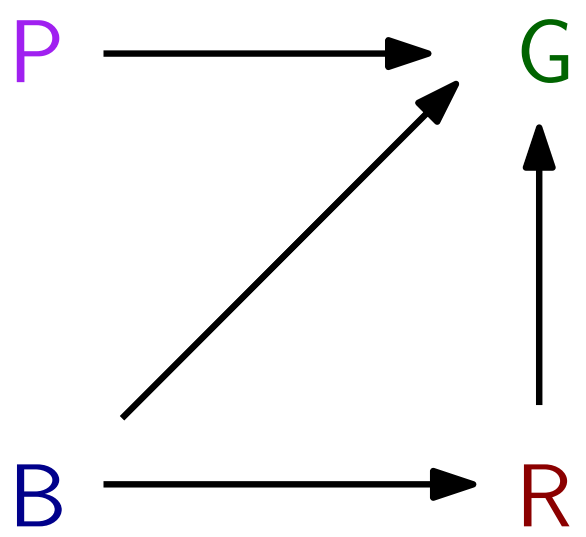


Order graph

Tubes



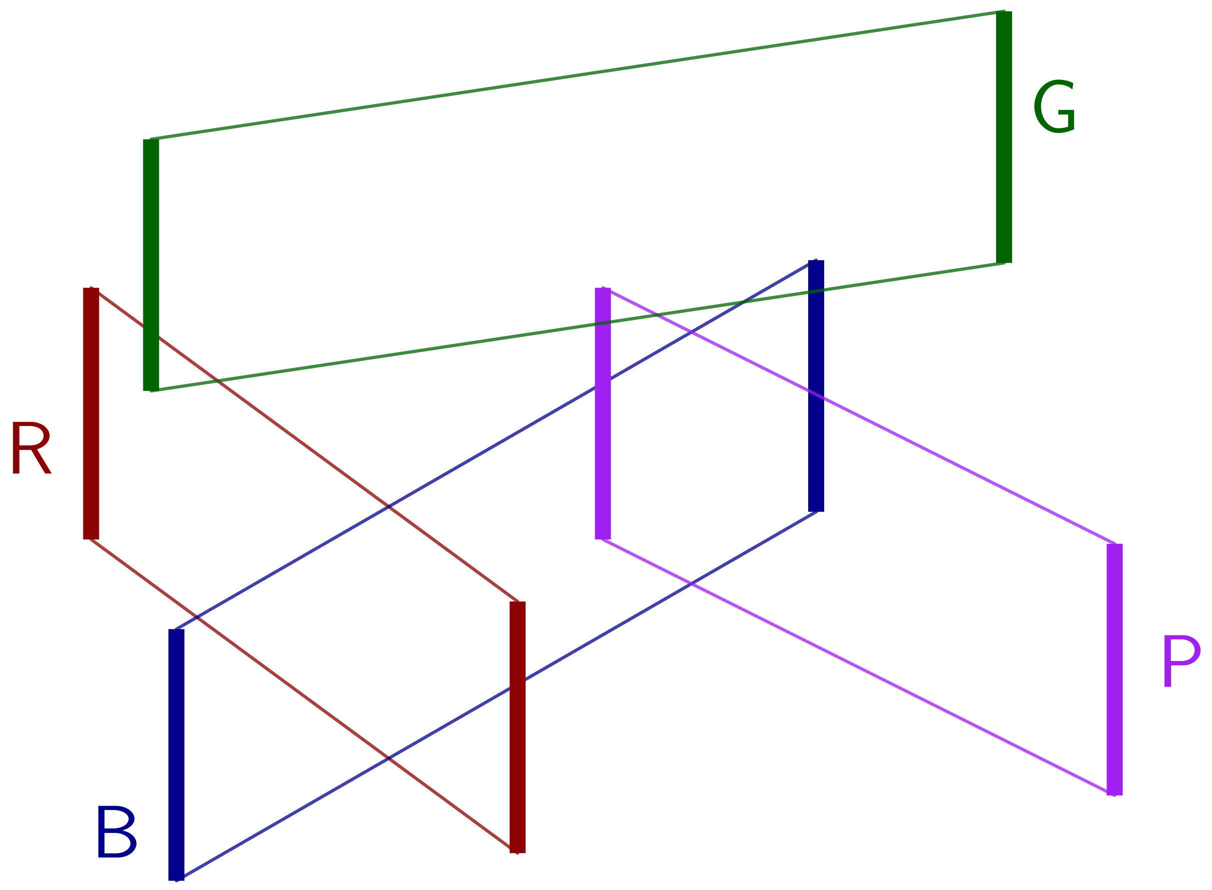
Order graph



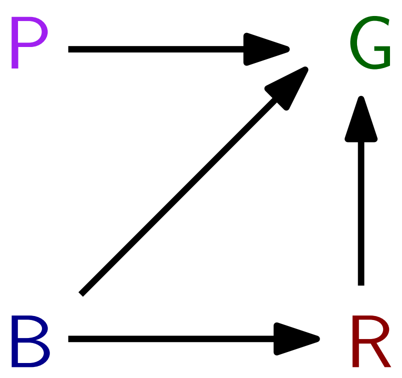
There is a solution if and only if the order graph has no directed cycles

Drawing the tubes in order

Tubes



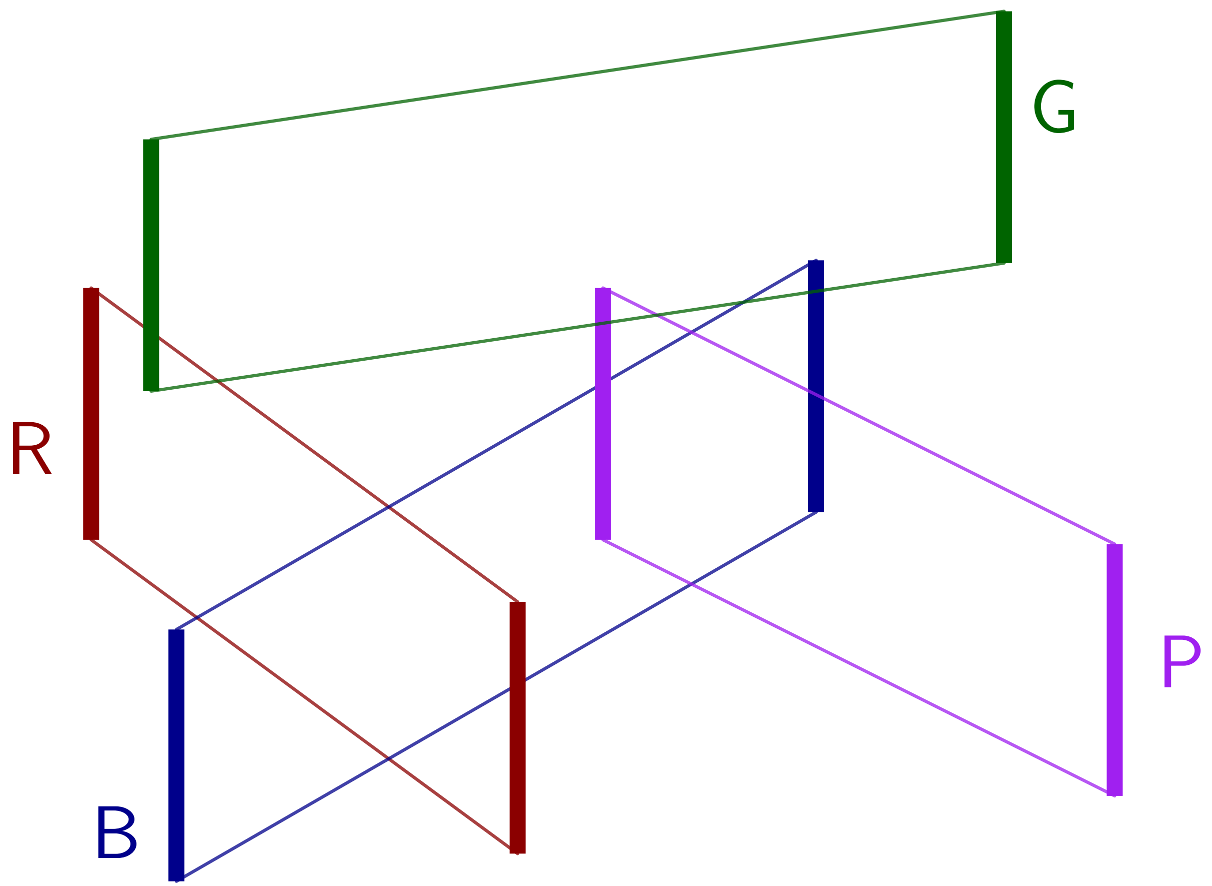
Order graph



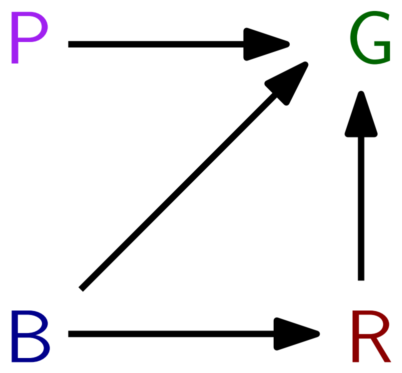
If no directed cycles:

Drawing the tubes in order

Tubes



Order graph

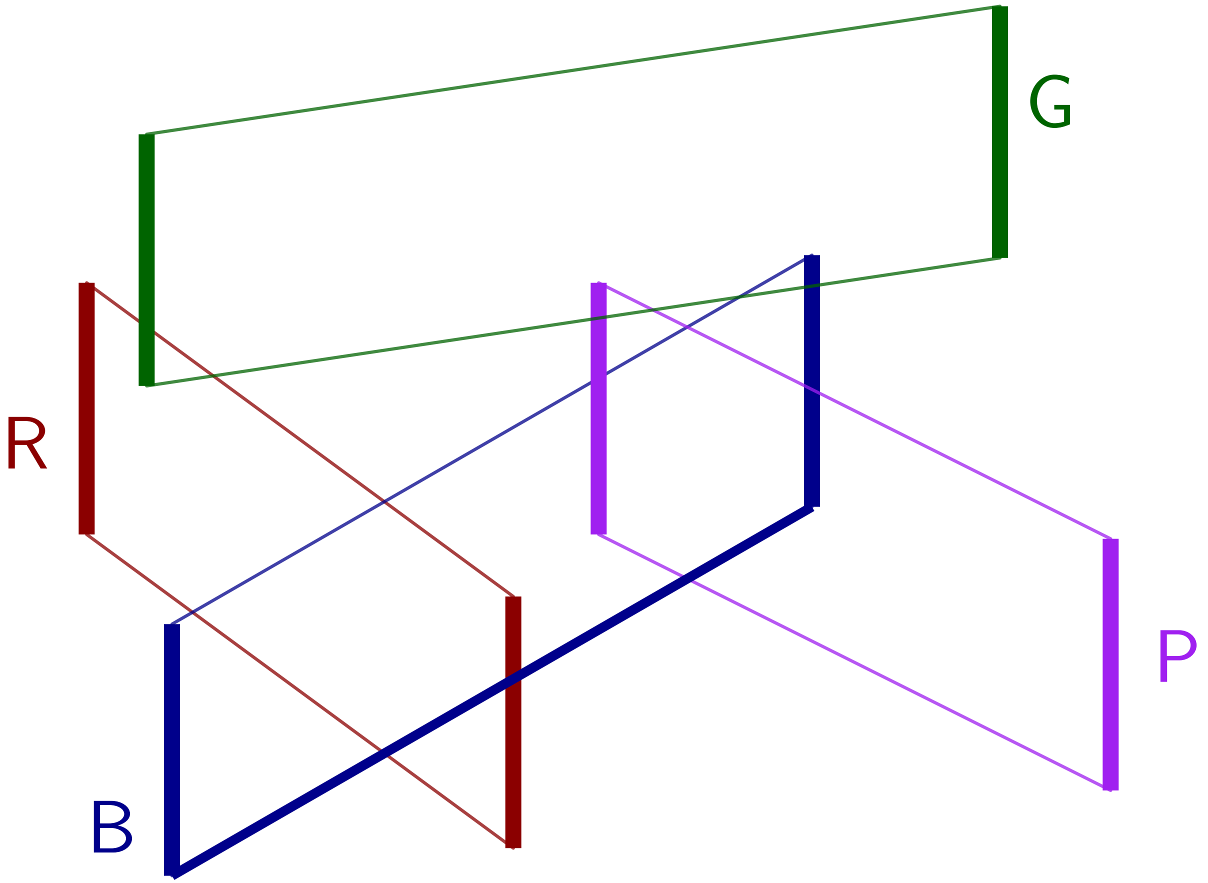


If no directed cycles:

- extract total order
  e.g., B R P G
- follow that order, drawing paths as low as possible

Drawing the tubes in order

Tubes



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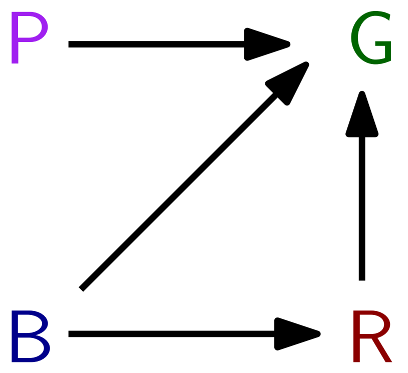

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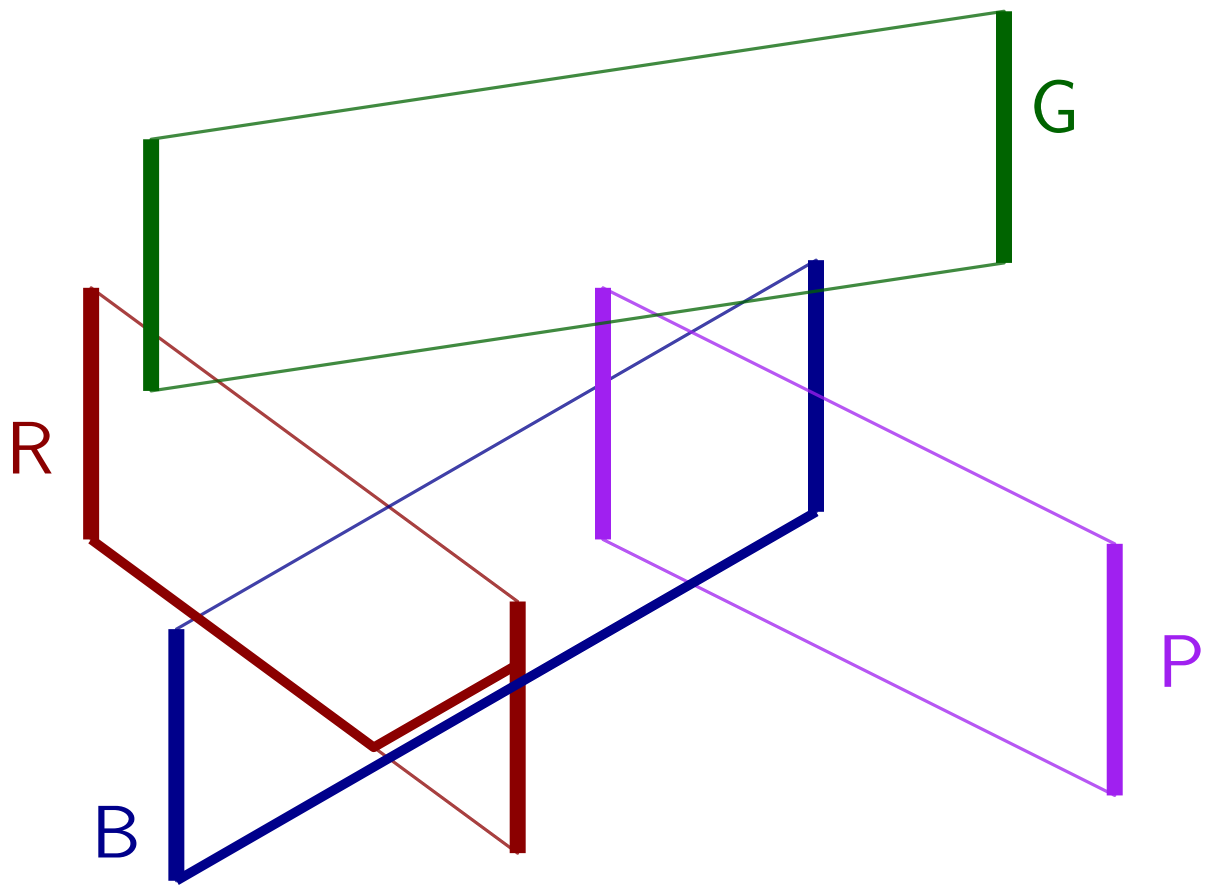

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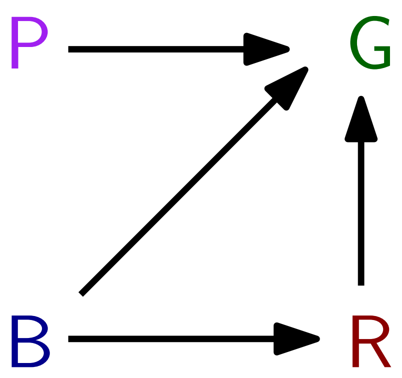

