FernUniversität in Hagen
Fakultät für Mathematik und Informatik

theoretische informatik

1-Fan-Bundle-Planar Drawings of Graphs

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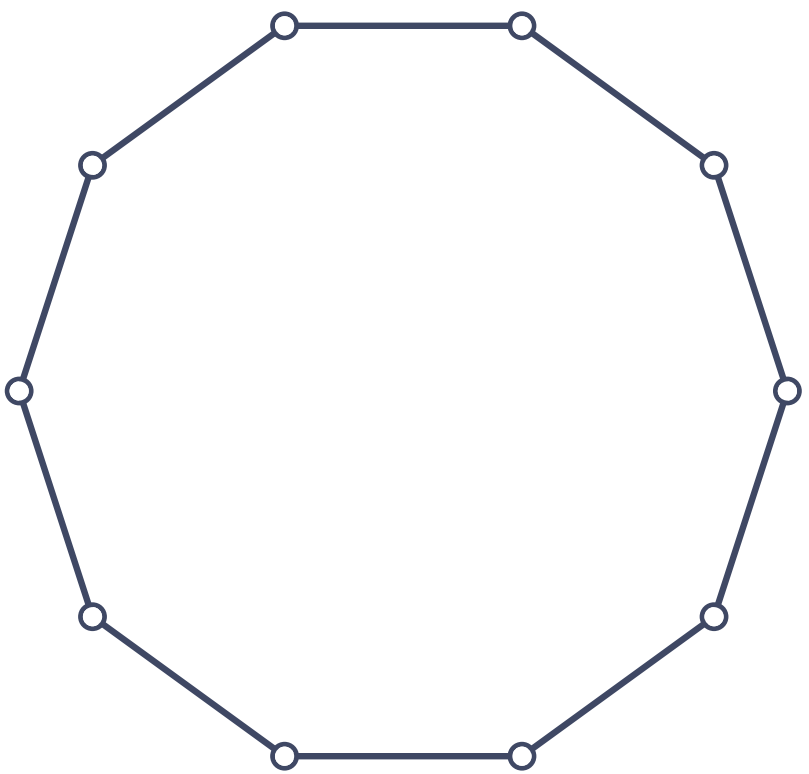
MATHEMATISCH-
NATURWISSENSCHAFTLICHE FAKULTÄT
ARBEITSBEREICH ALGORITHMIK

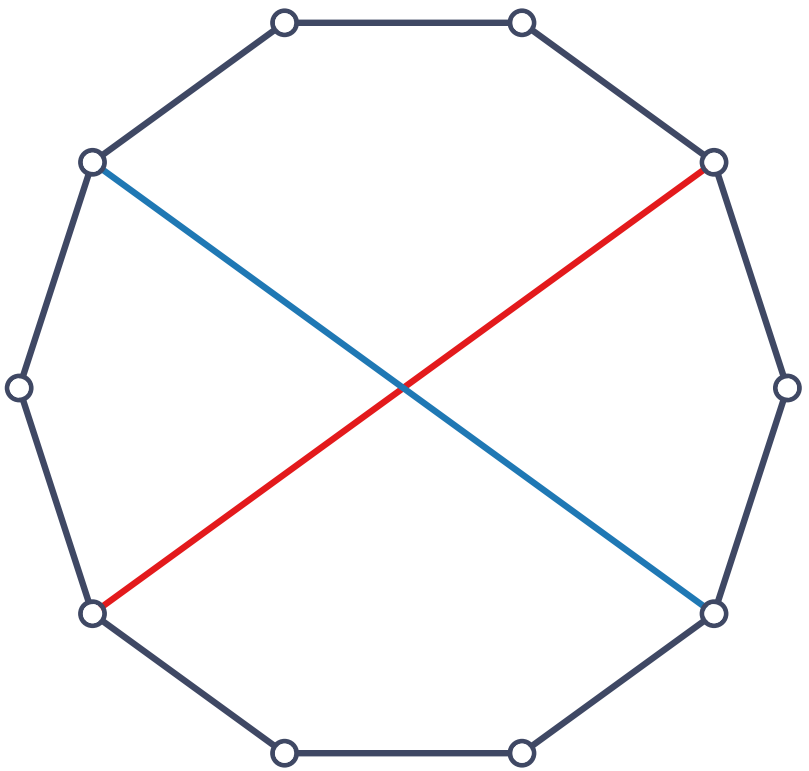
Beyond Planarity

Planar



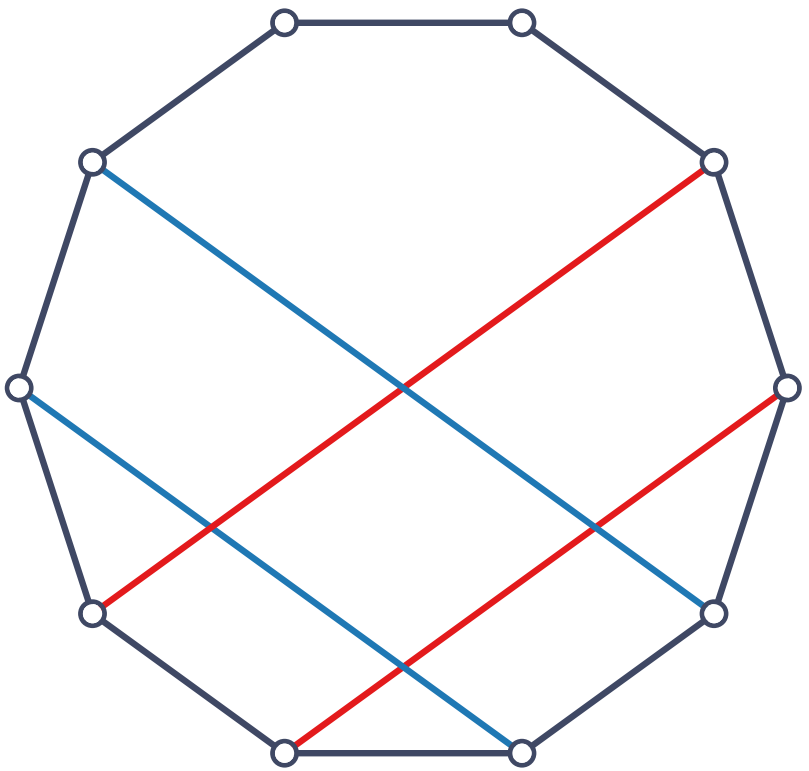
Beyond Planarity

1-planar



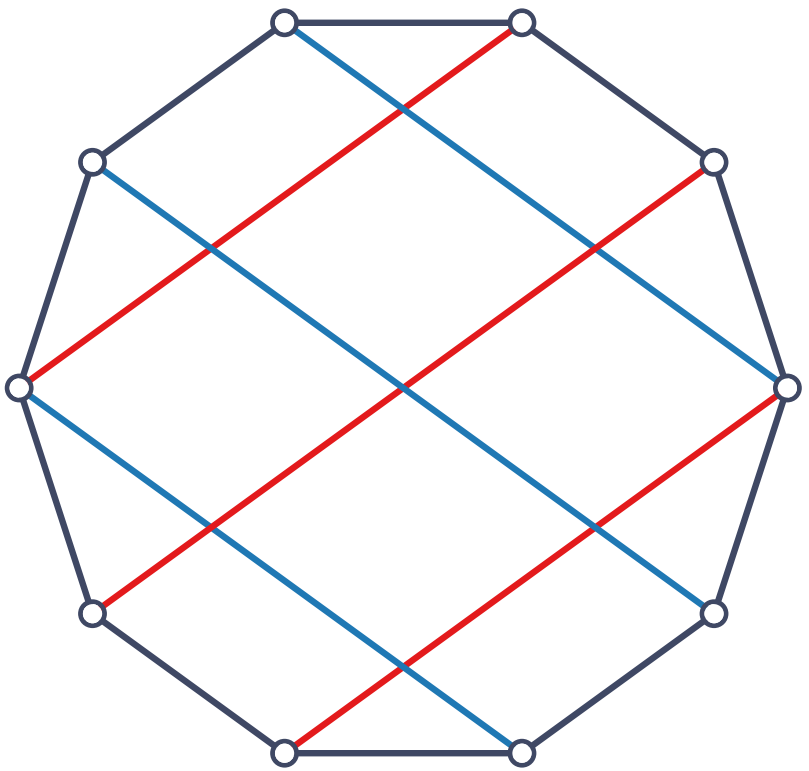
Beyond Planarity

2-planar



Beyond Planarity

k-planar



Beyond Planarity

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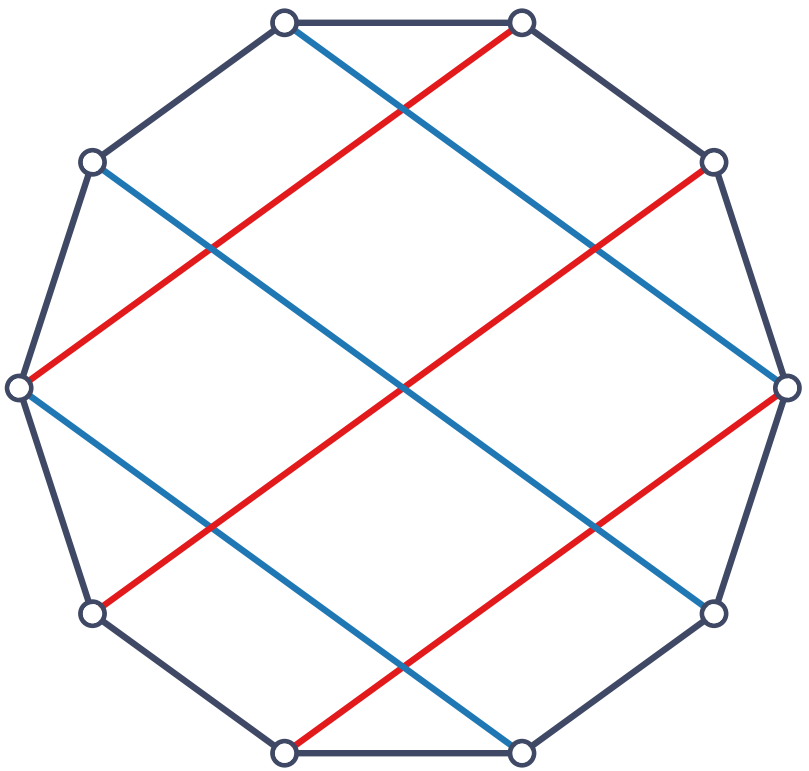