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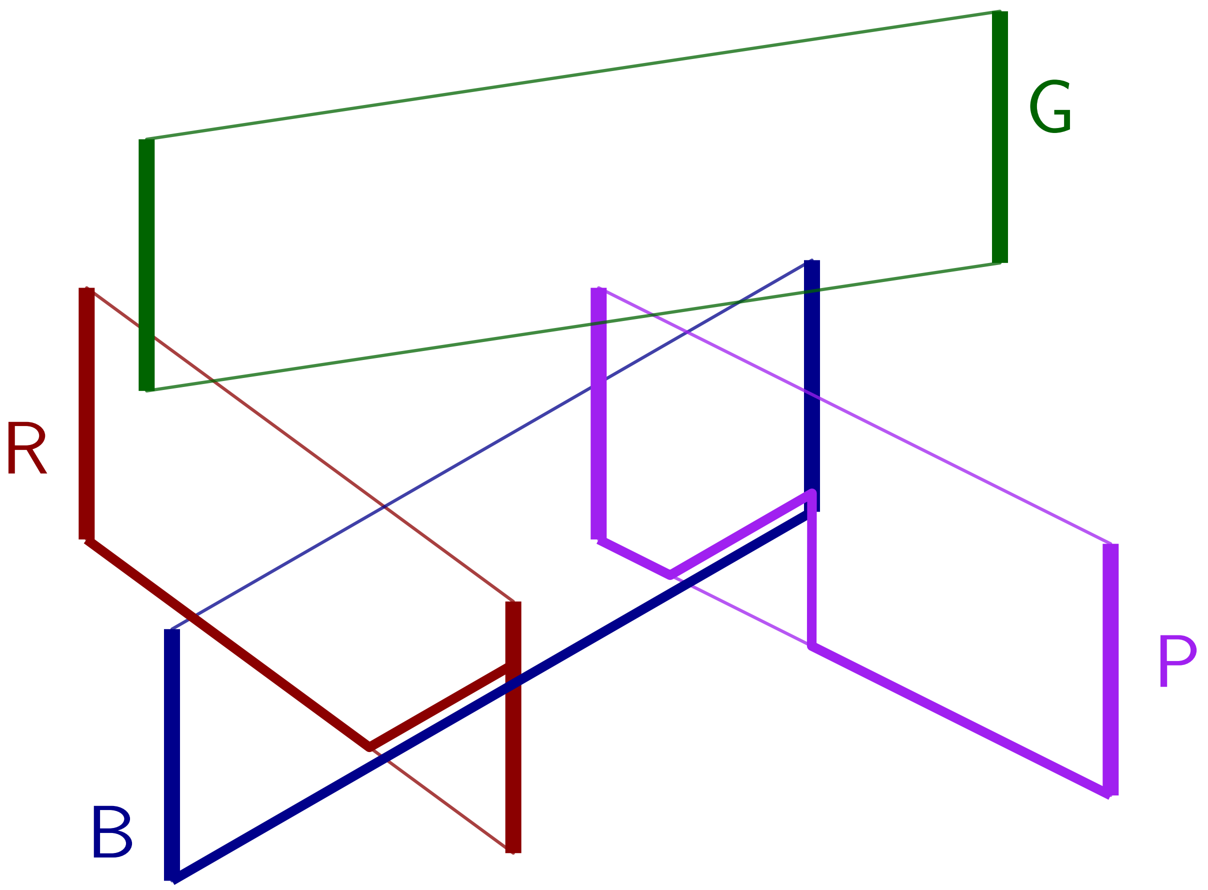

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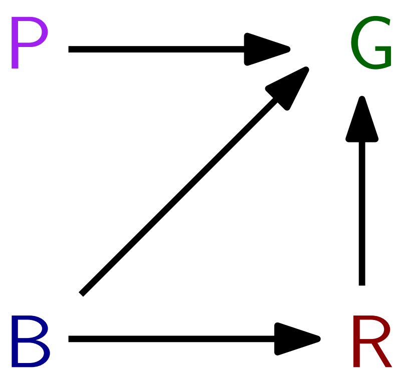

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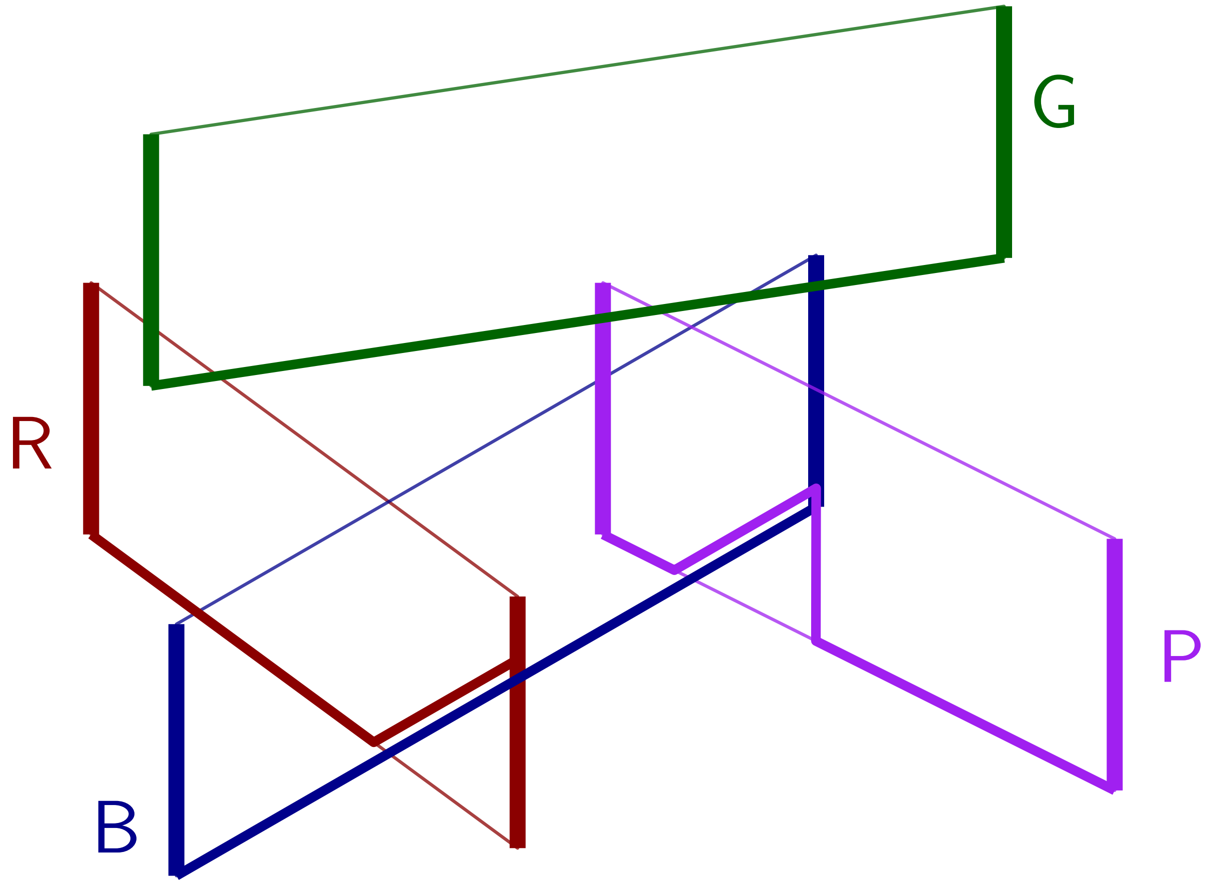
Drawing the tubes in order

Tubes



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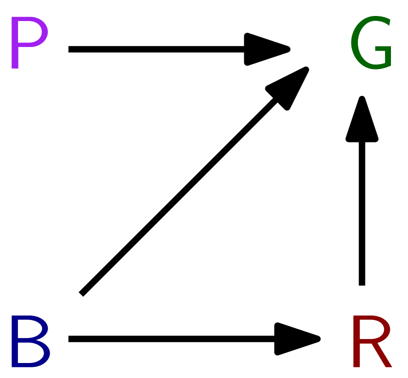

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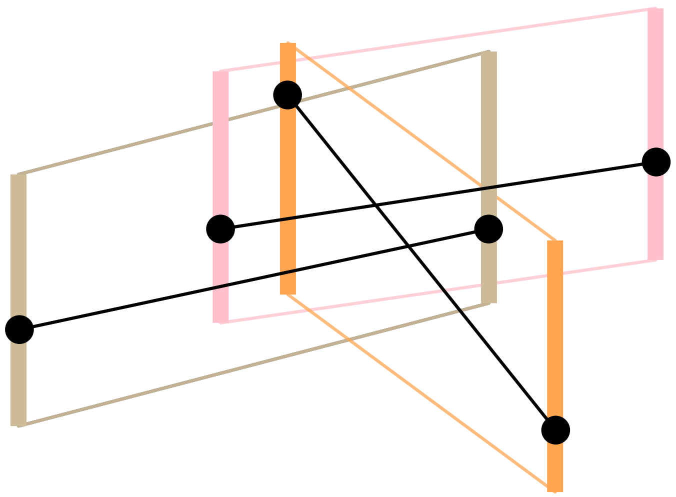
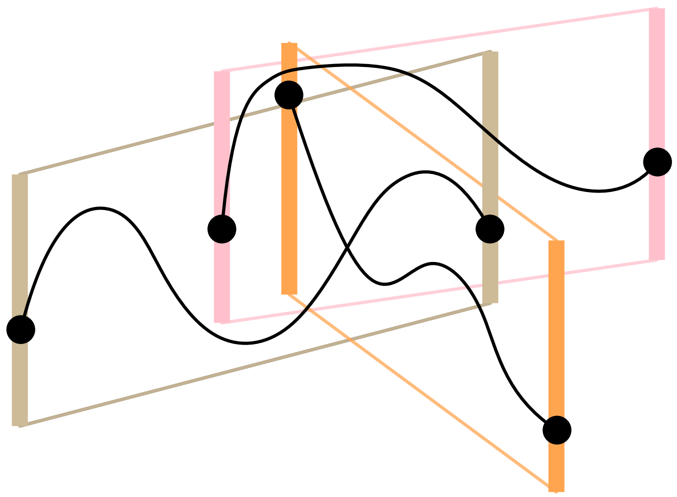
Our results

straight line paths



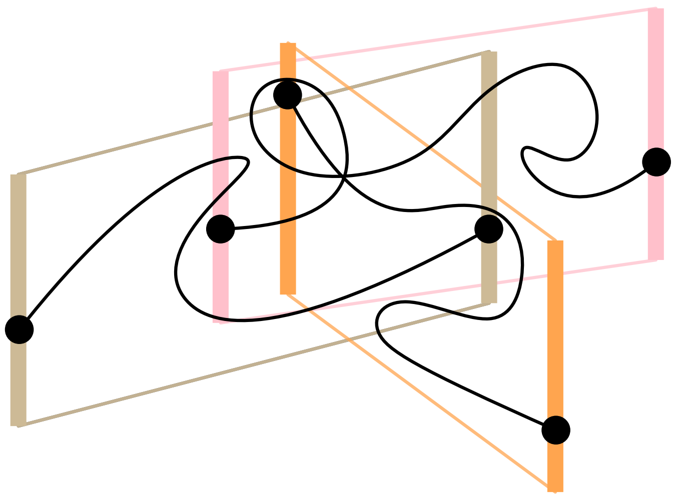
NP-complete

(x-)monotone paths



Polynomial

arbitrary paths

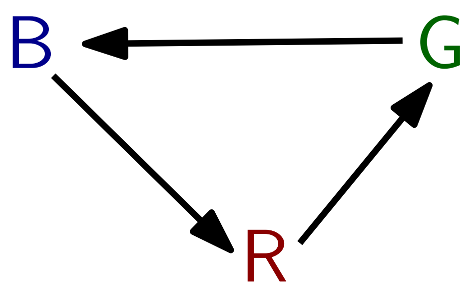


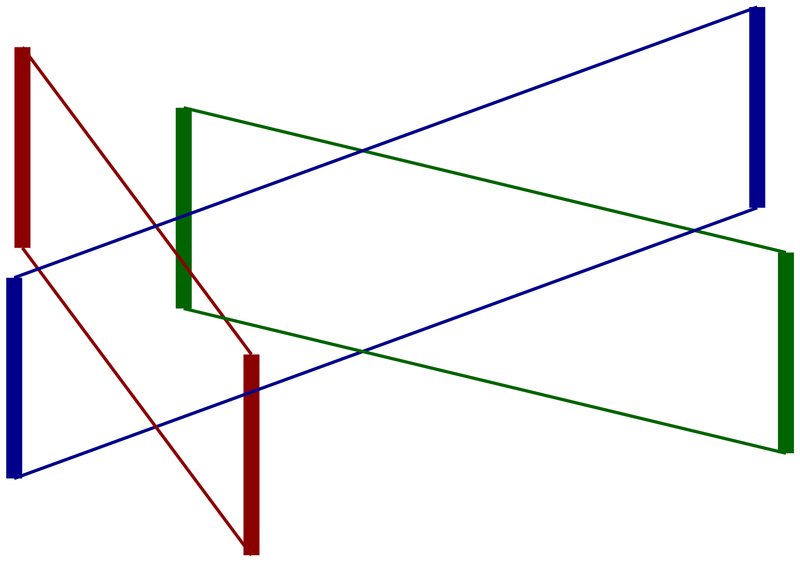
Polynomial under certain assumptions

Arbitrary paths

Arbitrary paths

Now, some cycles of directed edges can be solved:





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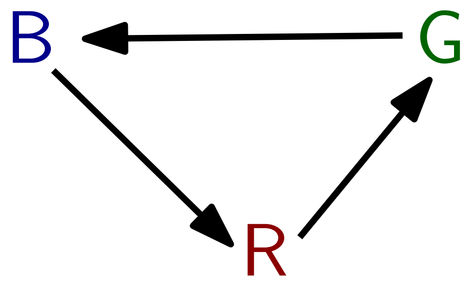

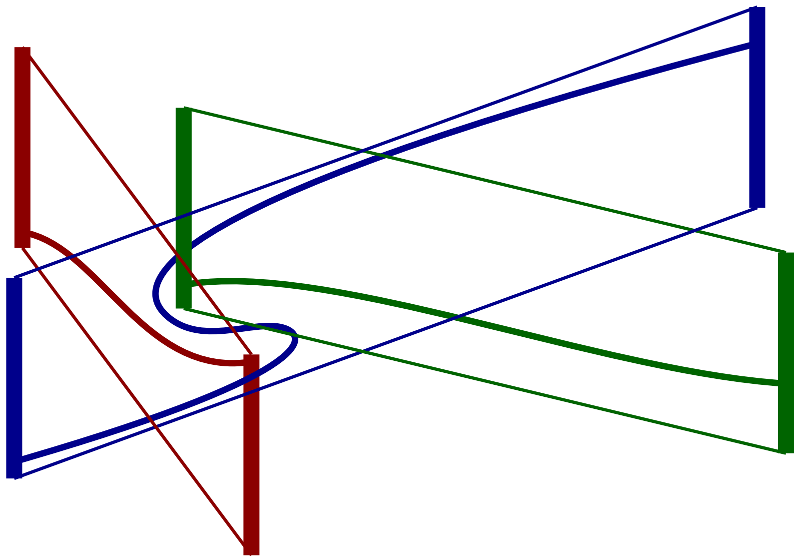

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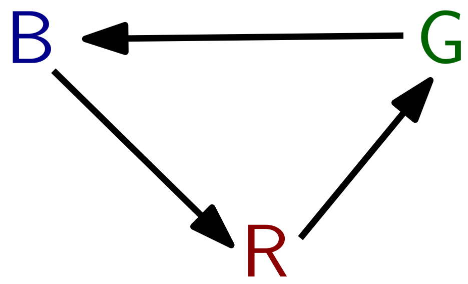

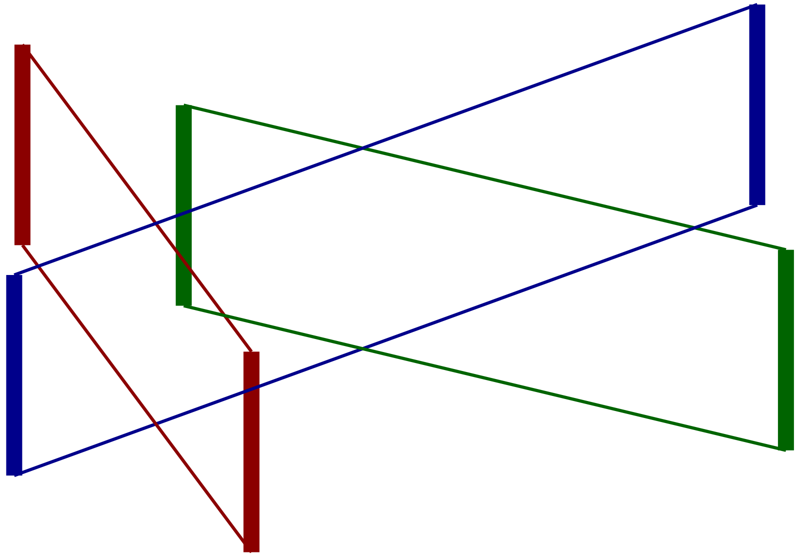

Related problem: *Non-crossing connectors* [Kratochvíl and Ueckerdt, 2013]

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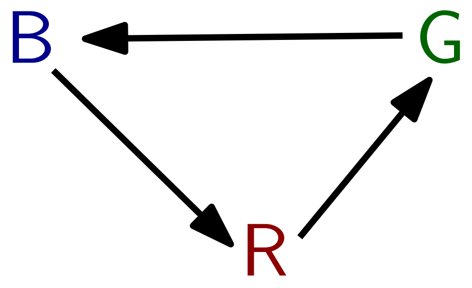

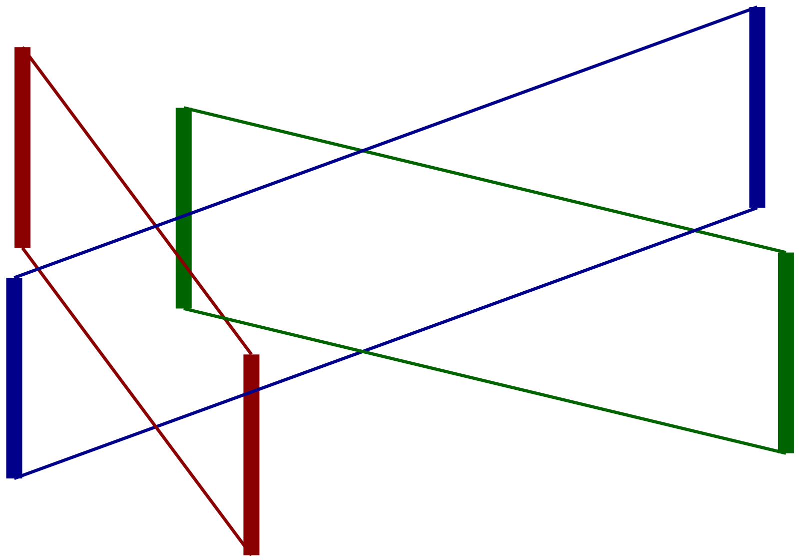

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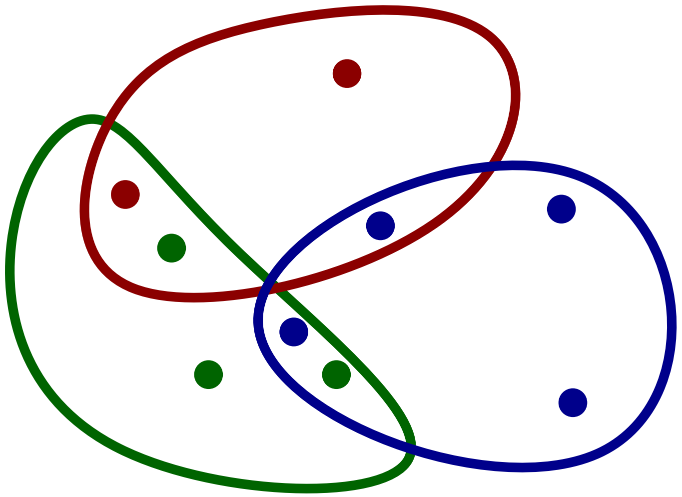

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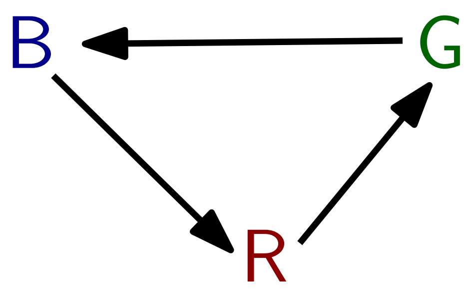

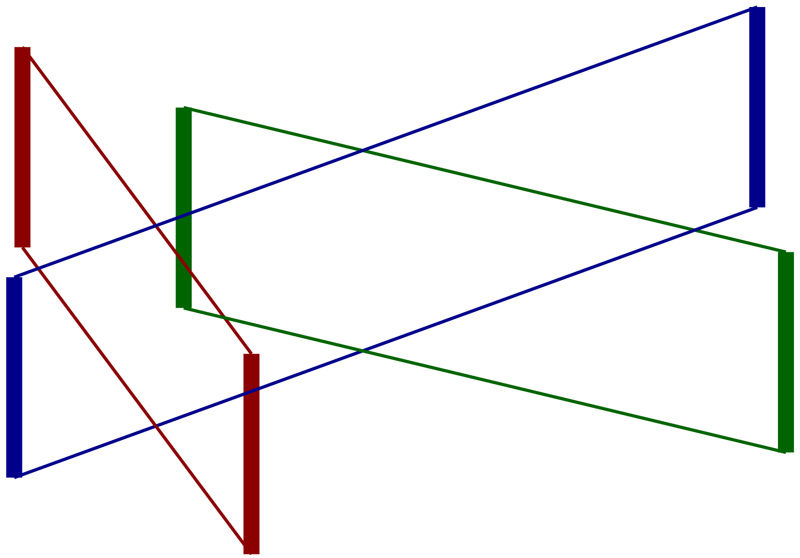

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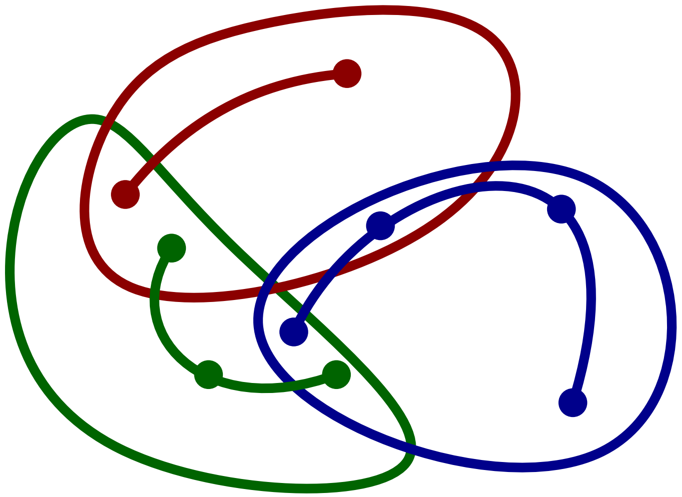

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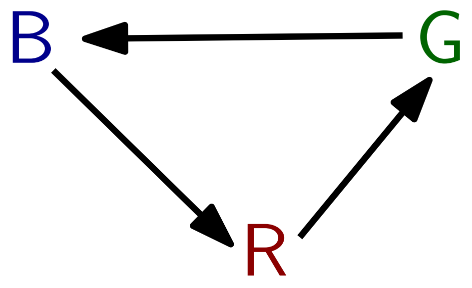

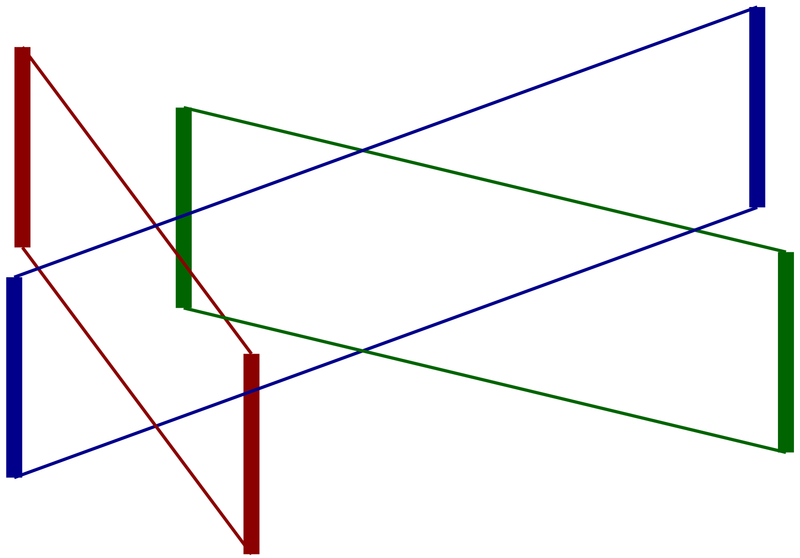

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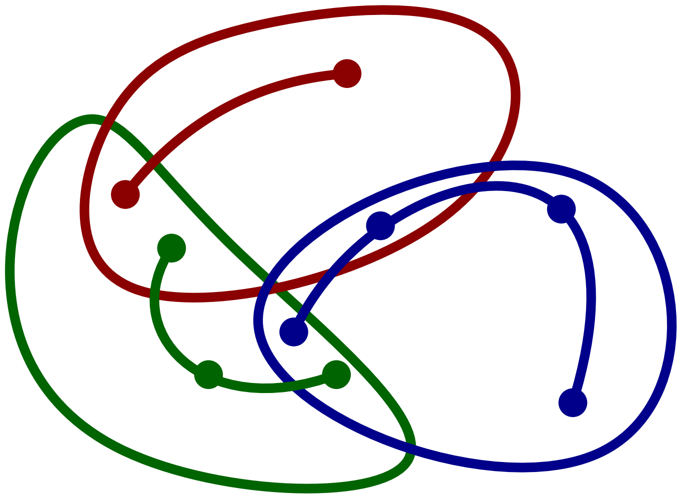
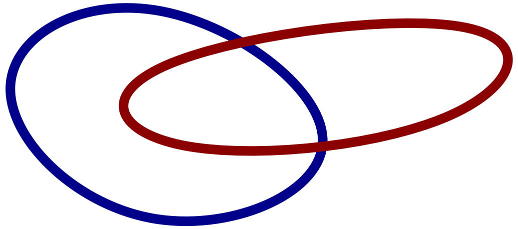


They prove: If the regions are pseudo-disks, there is always a solution

Are our tubes pseudo-disks?



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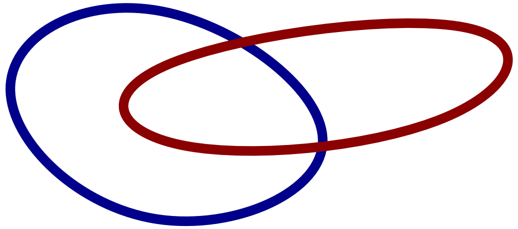

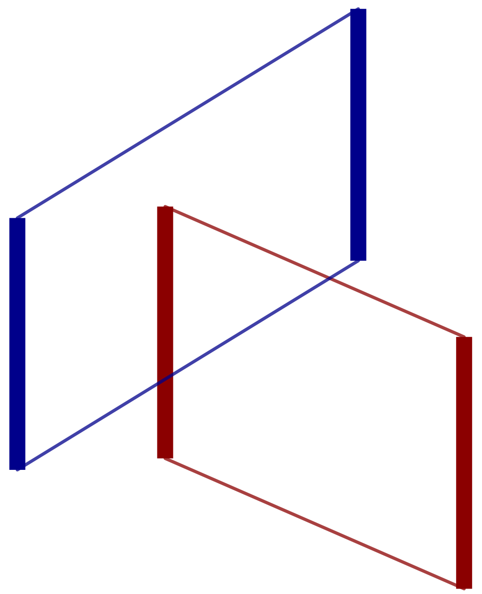

single intersection

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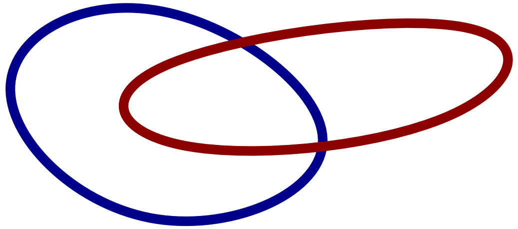

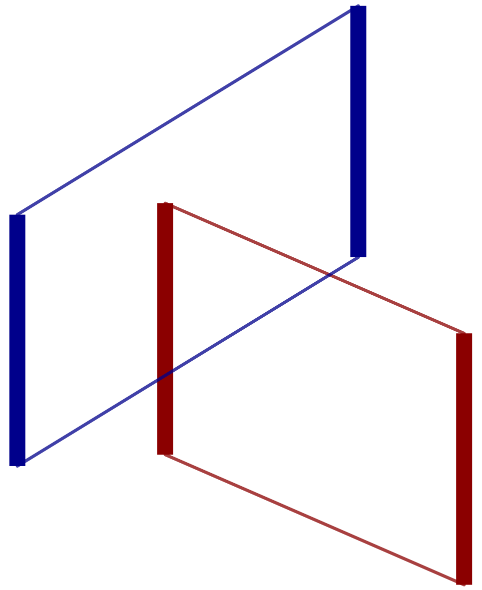

single intersection

pseudo-disks!

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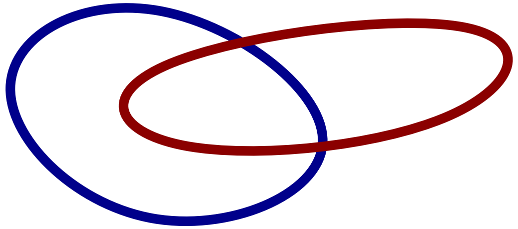

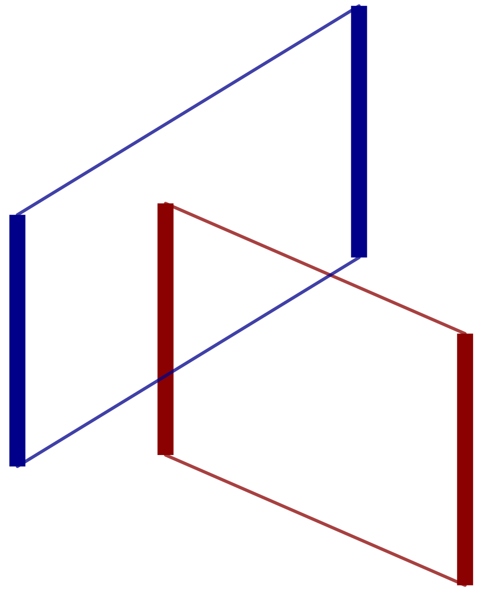

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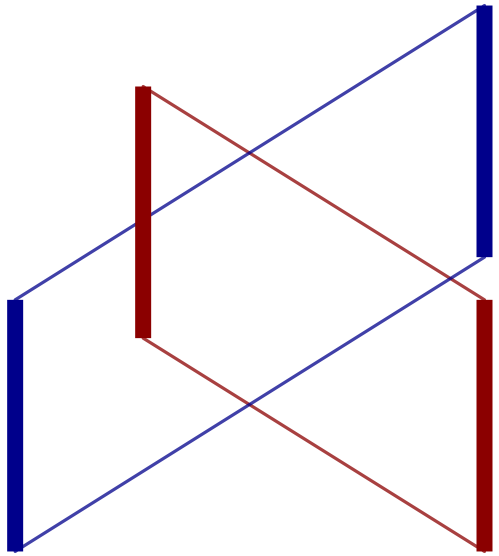

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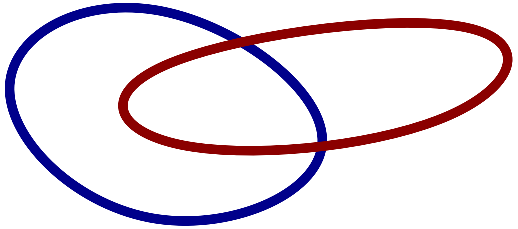

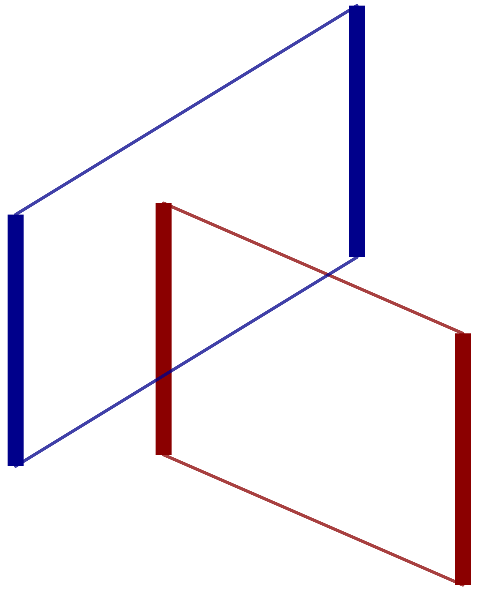

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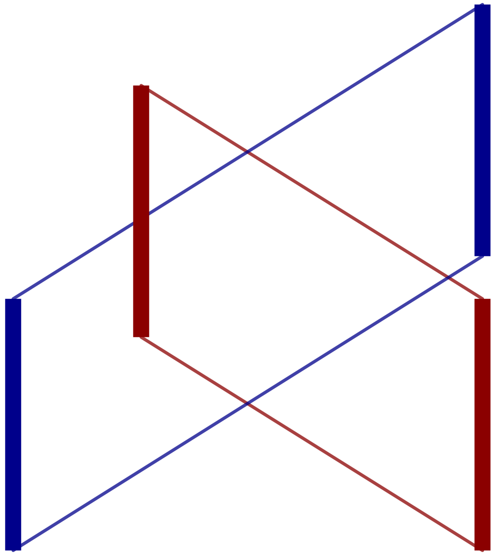
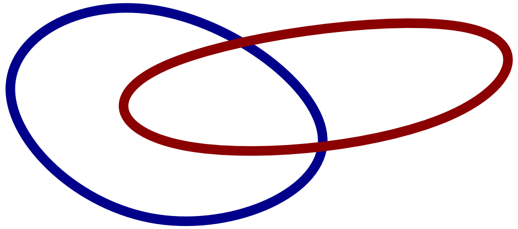
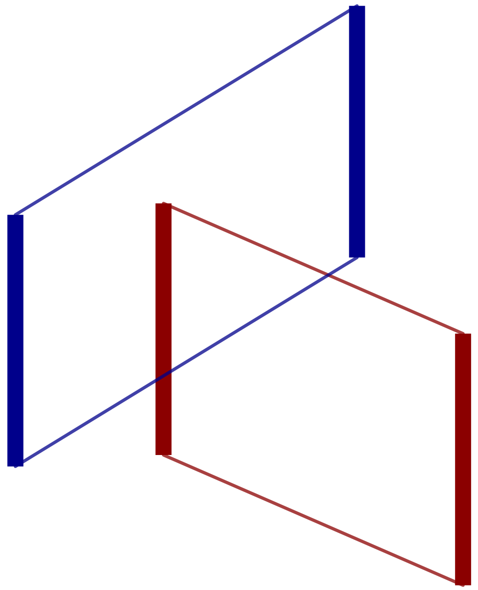


single intersection

not pseudo-disks

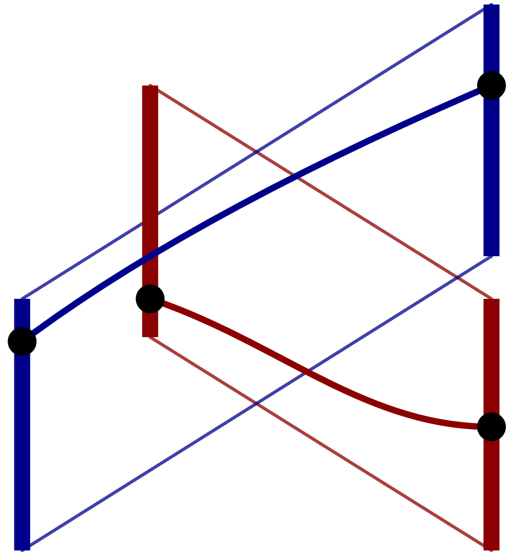
Are our tubes pseudo-disks?





single intersection

pseudo-disks!