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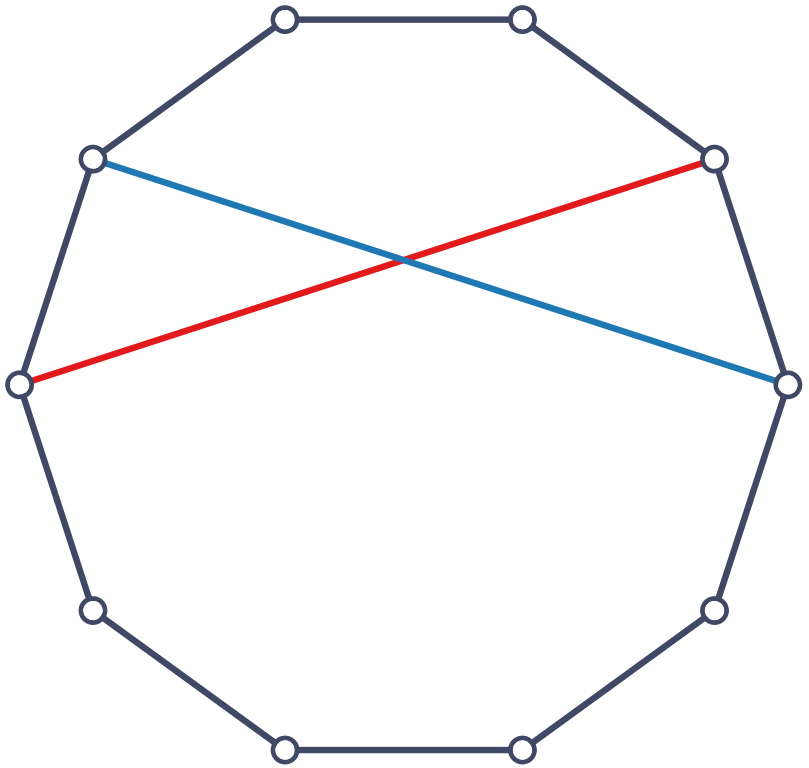

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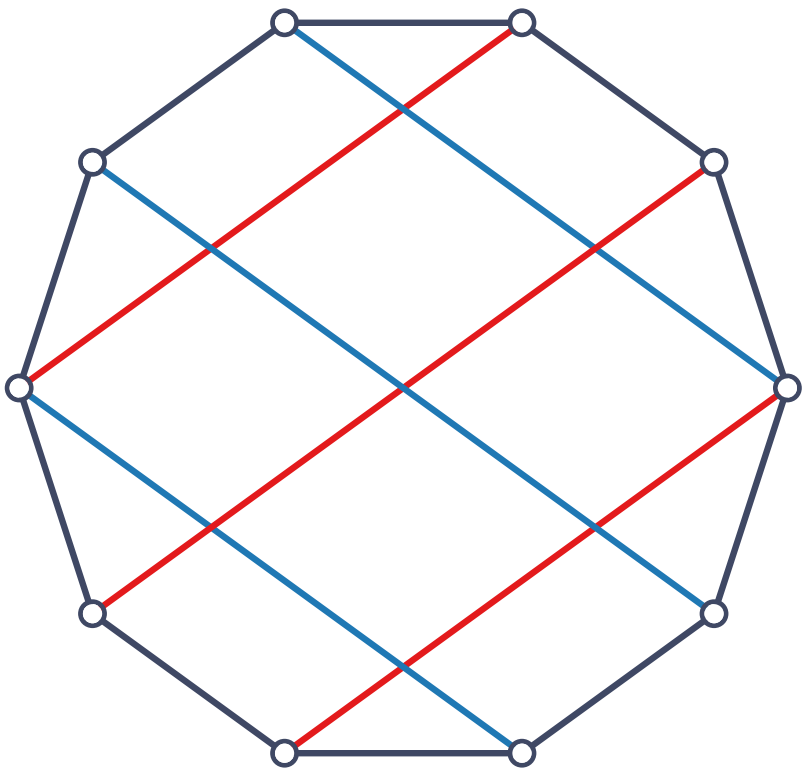

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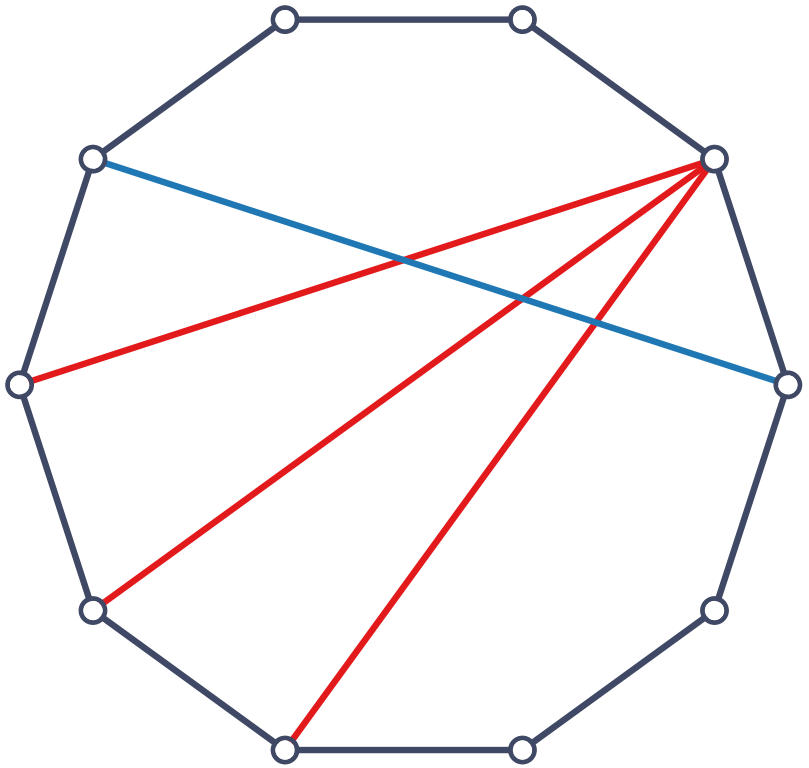

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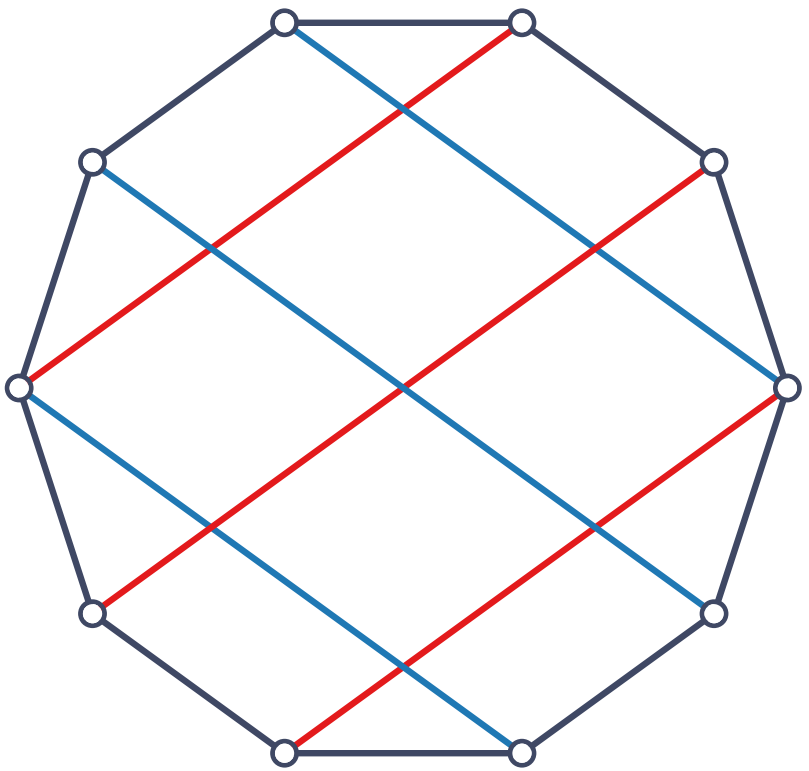

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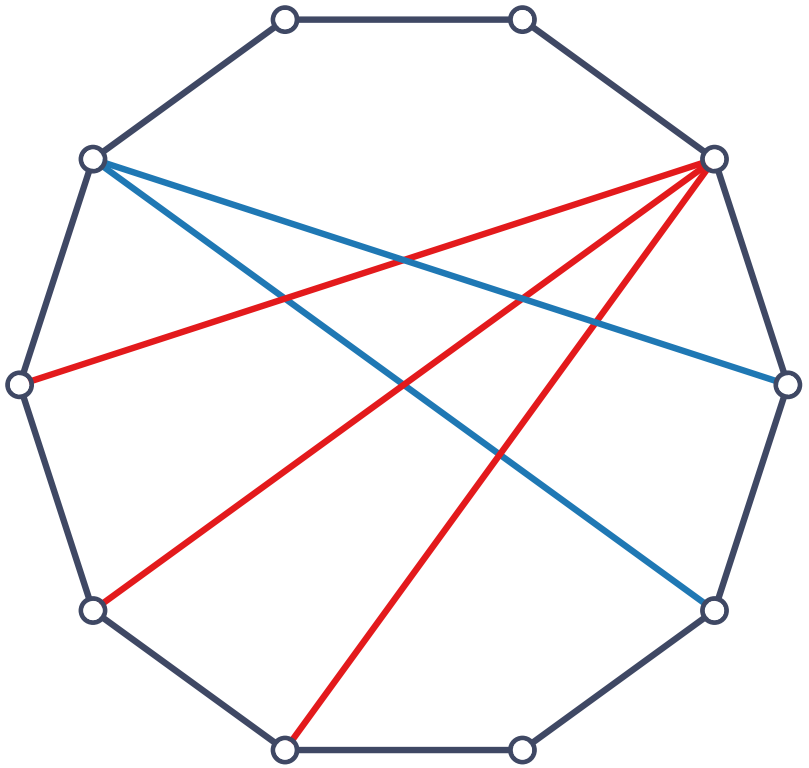

Beyond Planarity

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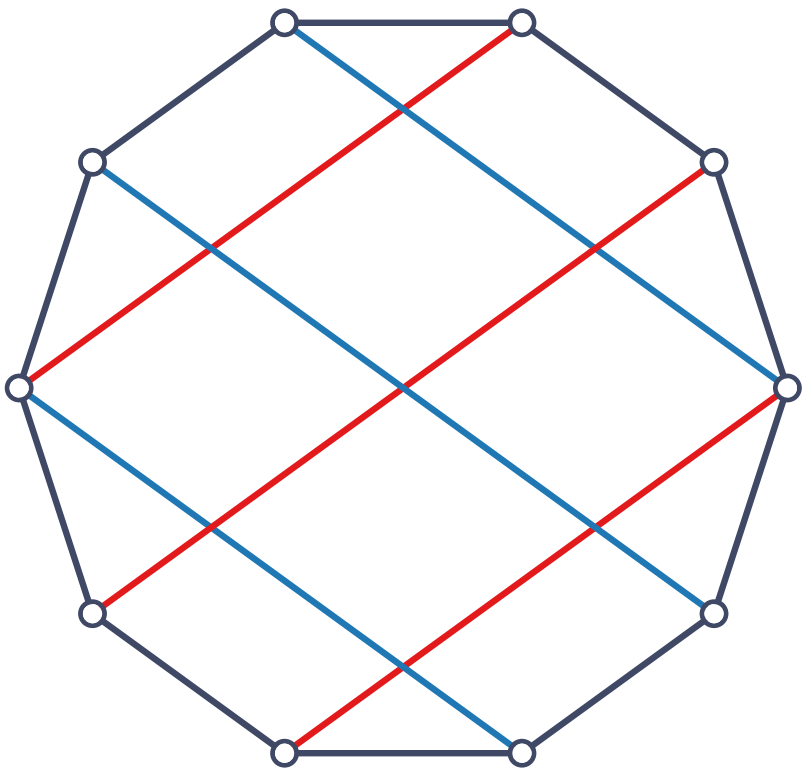

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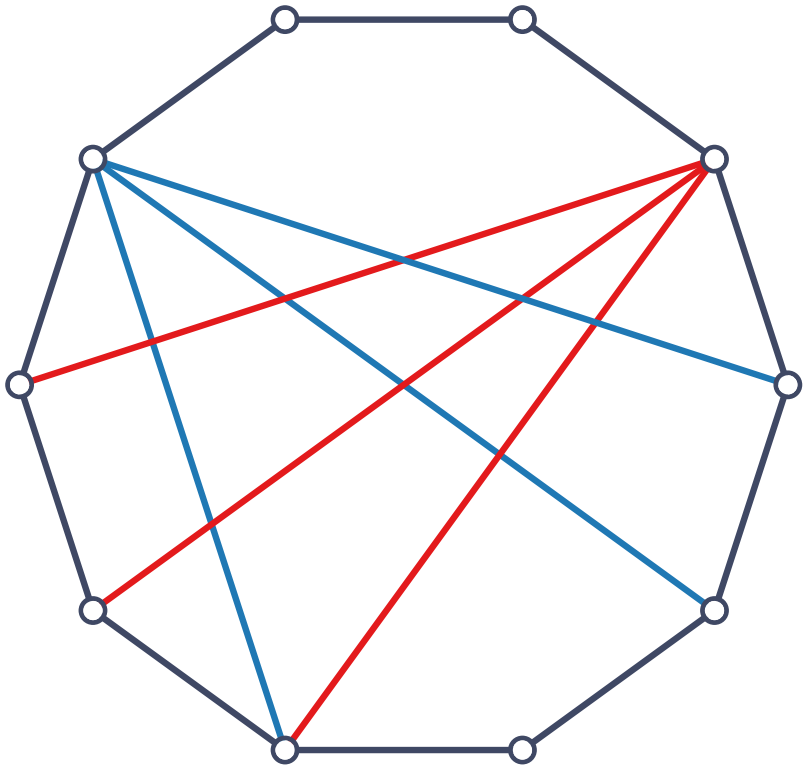
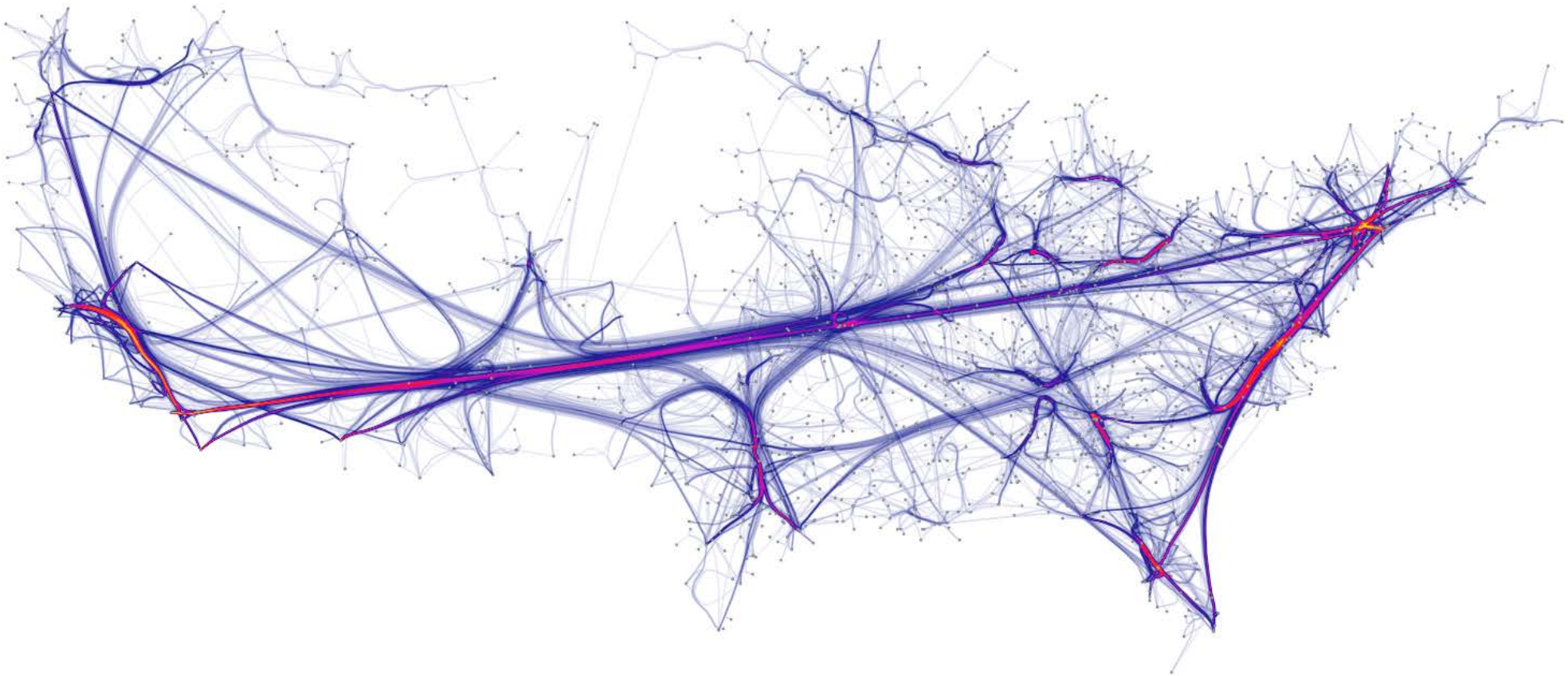
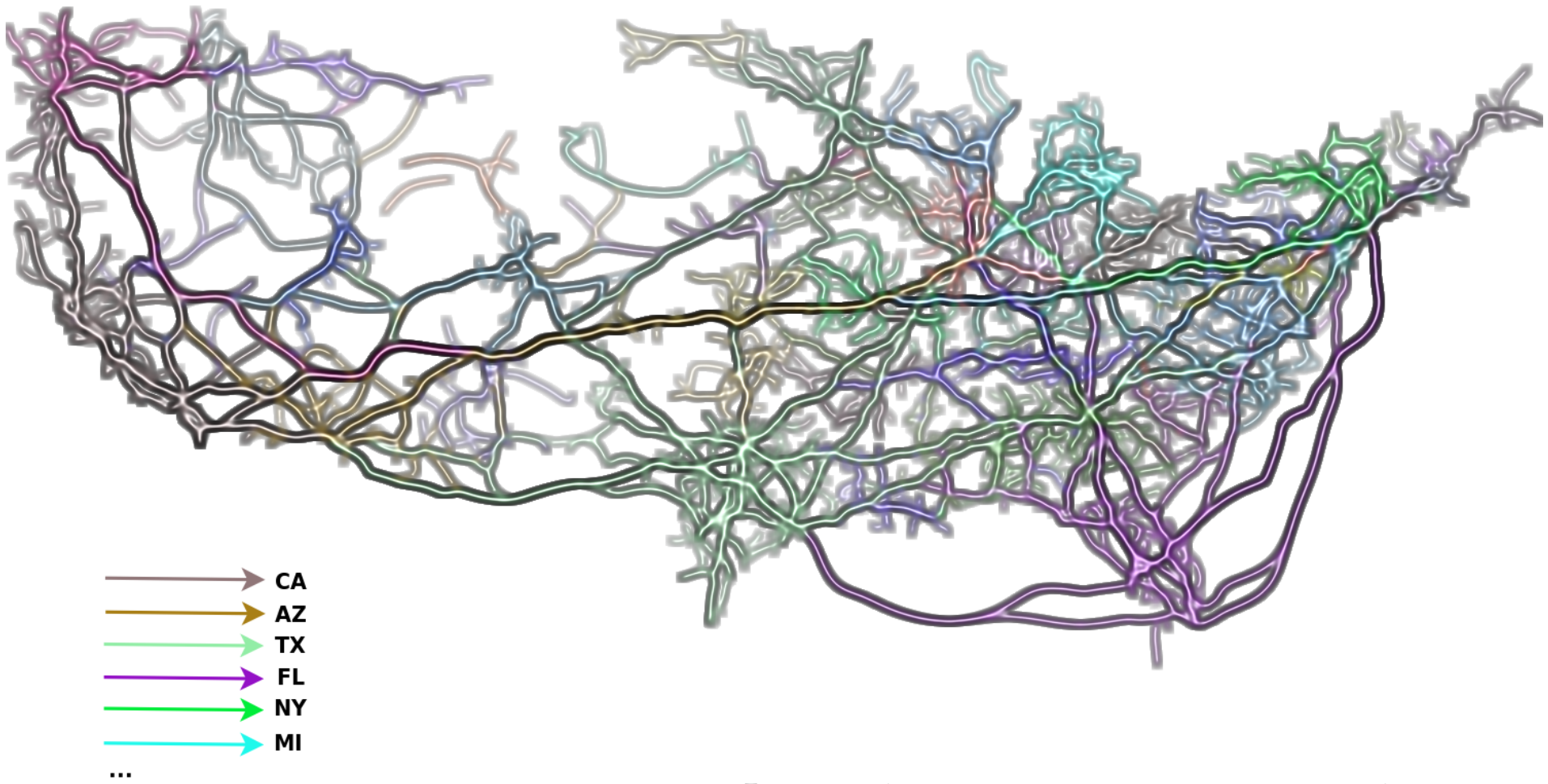