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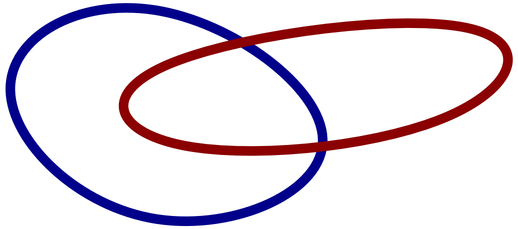

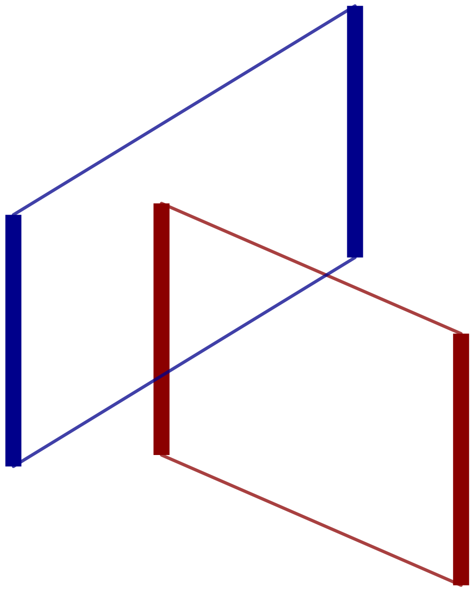

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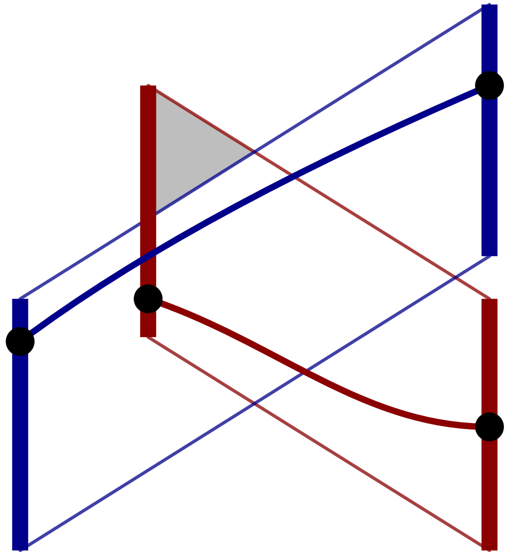

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not pseudo-disks

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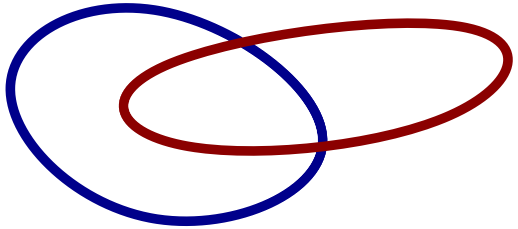
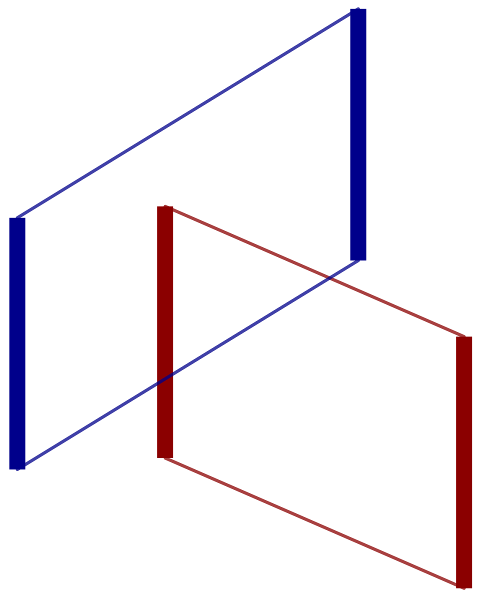
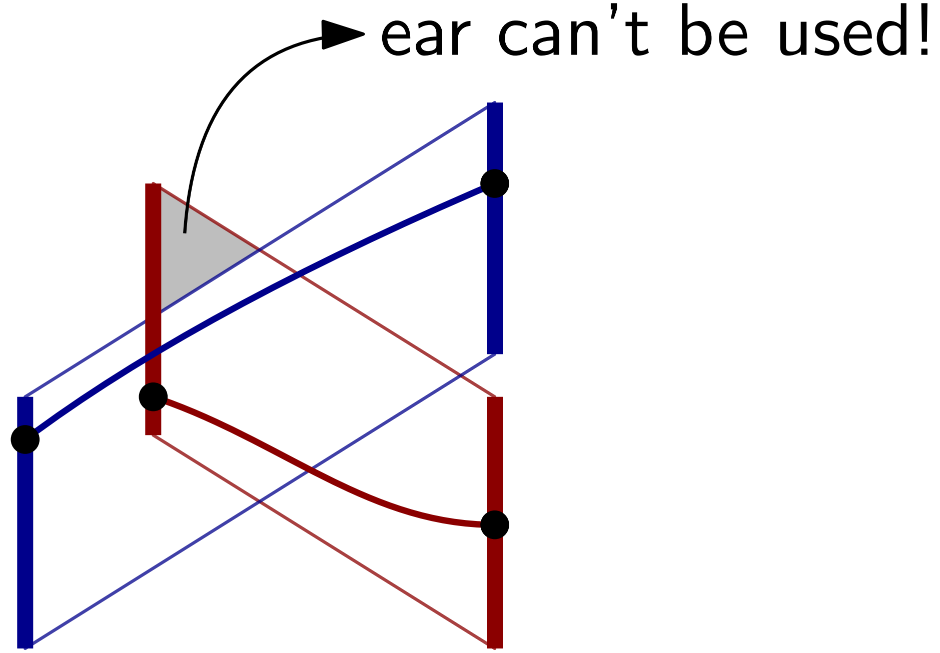




single intersection

pseudo-disks!



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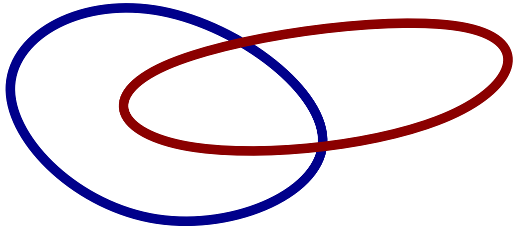

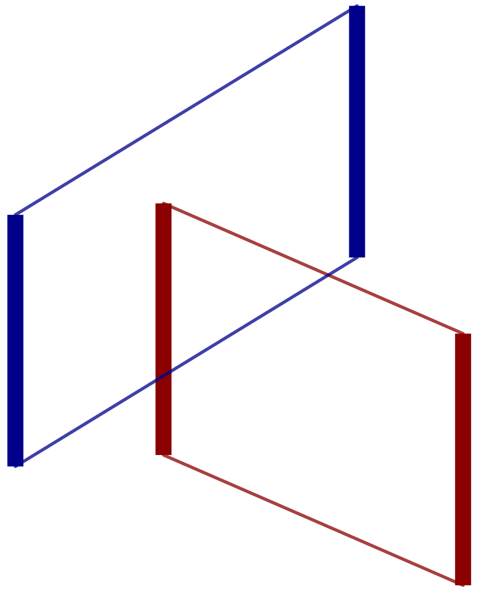

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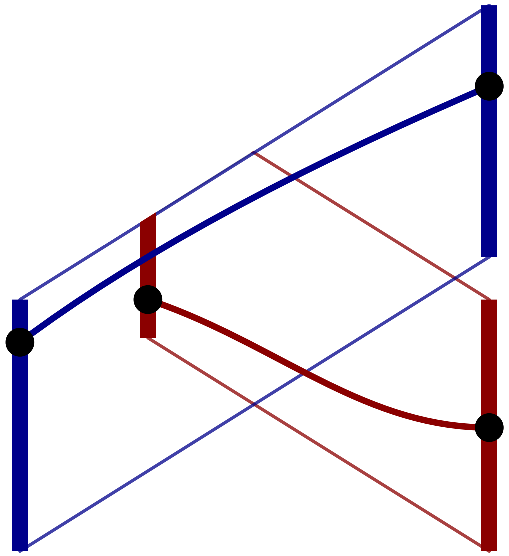

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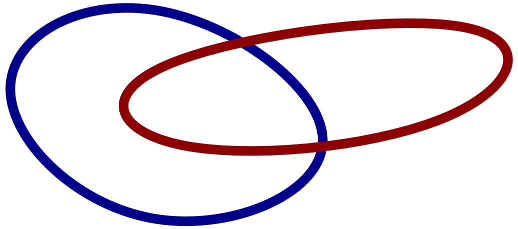

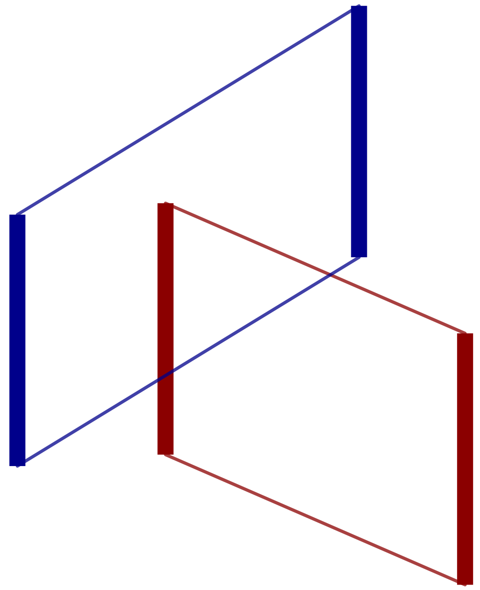

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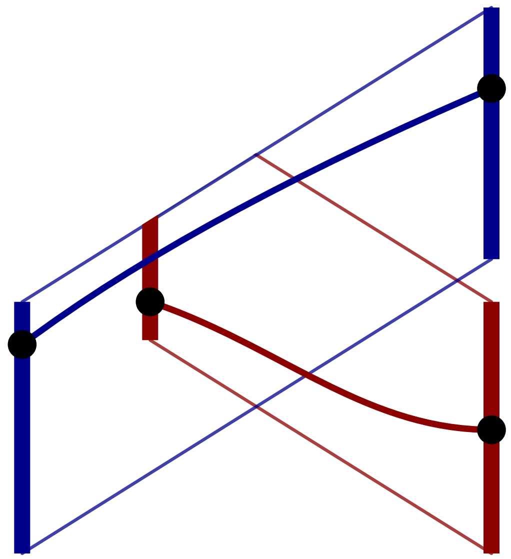

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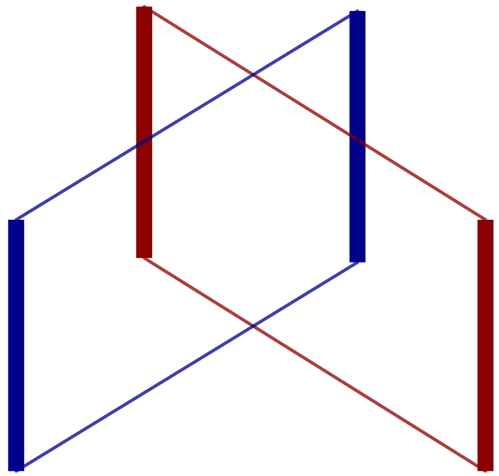

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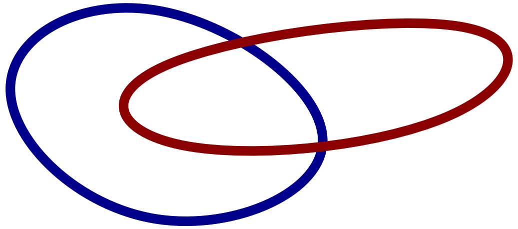

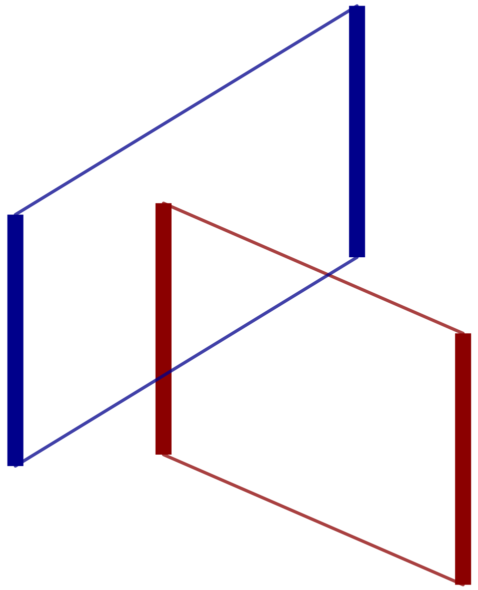

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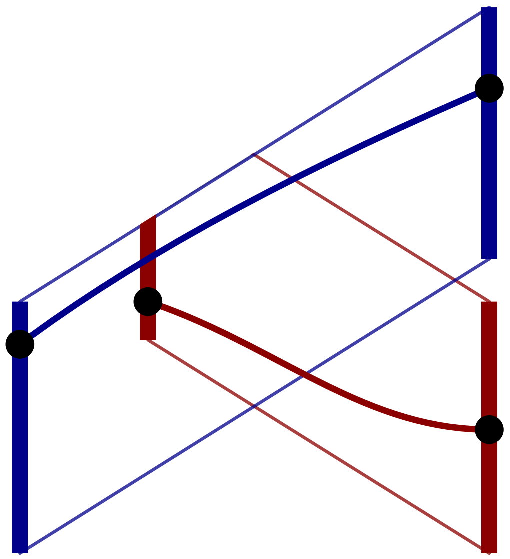

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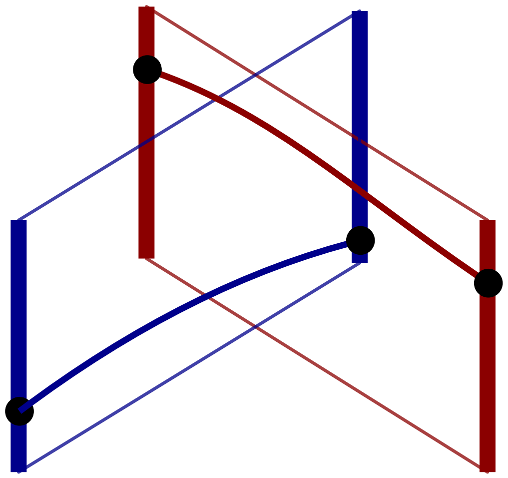

double intersection

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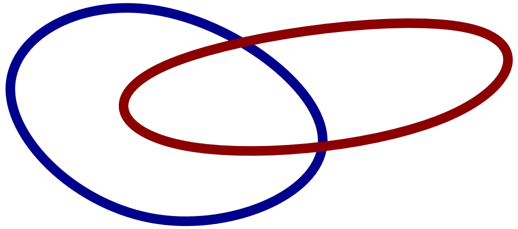