Bundled Edge Drawings



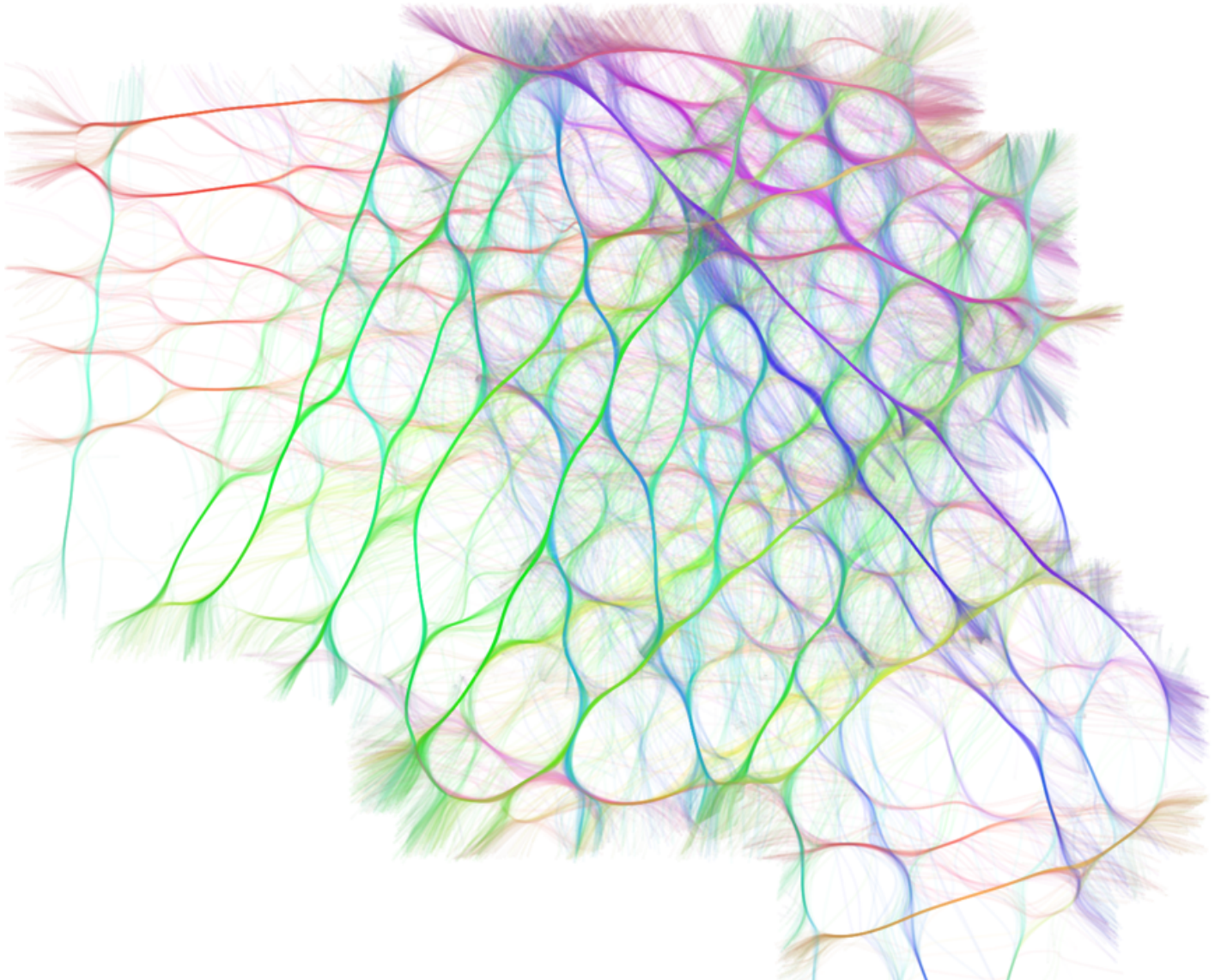
[Holten & van Wijk ’09]

Bundled Edge Drawings



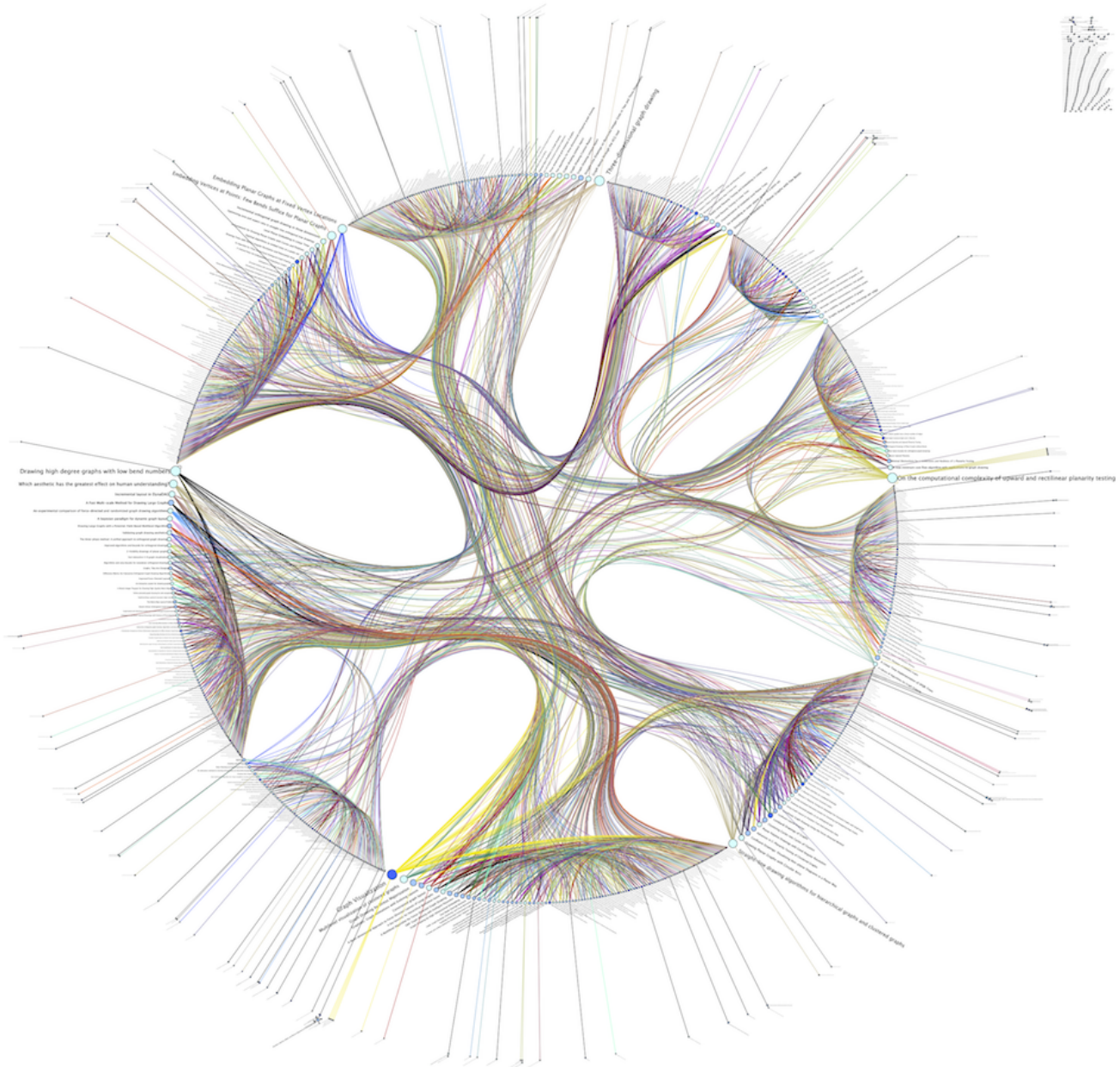
[Lambert, Bourqui &Auber ’10]

Bundled Edge Drawings



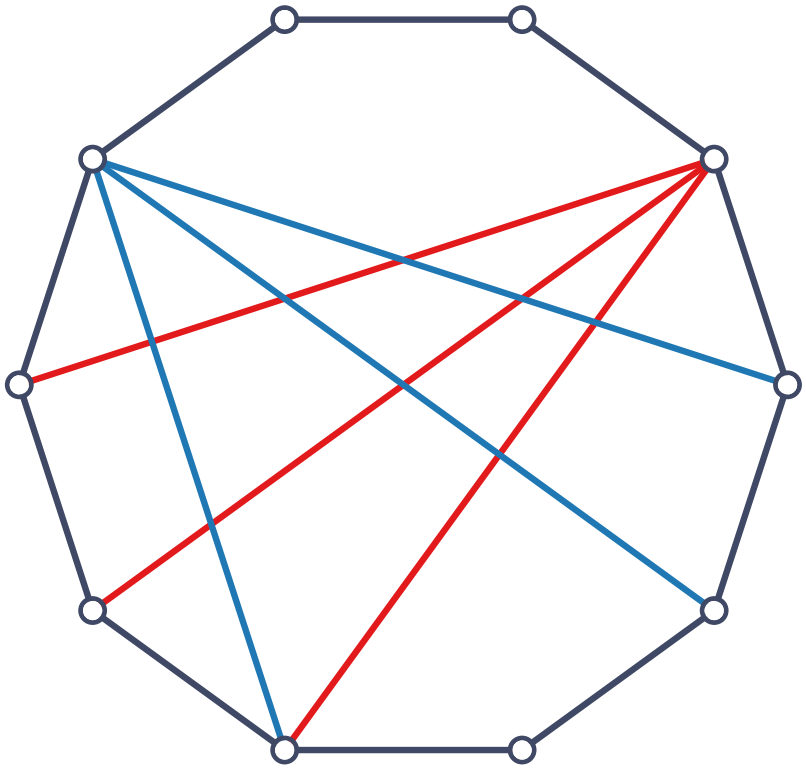
[Wu, Zeng, Zhu & Yu '17]