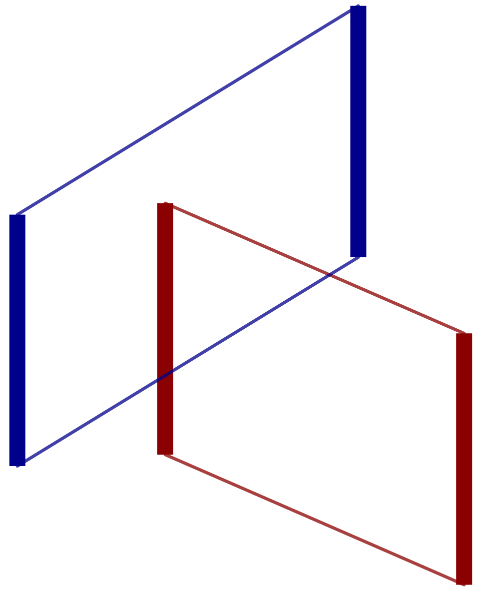

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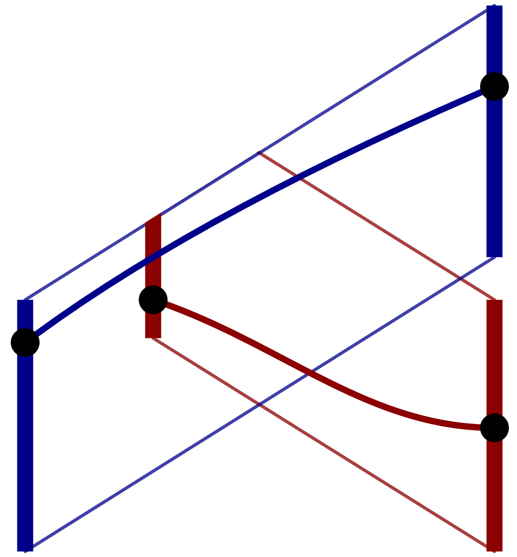

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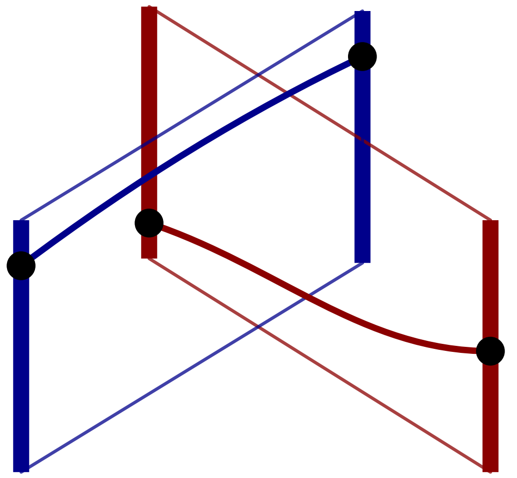

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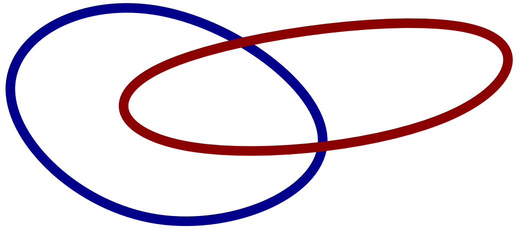

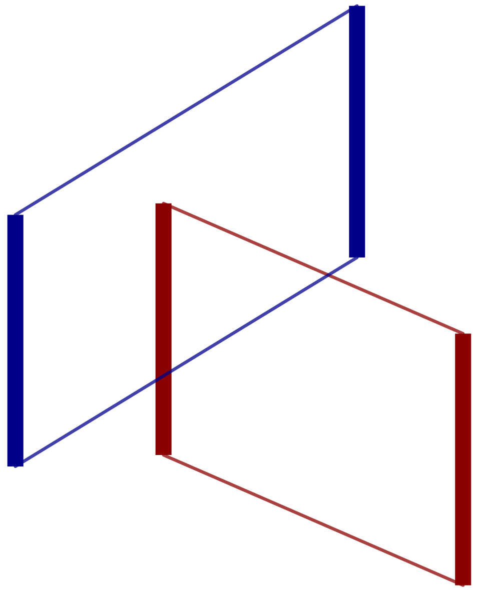

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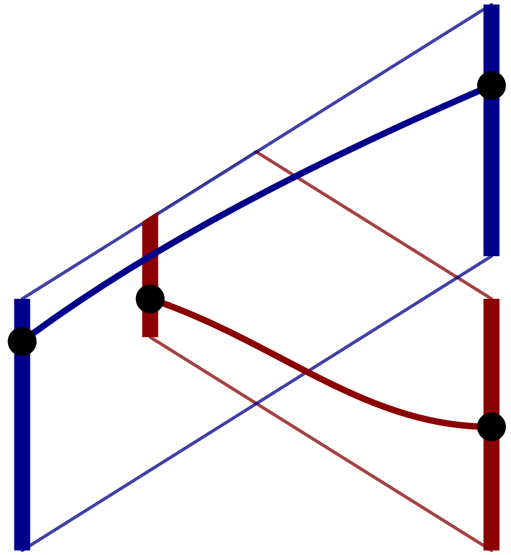

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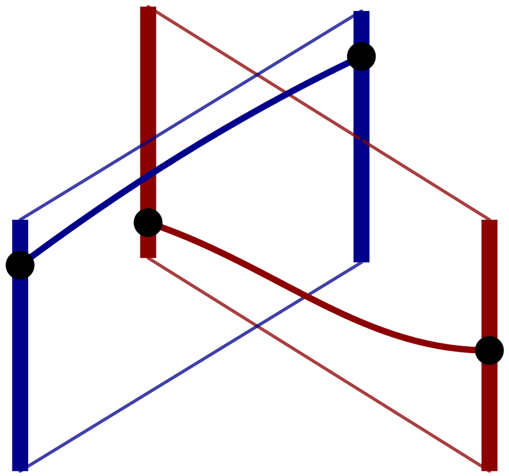
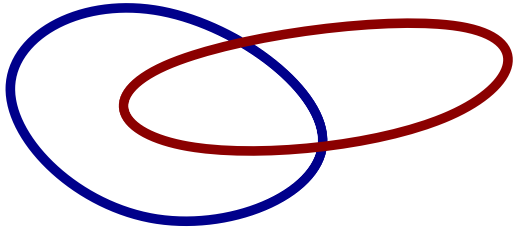


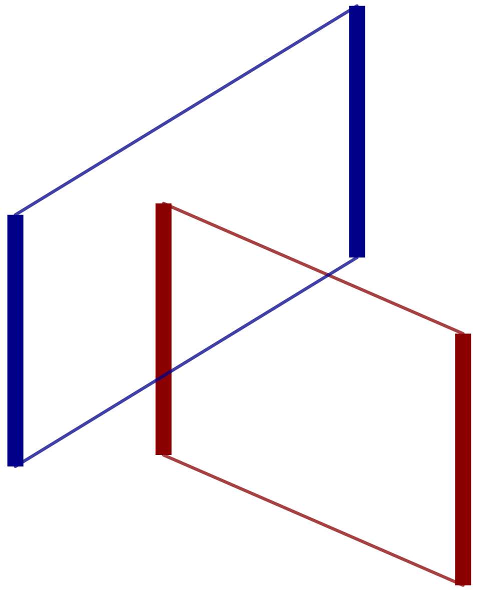
double intersection

not pseudo-disks

can't cut off ears

Are our tubes pseudo-disks?





single intersection

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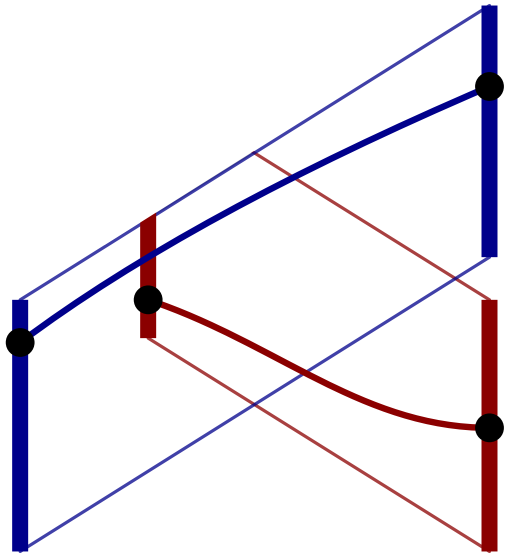

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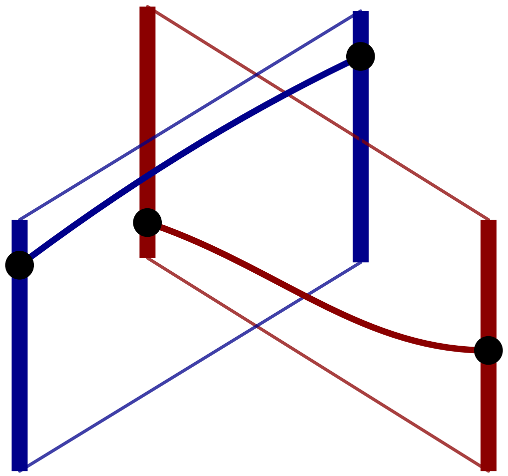

double intersection

We assume there are no double intersections

If no double intersections

Algorithm: while there is pair of tubes not pseudo-disks, cut off ear.

If no double intersections

Algorithm: while there is pair of tubes not pseudo-disks, cut off ear.

Result: set of pruned tubes that are pseudo-disks

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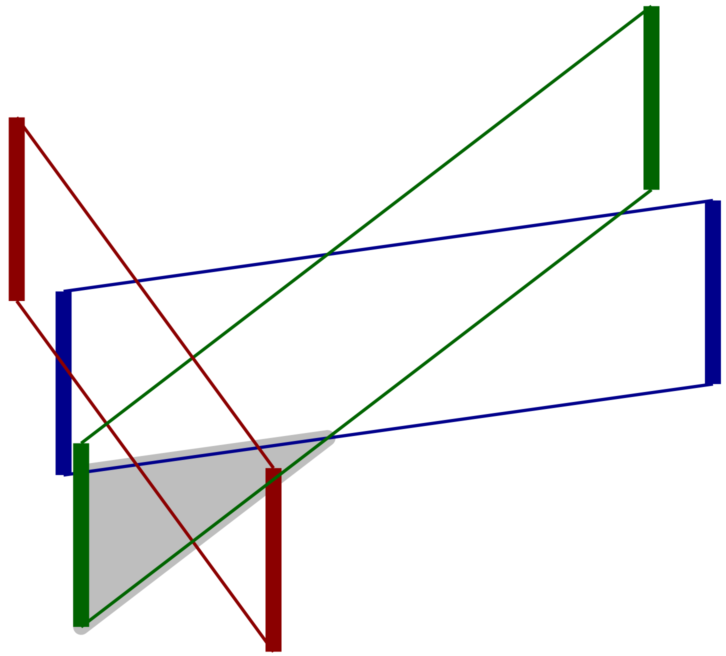

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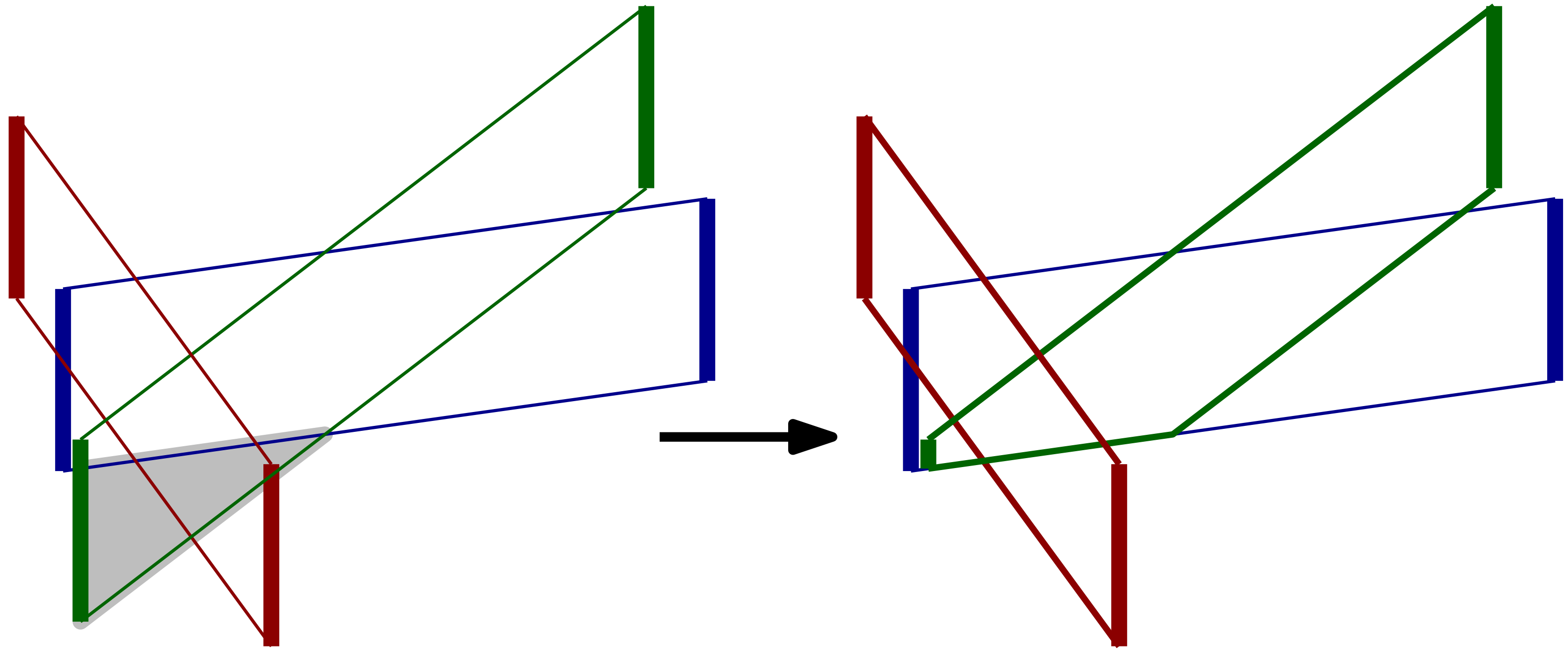

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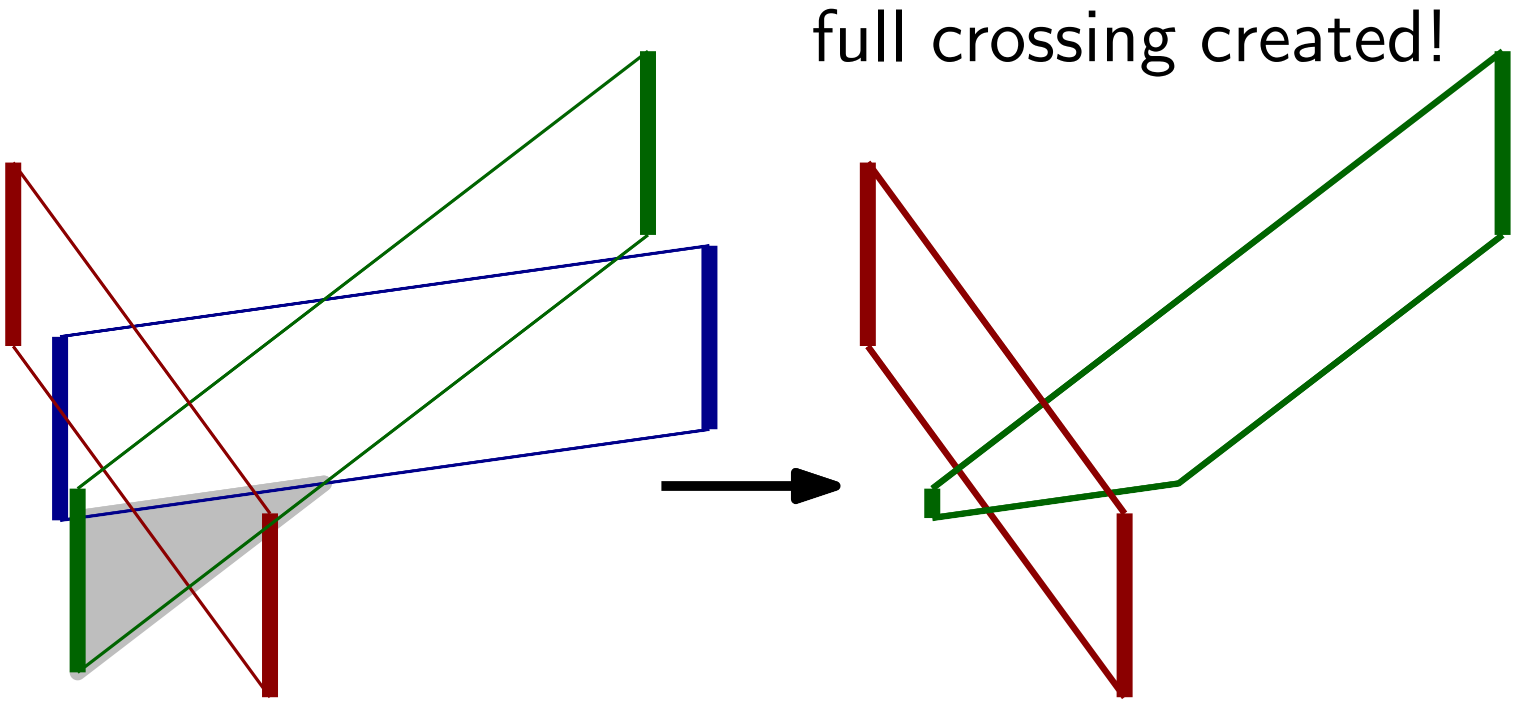