Bundled Edge Drawings



[Ye ’17]

k-Fan-Bundle-Planarity



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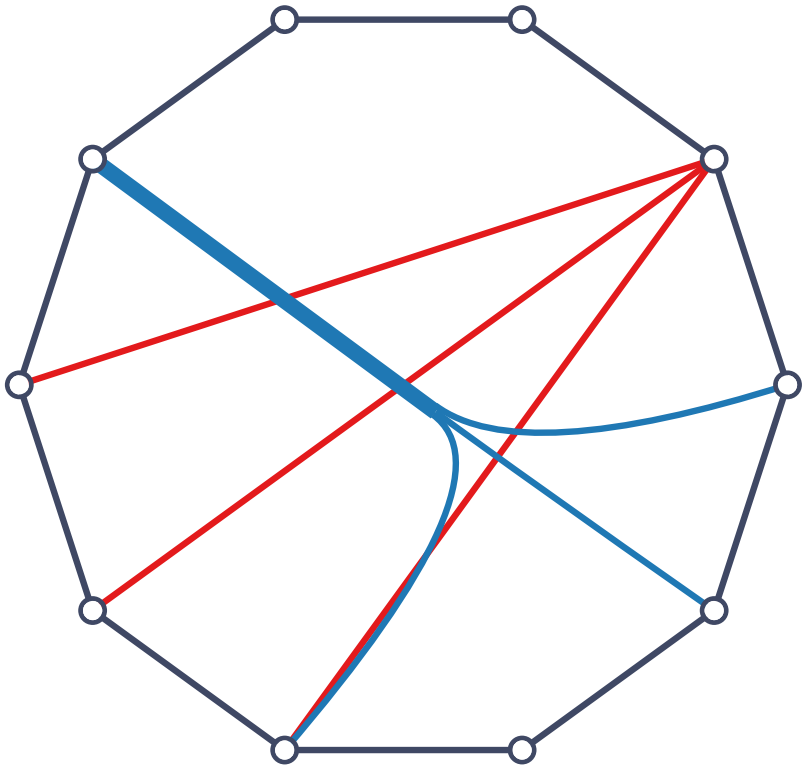

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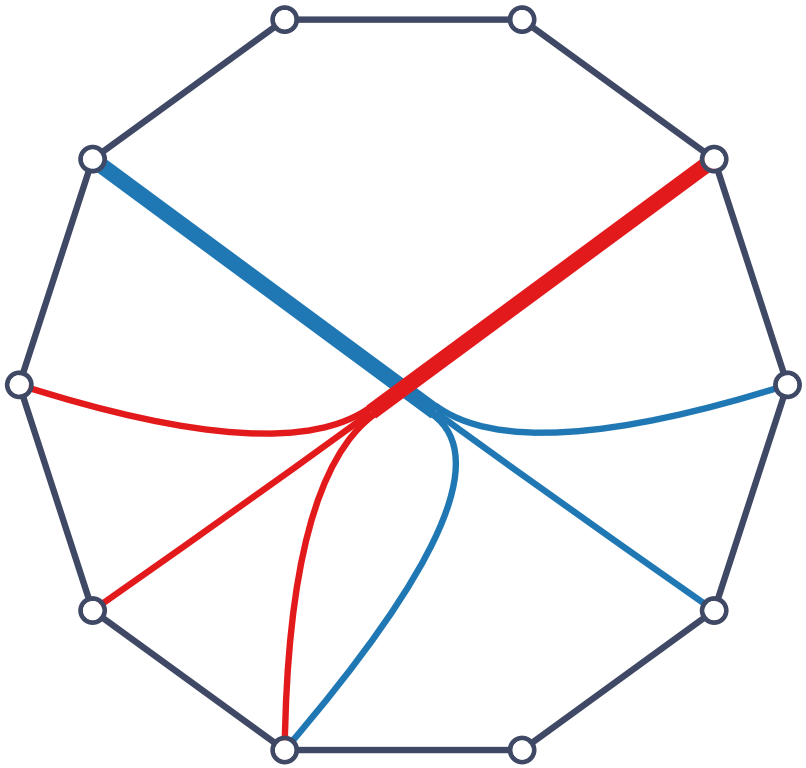

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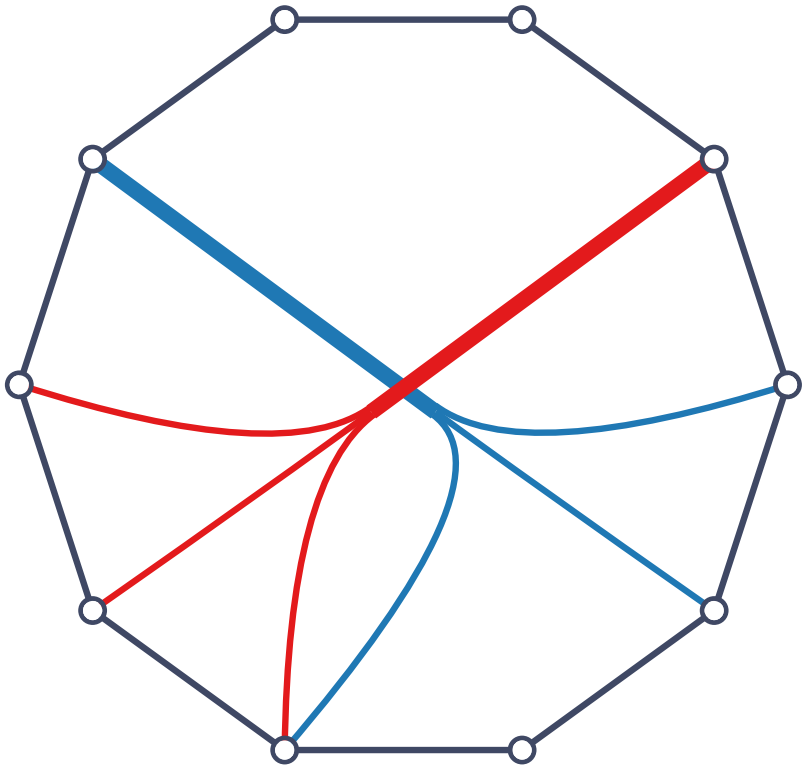

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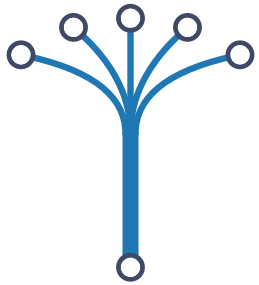

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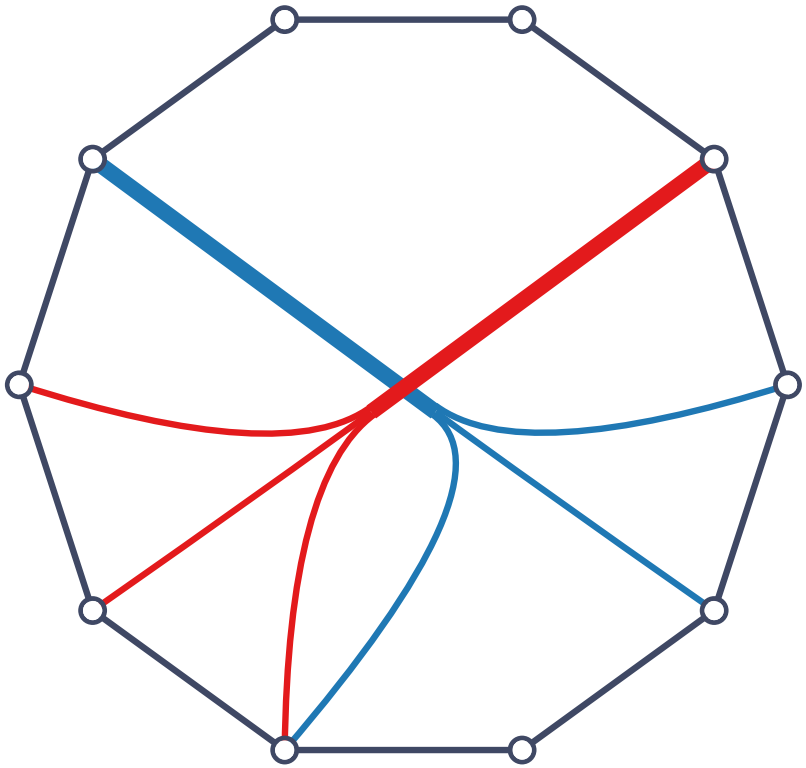

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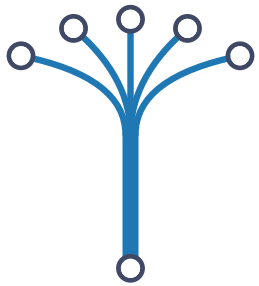

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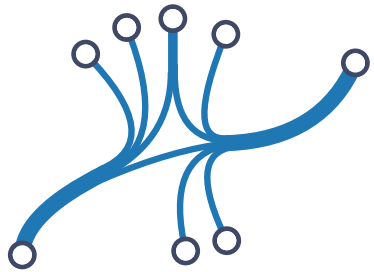

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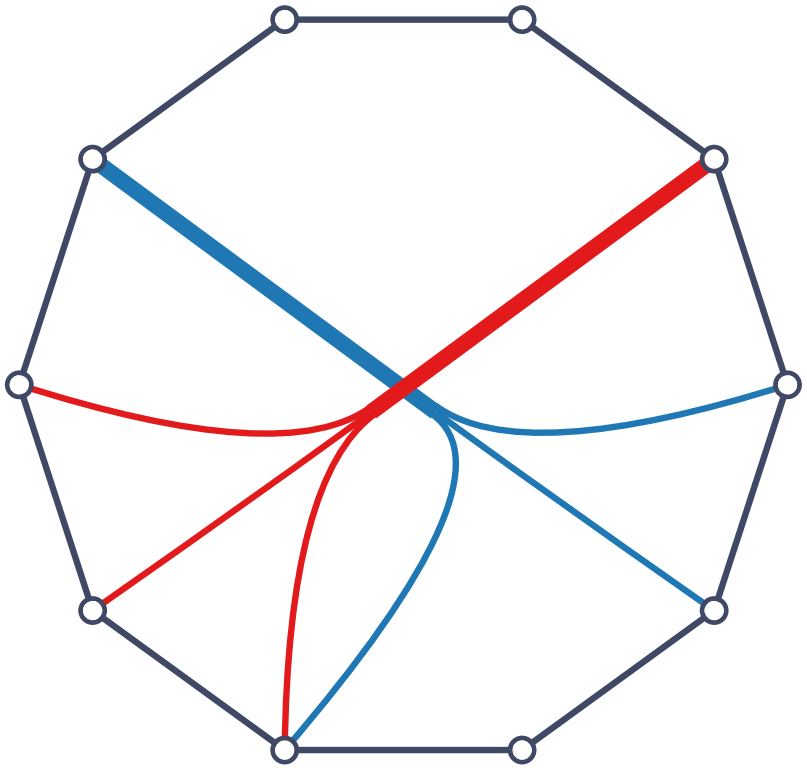

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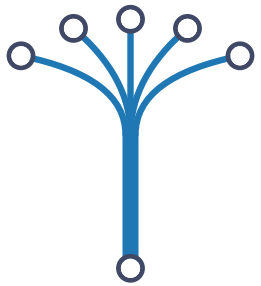

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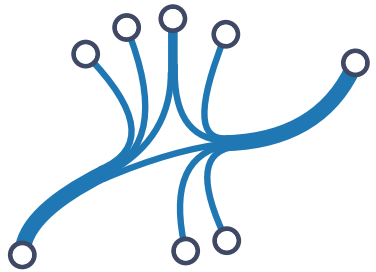
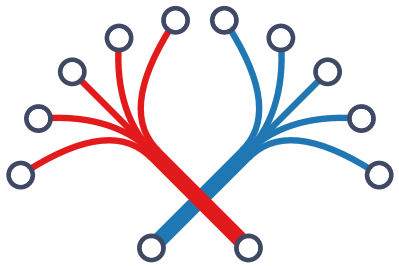
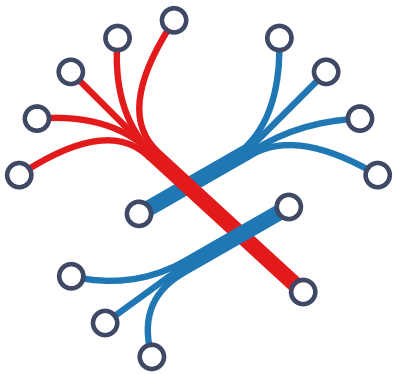


k crossings per bundle





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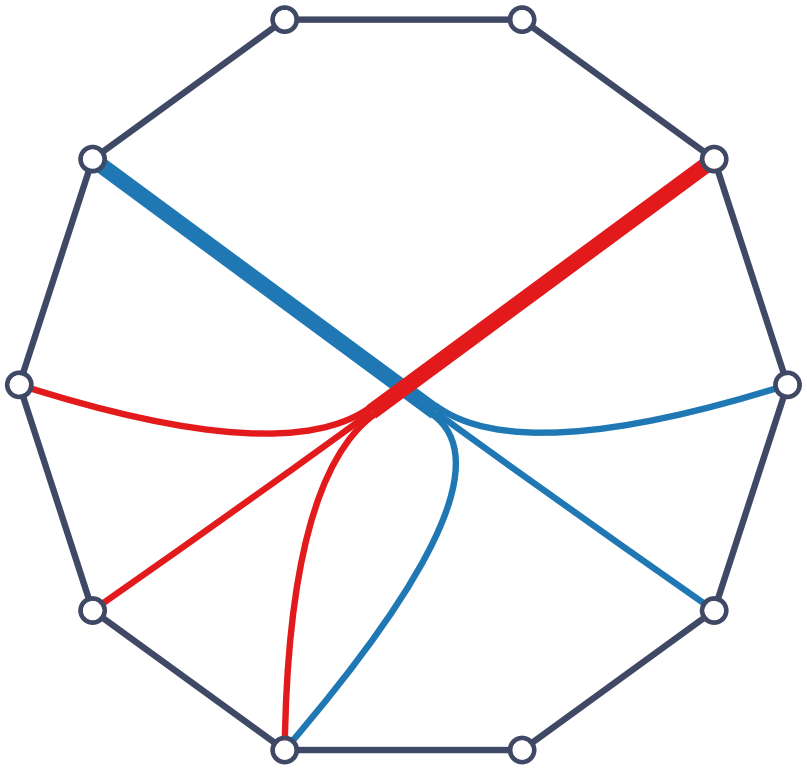
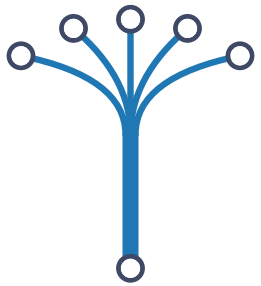


1-Fan-Bundle-Planarity

1-sided



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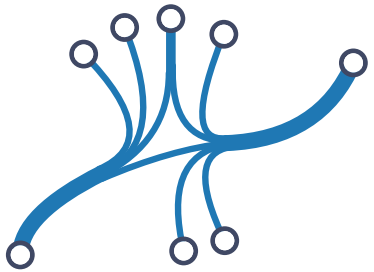
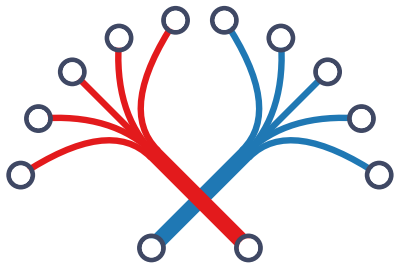
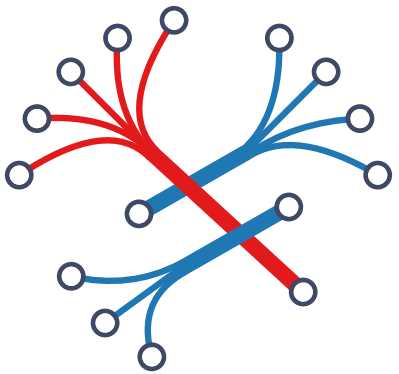
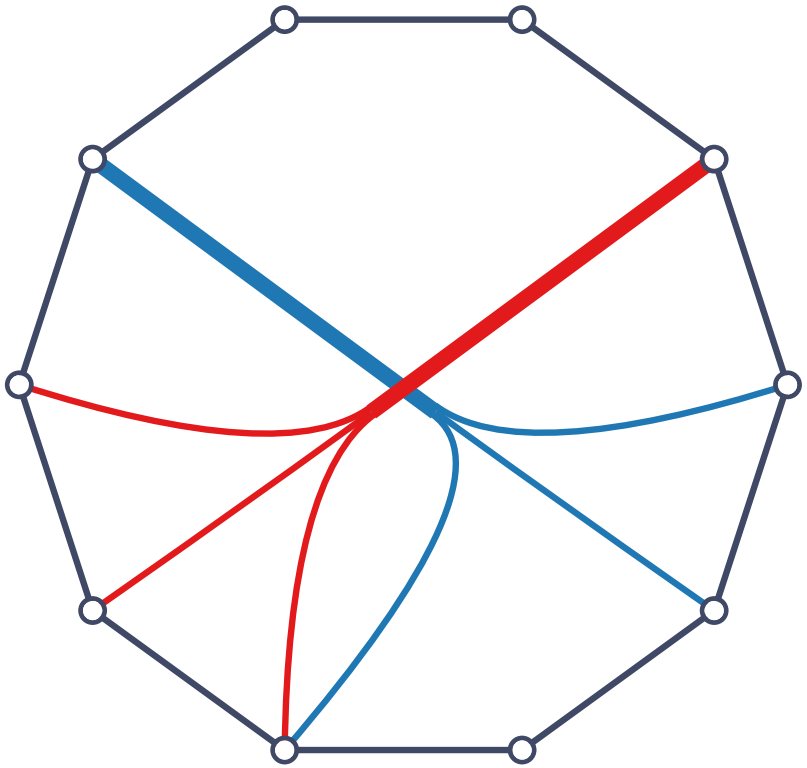


k crossings per bundle





k-Fan-Bundle-Planarity



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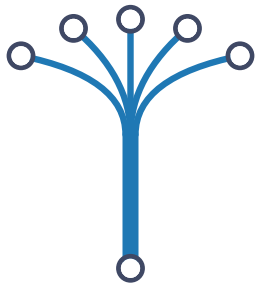

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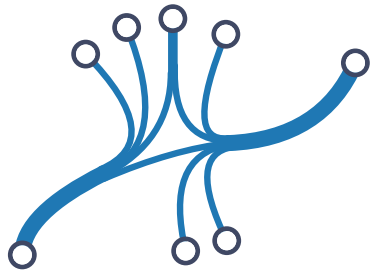
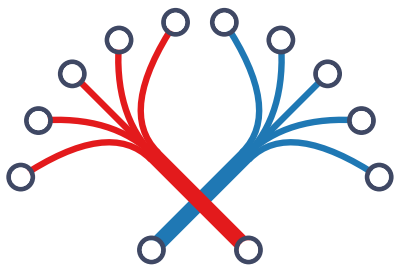
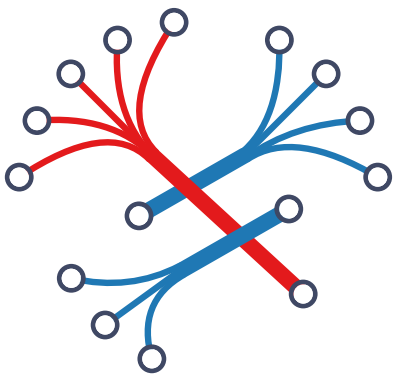


k crossings per bundle

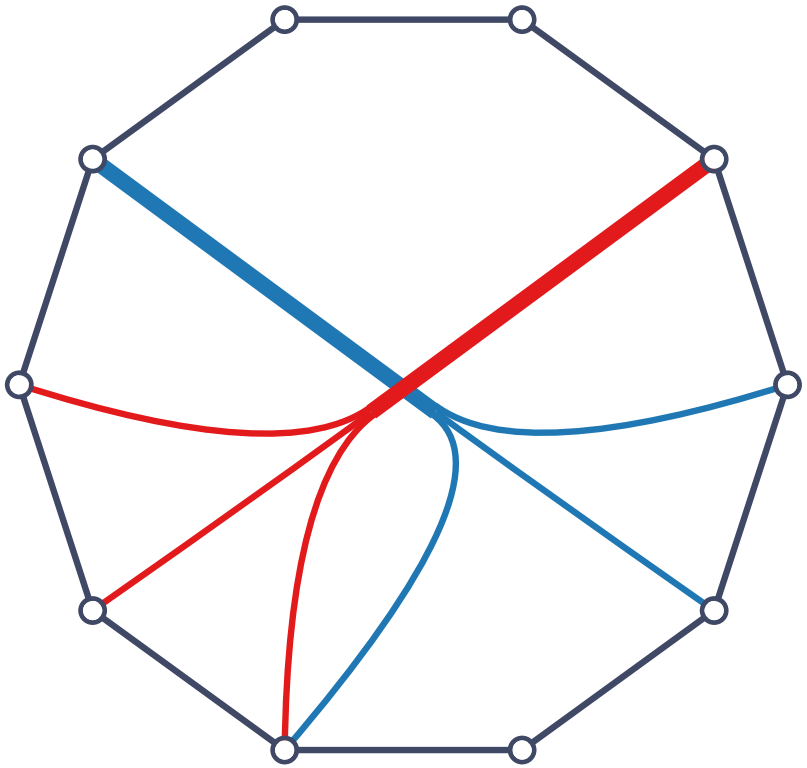




1-Fan-Bundle-Planarity

↳ Density

k-Fan-Bundle-Planarity



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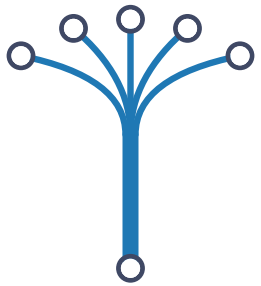