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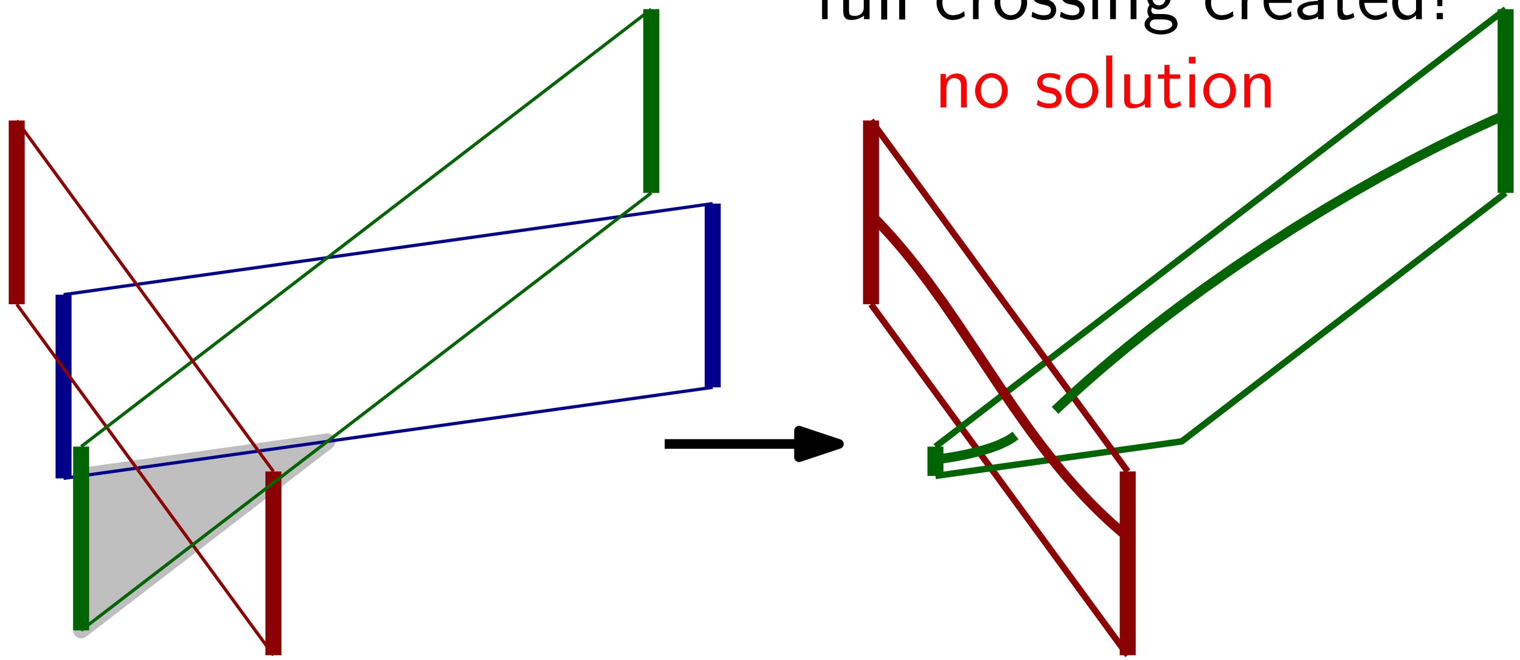
... then there is always a solution?

full crossing created!

no solution



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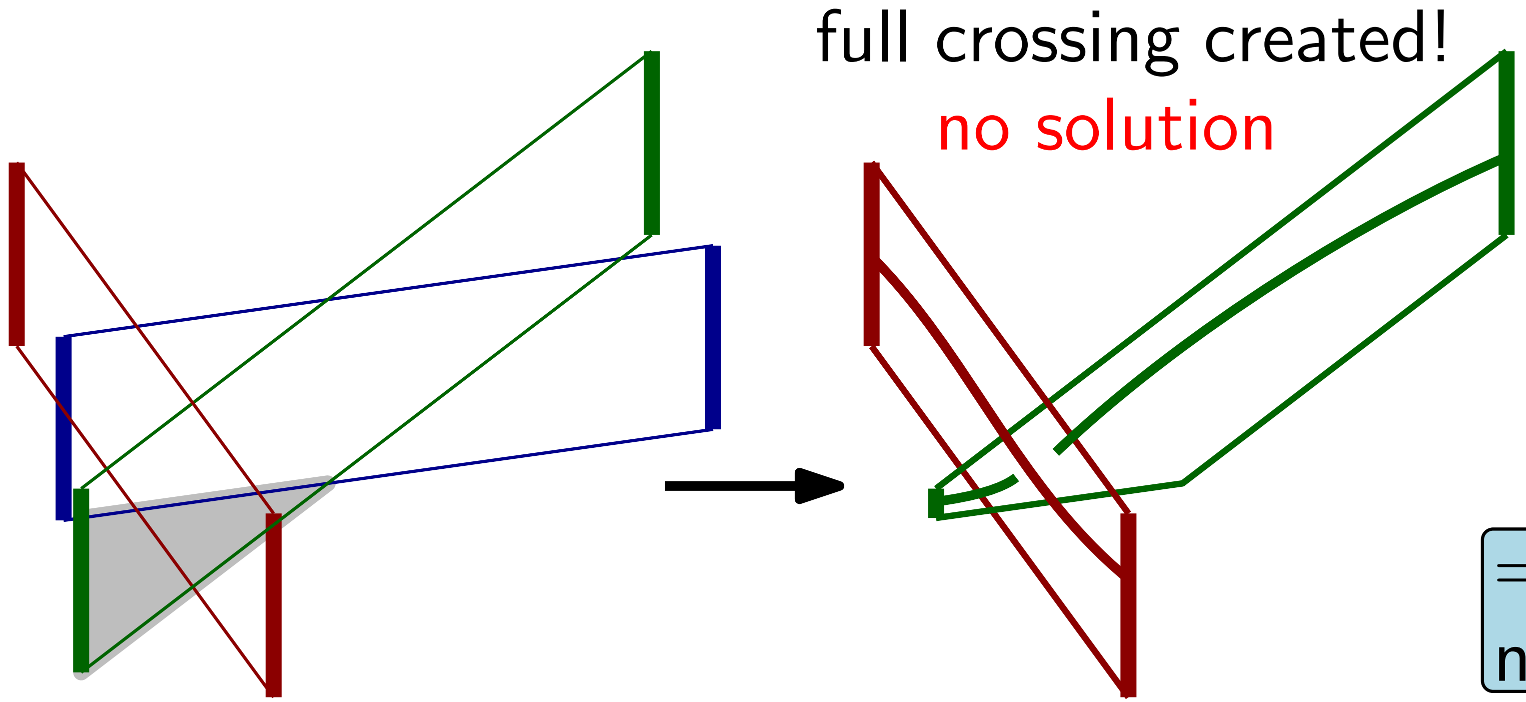


⇒ original instance had no solution either

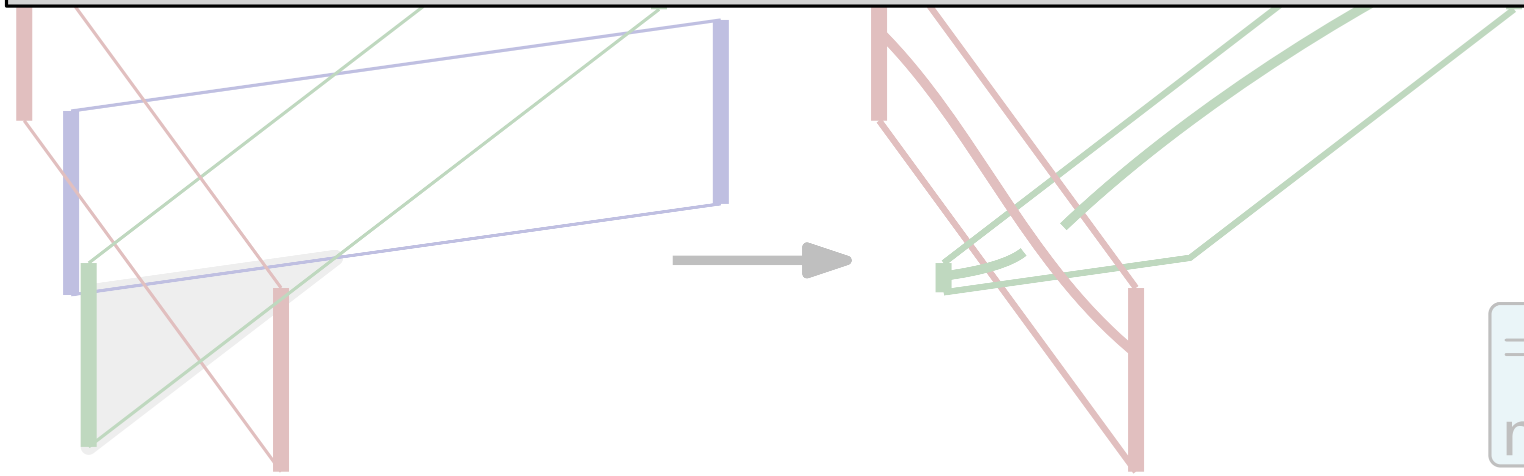
If no double intersections

Algorithm: while there is pair of tubes not pseudo-disks, cut off ear.

Result: set of pruned tubes that are pseudo-disks

... then there is always a solution?

If there are no double intersections, one can determine if all the tubes can be connected using arbitrary paths in polynomial time

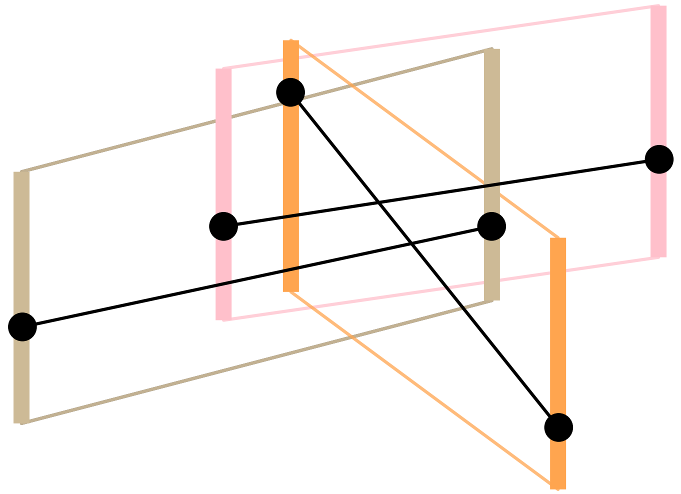


⇒ original instance had no solution either

Summary

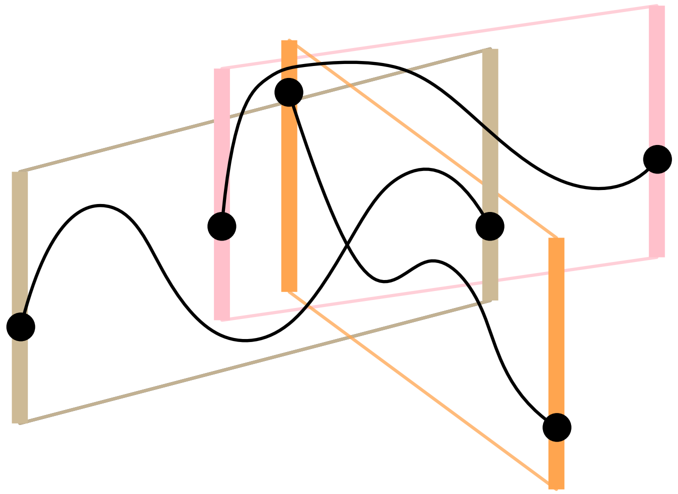
Summary

straight-line paths



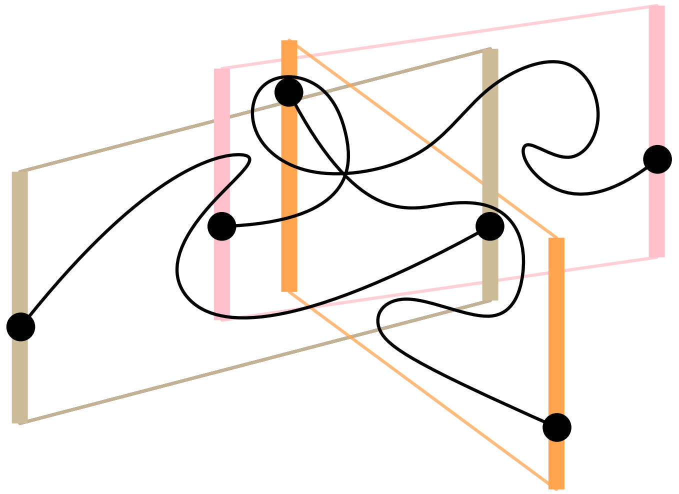
NP-complete

(x-)monotone paths



Polynomial

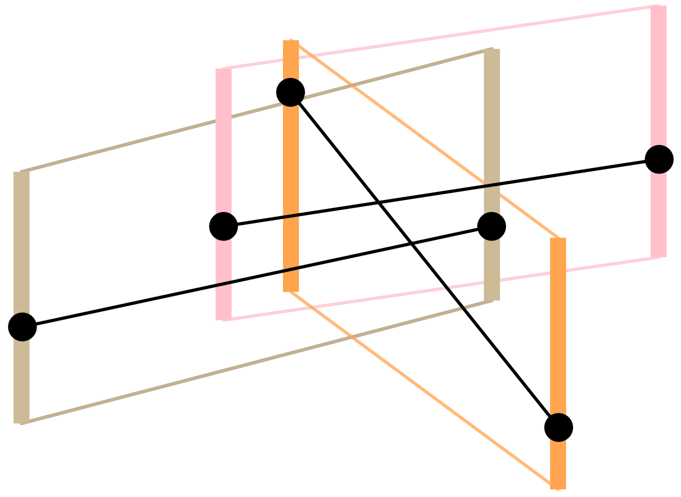
arbitrary paths



Polynomial if no double intersections

Summary

straight-line paths



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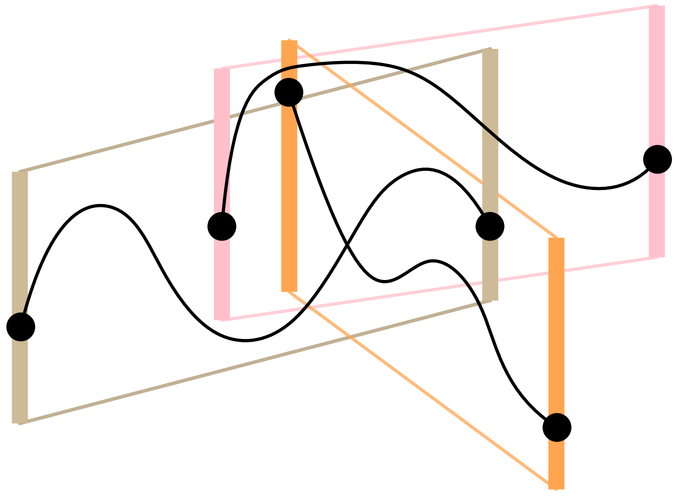

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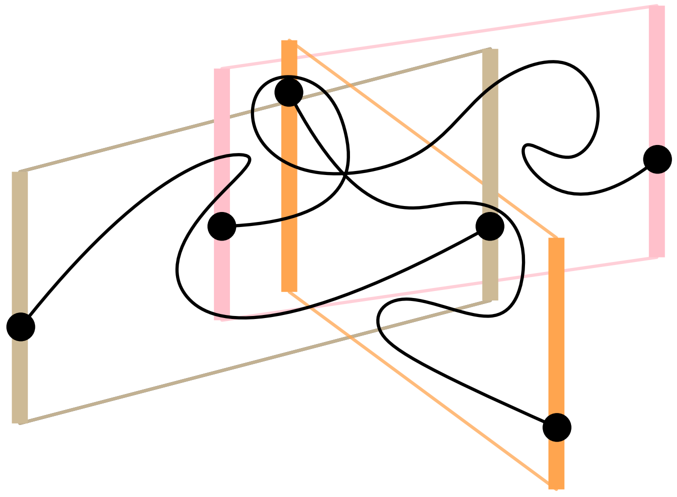