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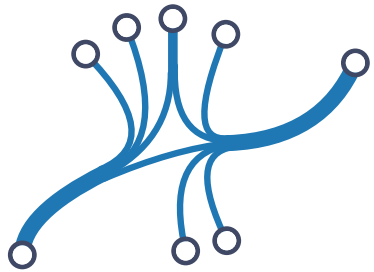
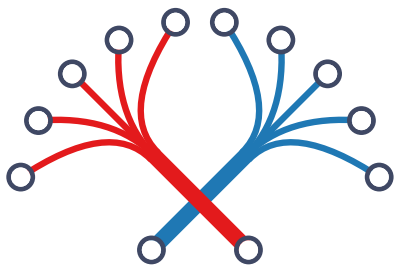
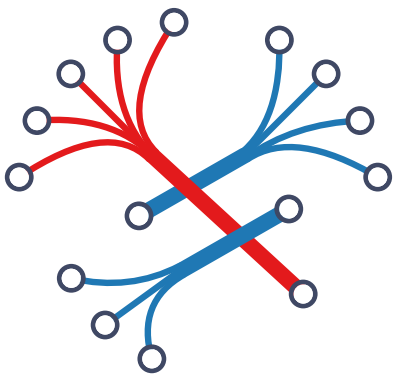


k crossings per bundle

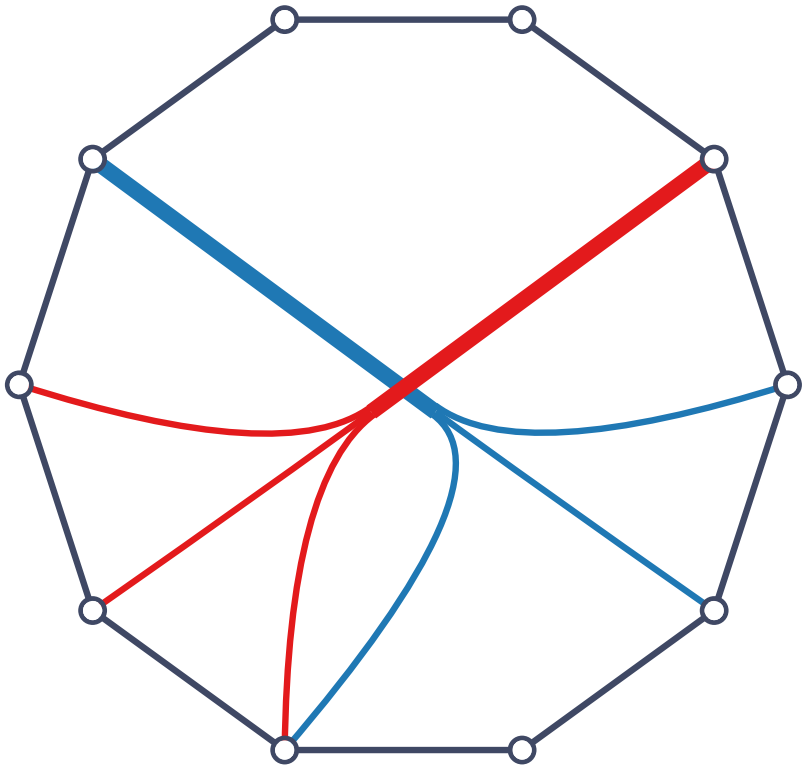




1-Fan-Bundle-Planarity

- Density
- Relationships

k-Fan-Bundle-Planarity



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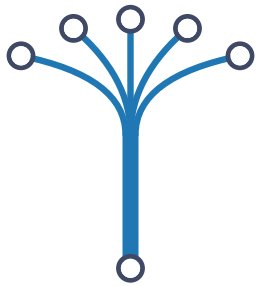

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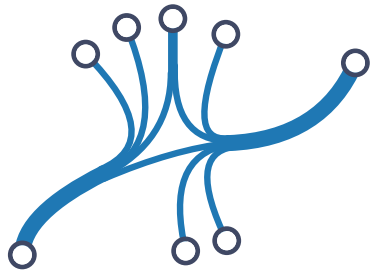

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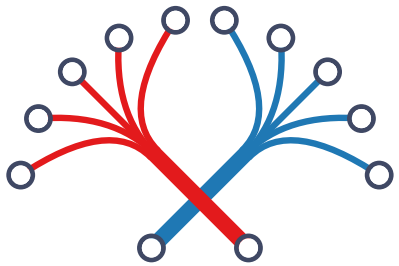
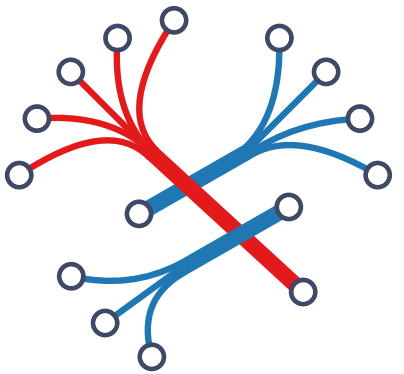




1-Fan-Bundle-Planarity

- Density
- Relationships
- Recognition

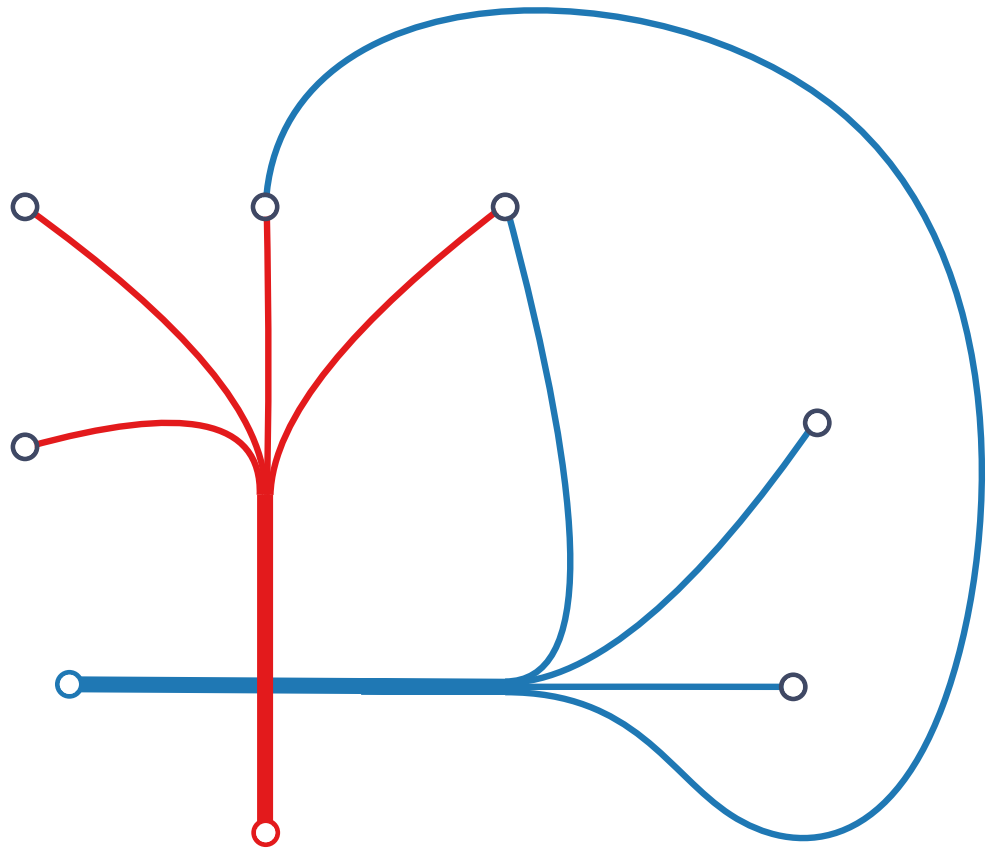
Density

Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar

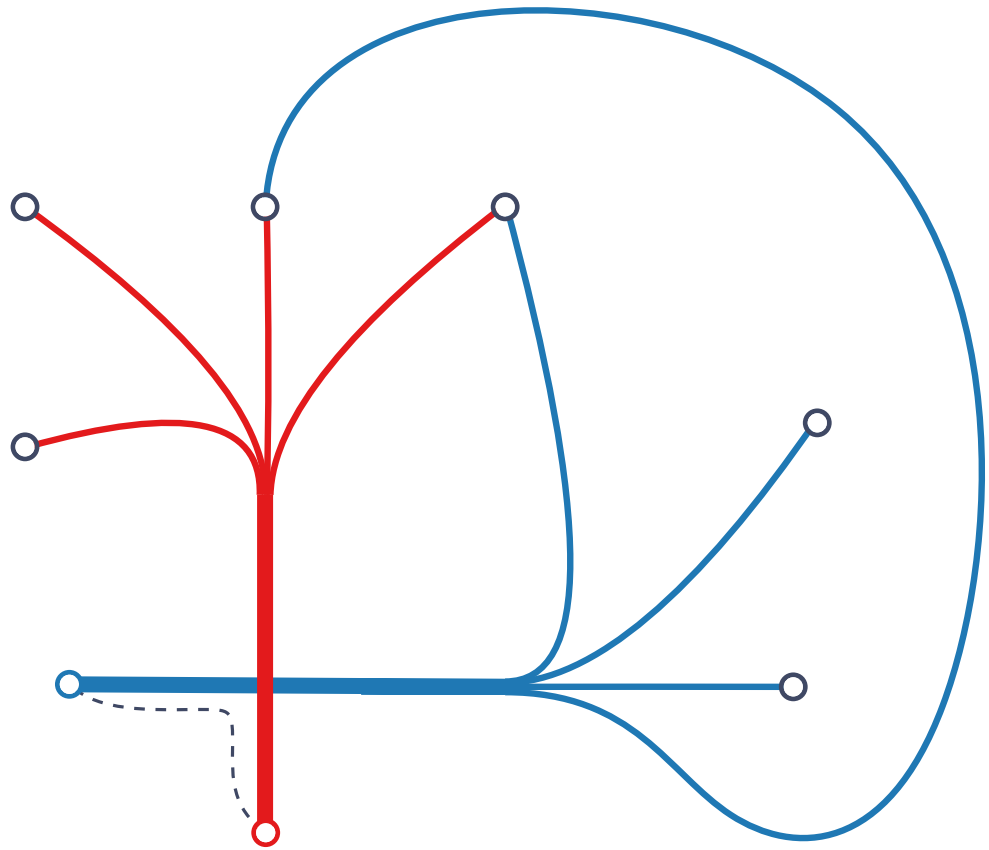
Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