Polynomial if no double intersections

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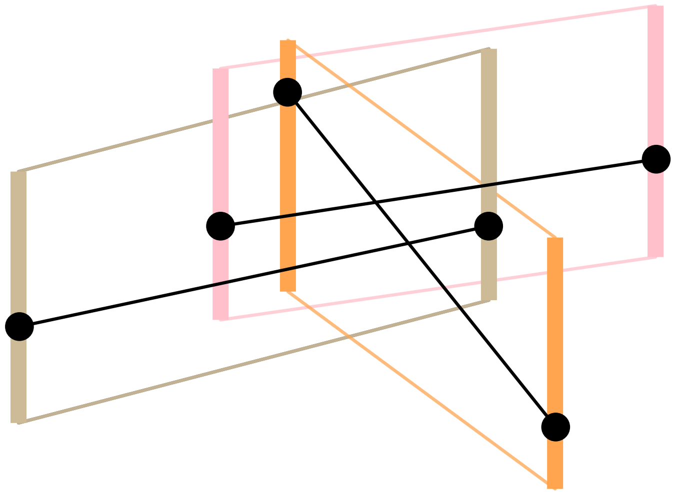
straight-line paths



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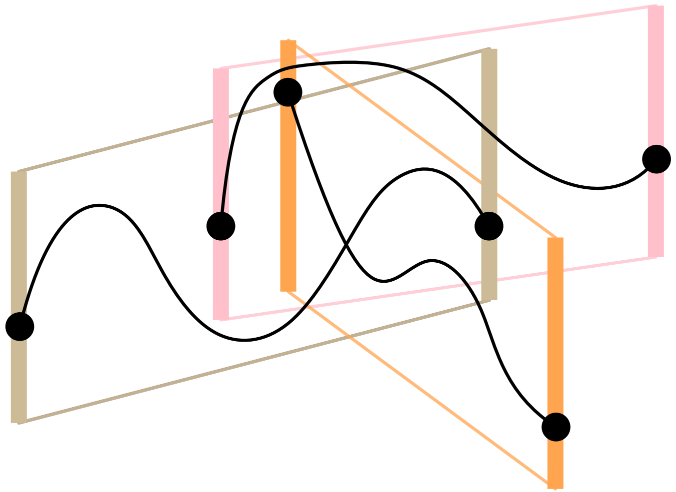

Polynomial

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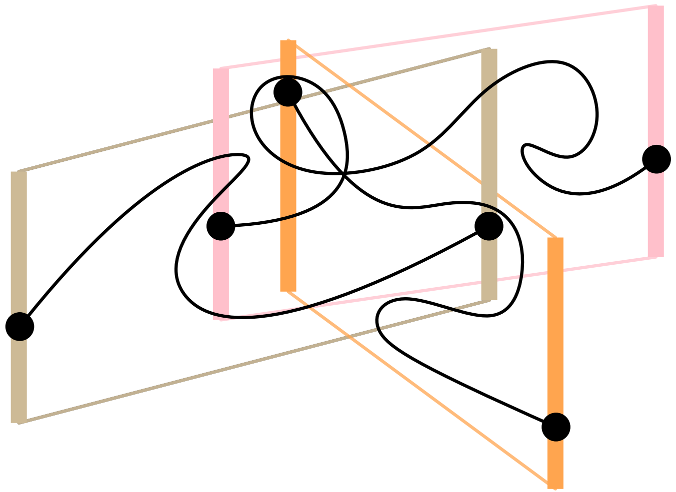
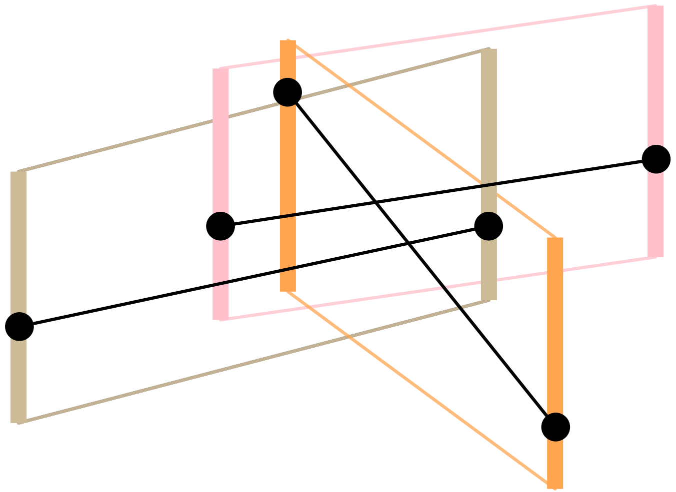


Polynomial if no double intersections

Is it also polynomial if there are double intersections?

Summary

straight-line paths



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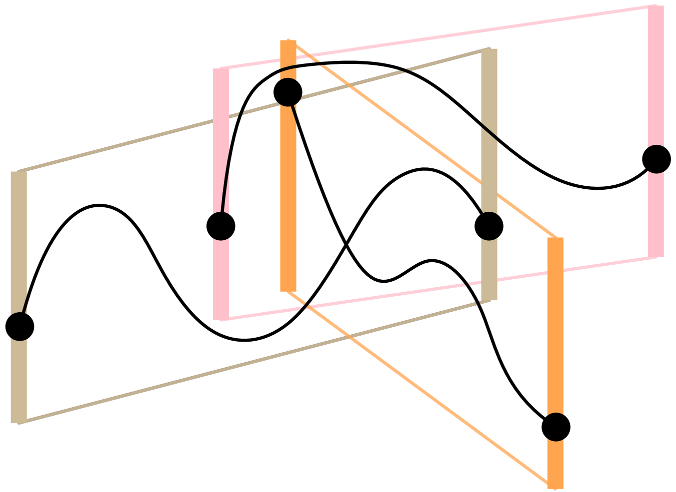
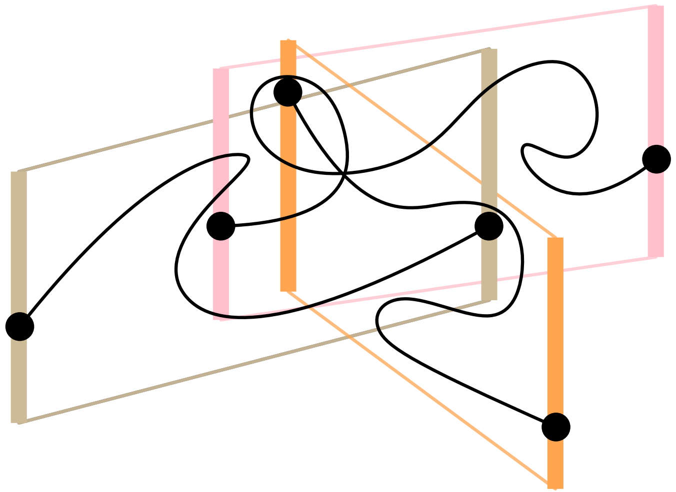


Polynomial

arbitrary paths



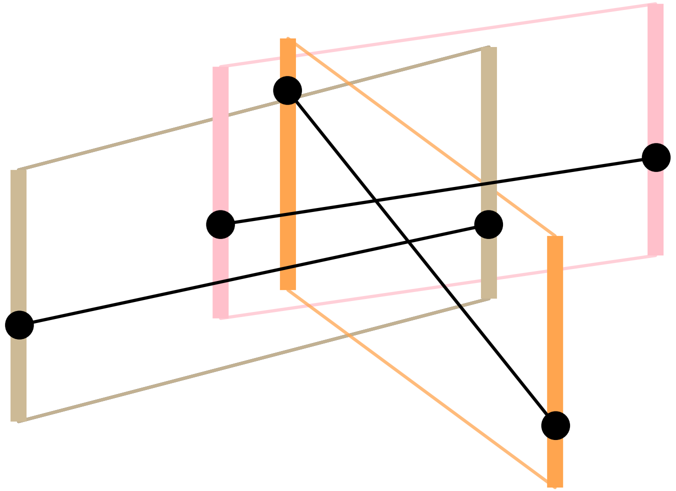
Polynomial if no double intersections

Is it also polynomial if there are double intersections?

We conjecture the answer is **Yes**

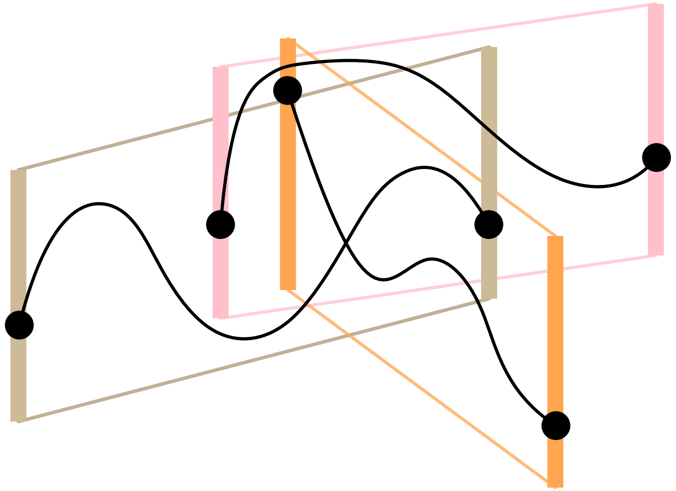
Our results

straight line paths



NP-complete

(x-)monotone paths



Polynomial

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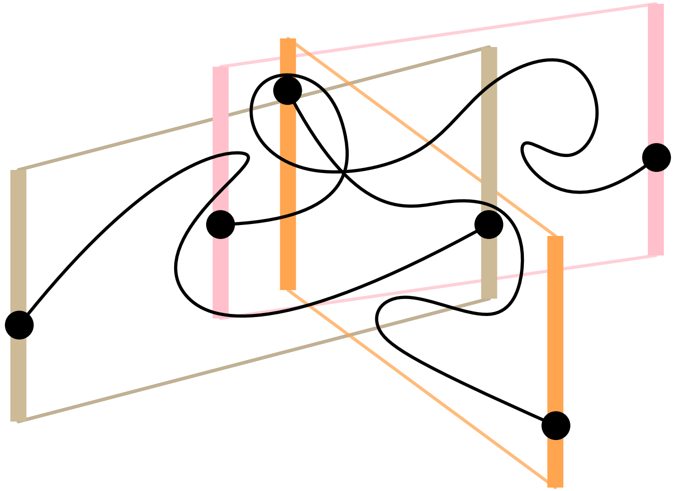
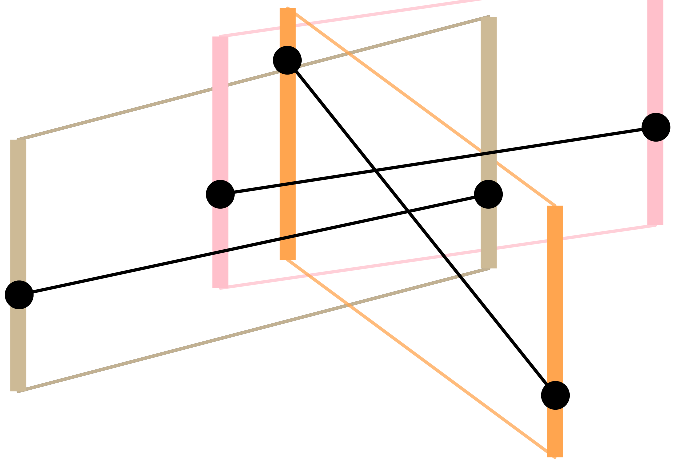


Polynomial under certain assumptions

Our results

straight line paths



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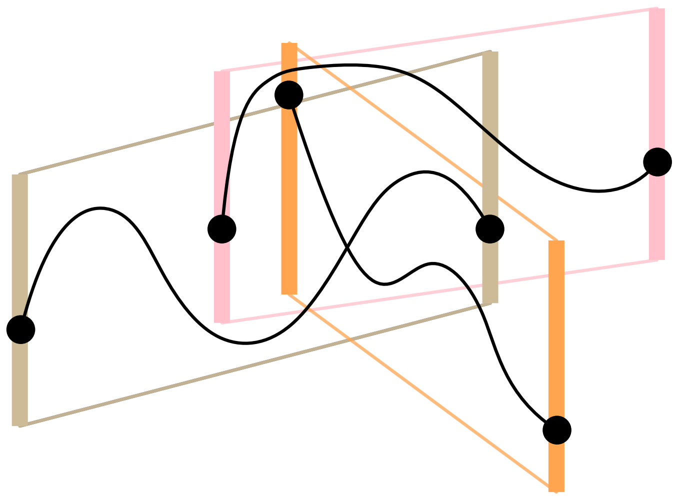

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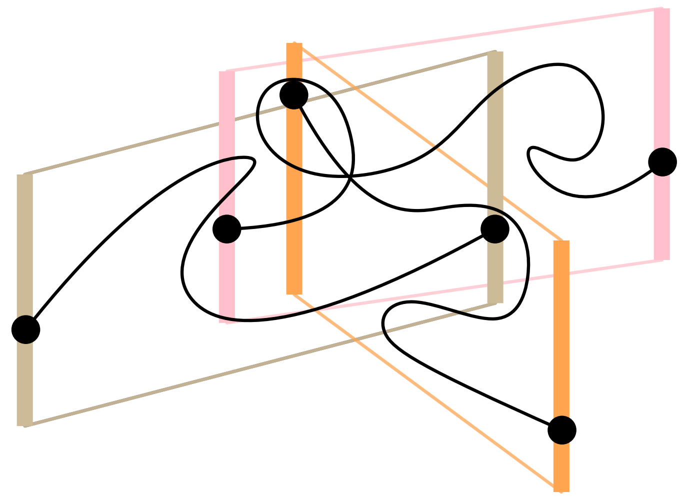

Polynomial under certain assumptions

Straight line paths

Given n tubes defined by unit vertical segments, deciding if the tubes can be connected with straight line segments is NP-Complete

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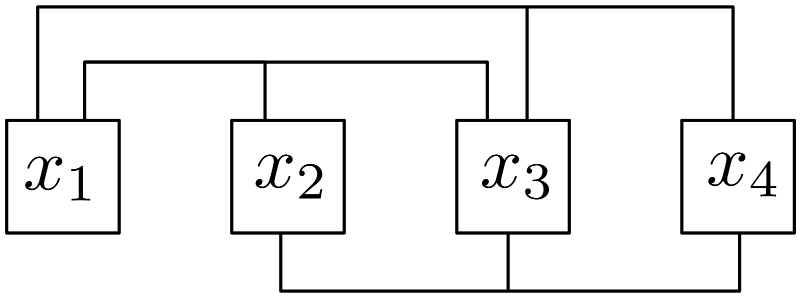
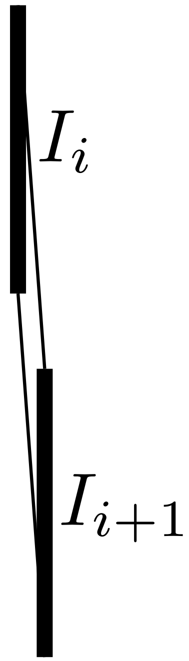


Reduction from Rectilinear Planar 3-SAT

Straight line paths

Given n tubes defined by unit vertical segments, deciding if the tubes can be connected with straight line segments is NP-Complete

Two useful gadgets:



Blocker

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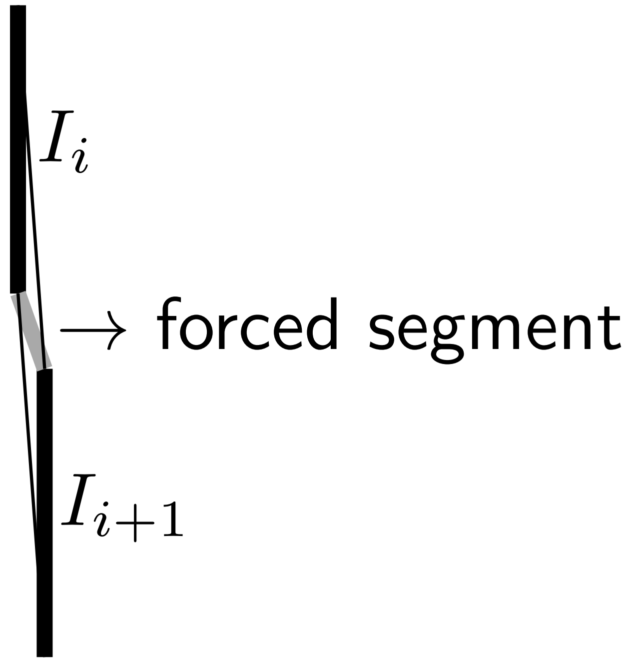

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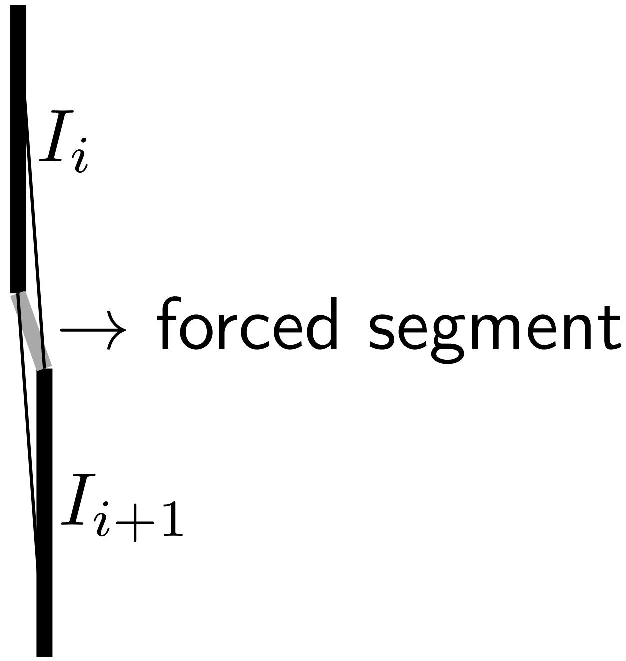

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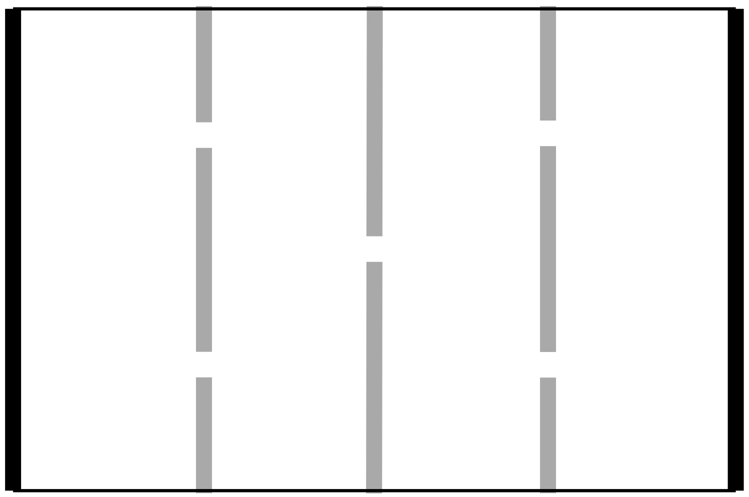

Variable

Straight line paths

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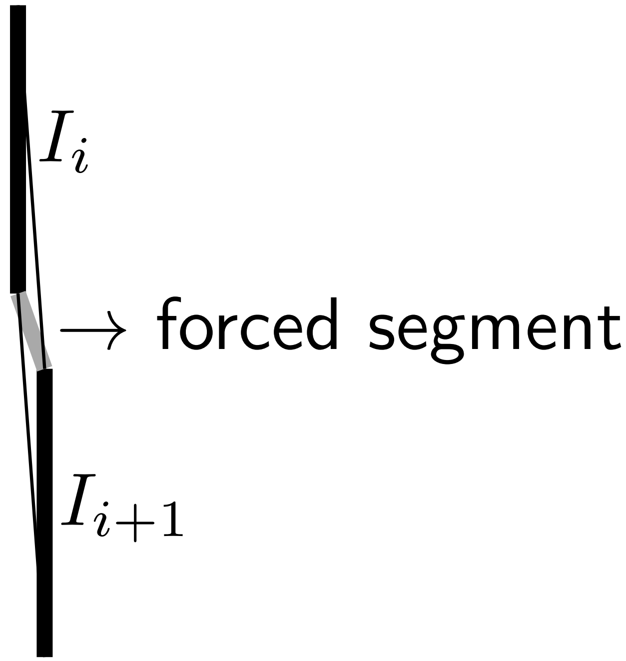

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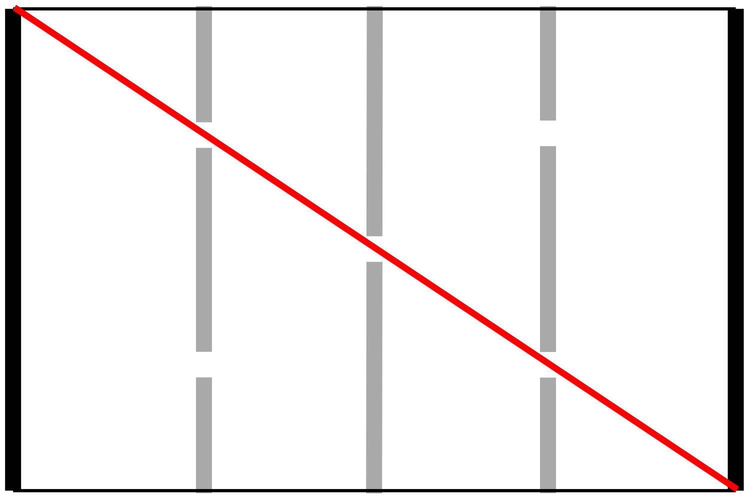

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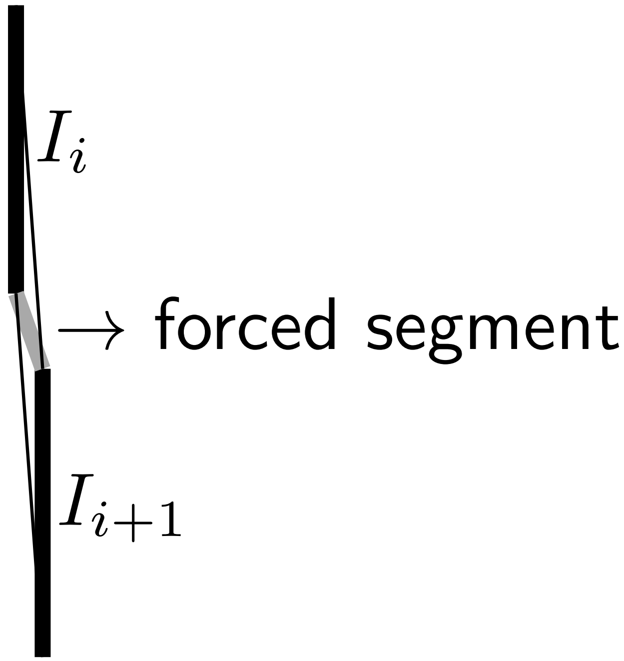

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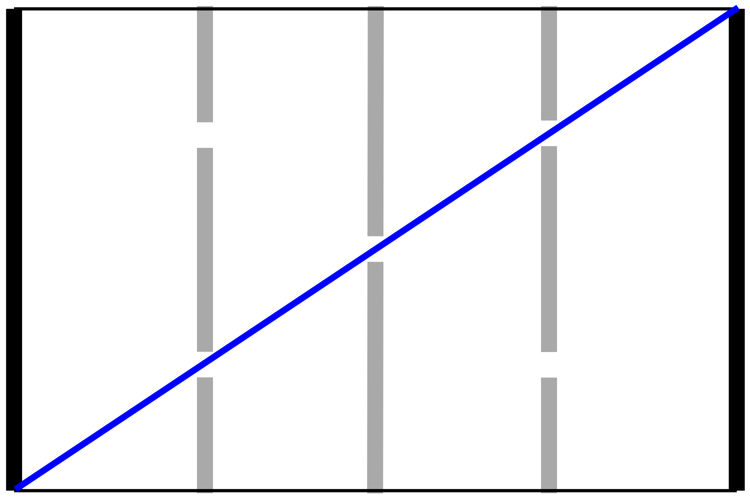
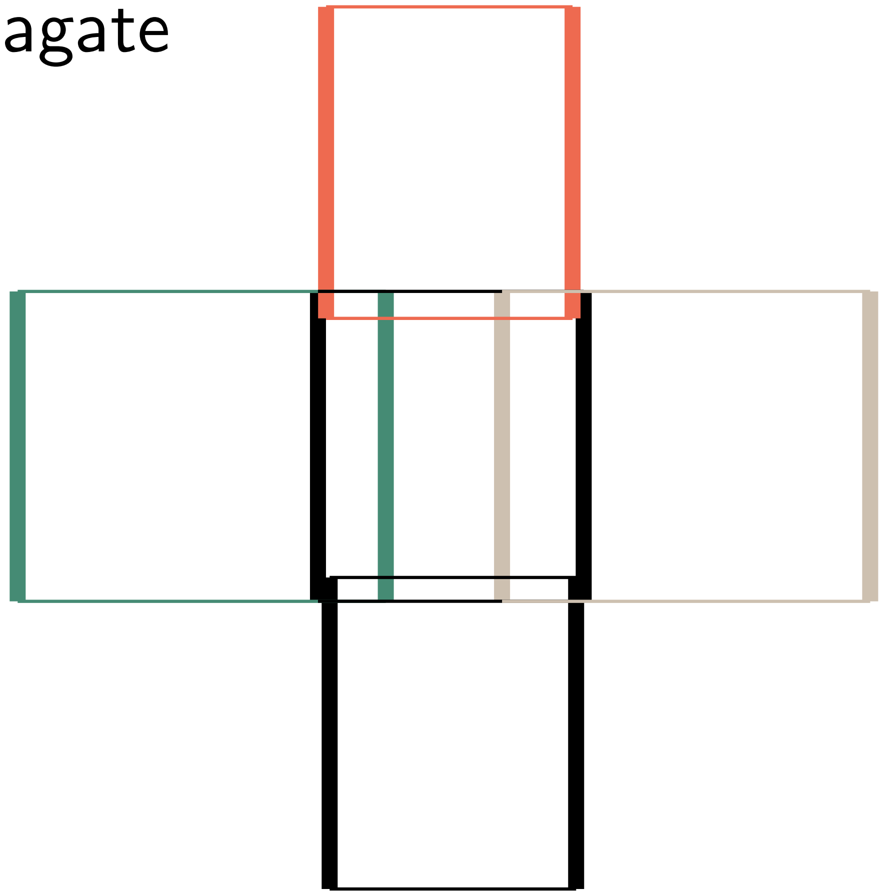


Variable

Straight line paths

Given n tubes defined by unit vertical segments, deciding if the tubes can be connected with straight line segments is NP-Complete

We can now propagate truth values...



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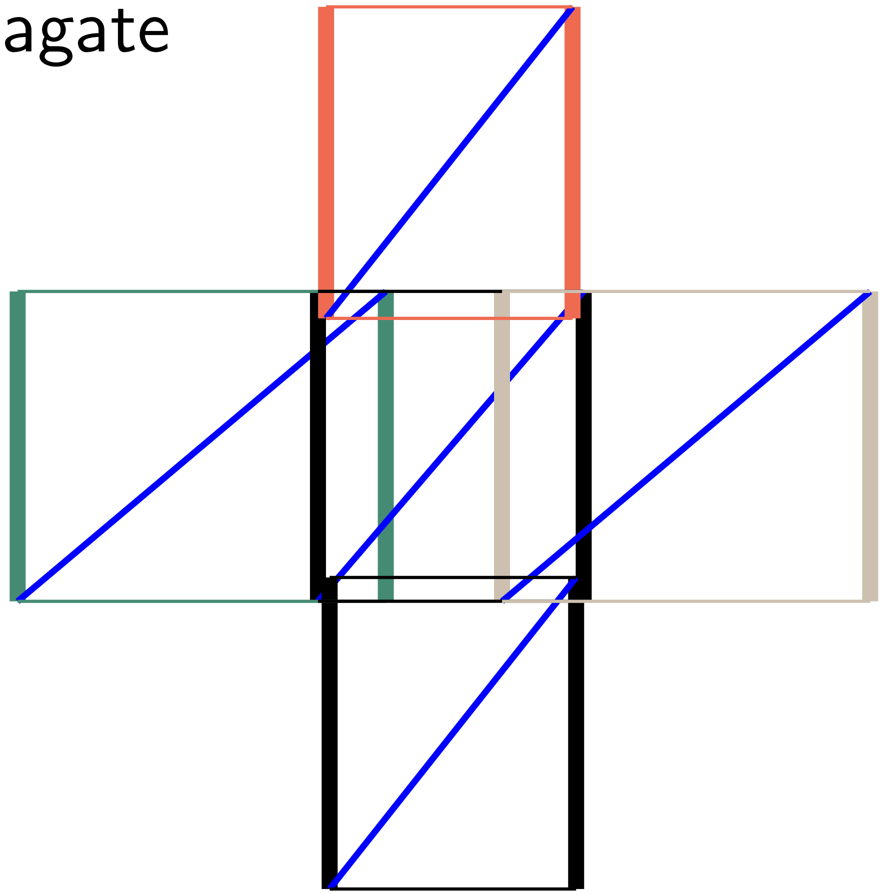

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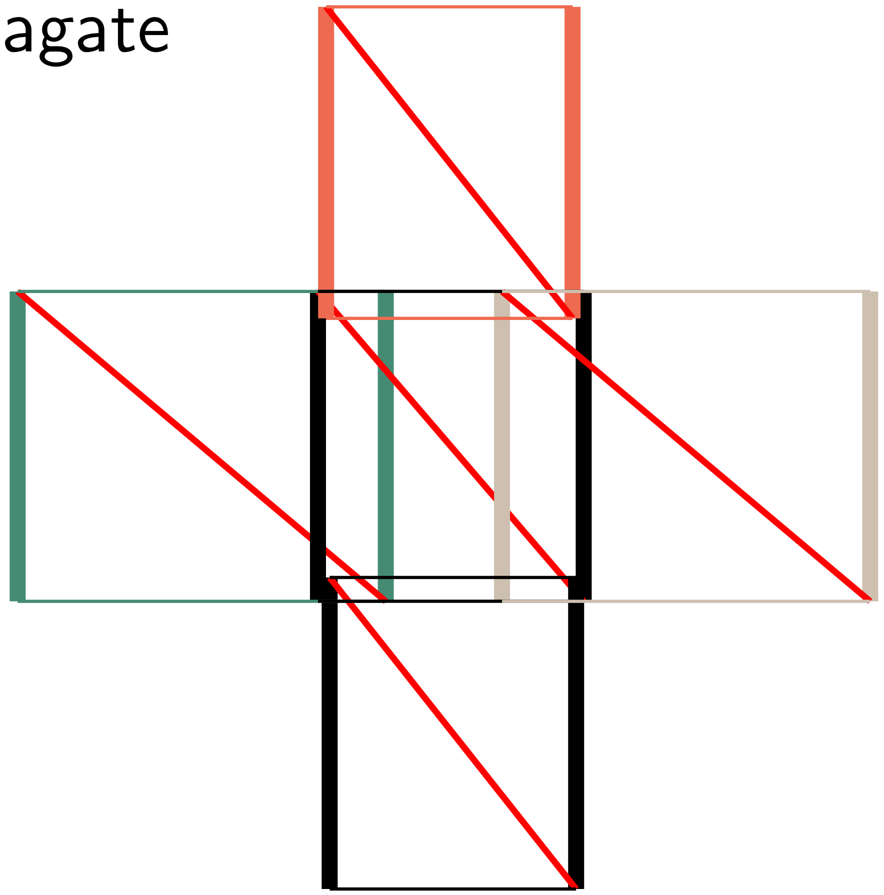

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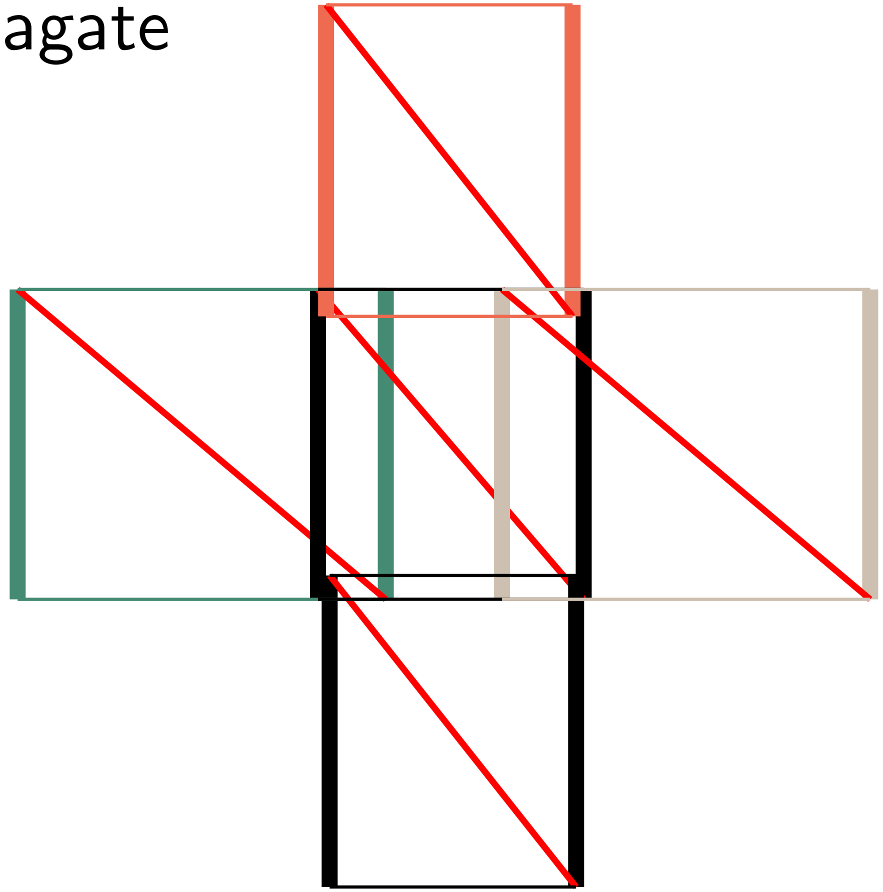

...as well as representing negations and clauses, so that the reduction works