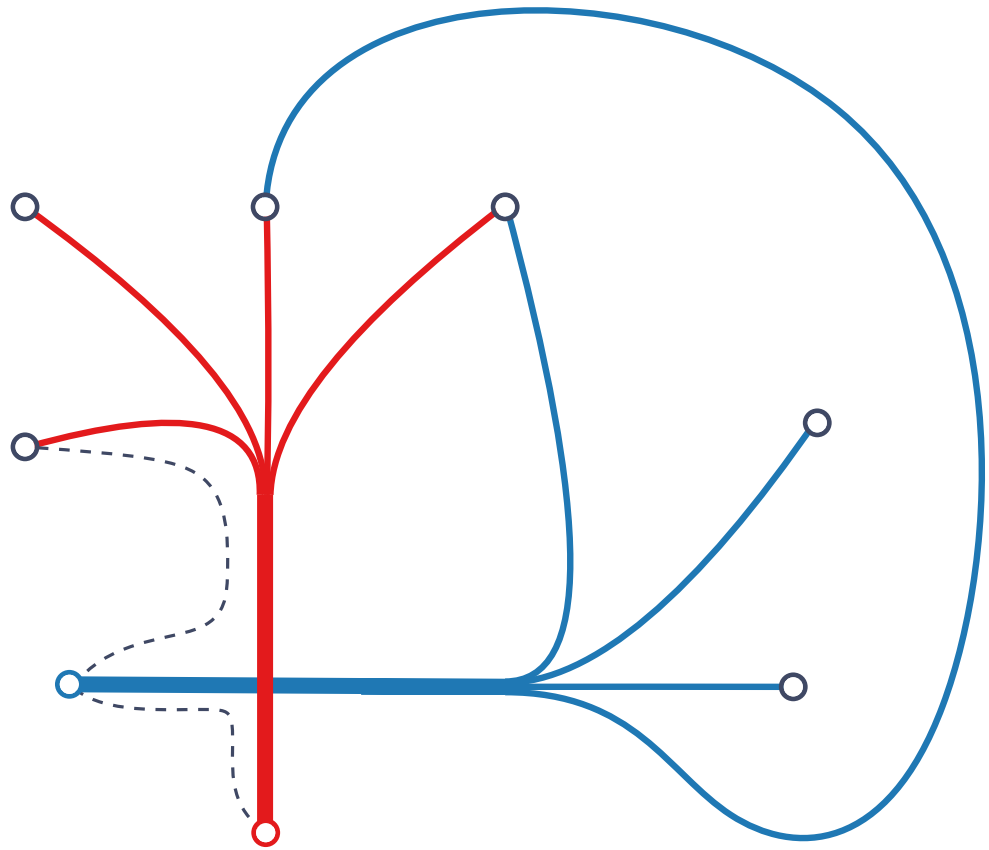
Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



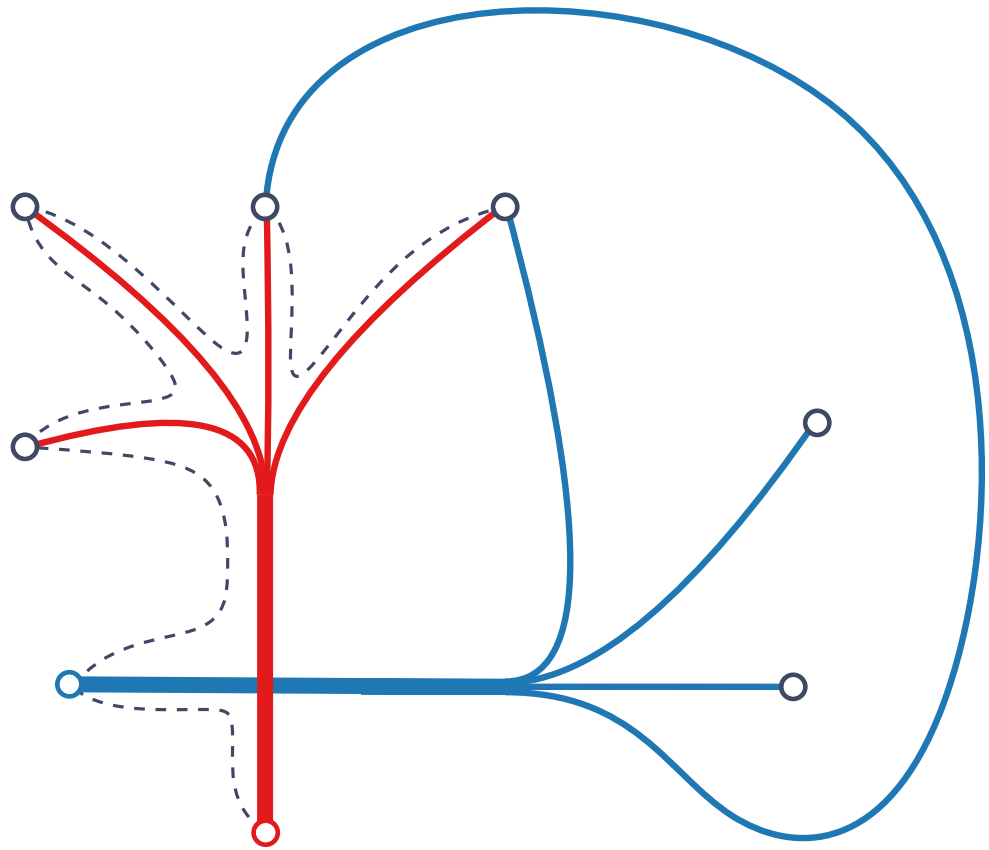
Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



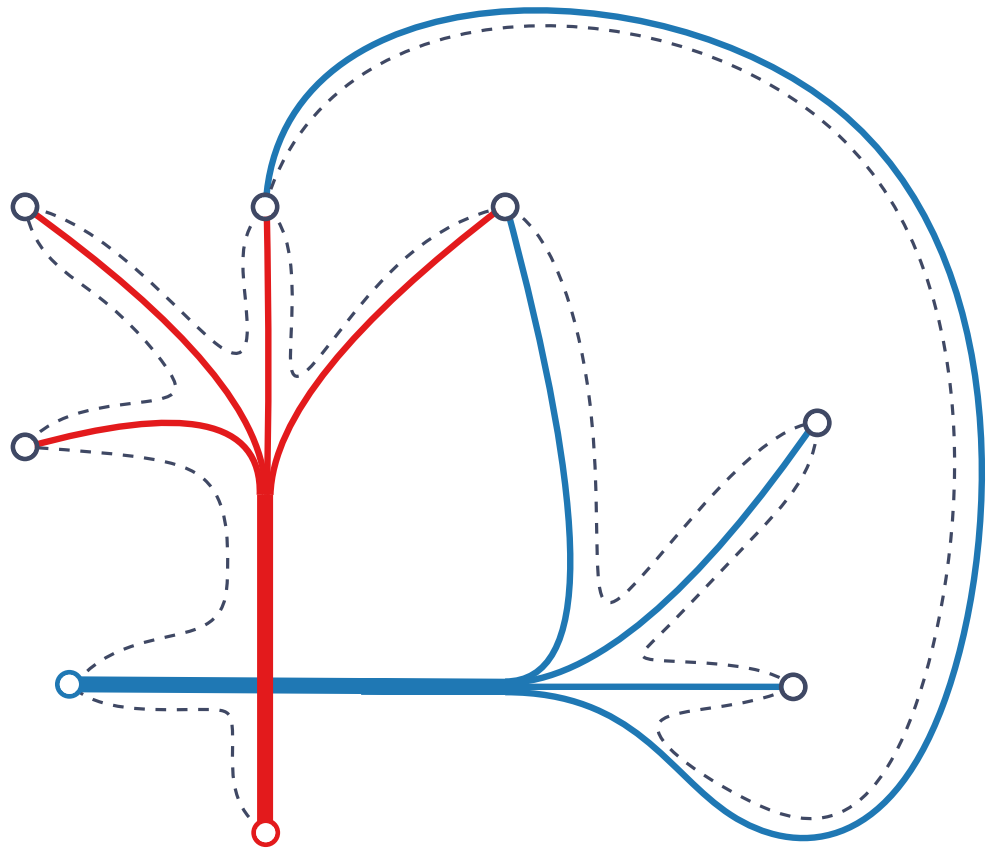
Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



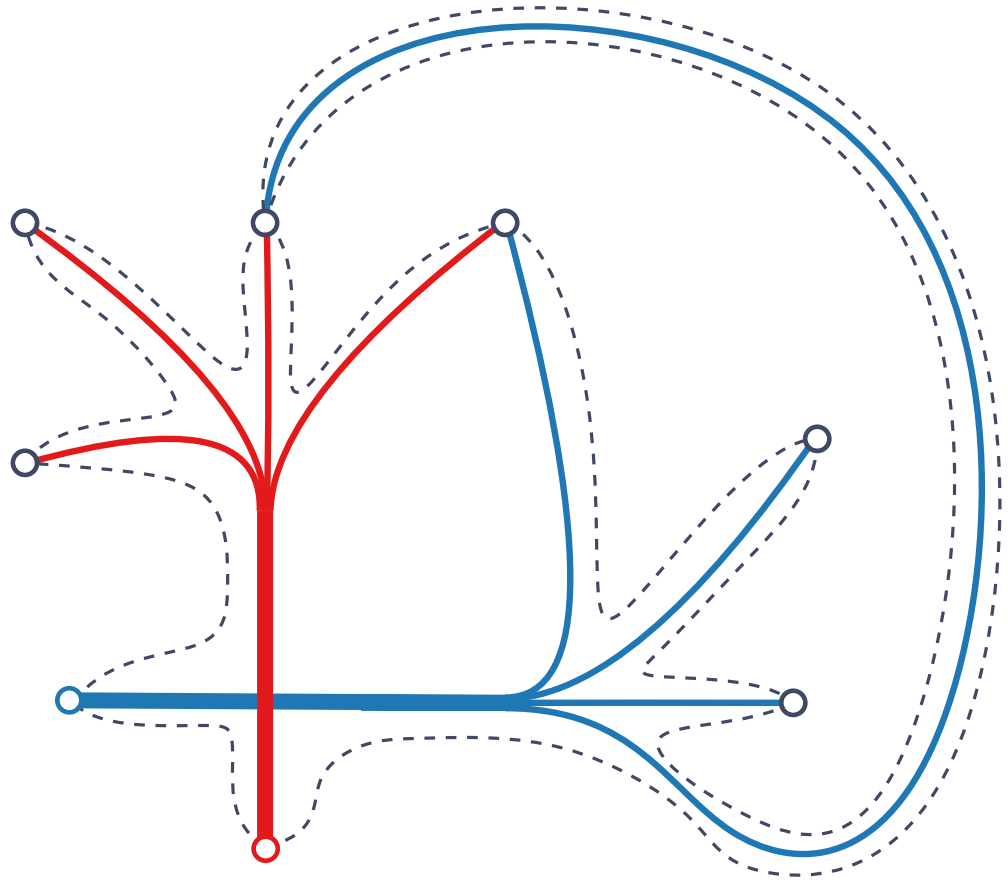
Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



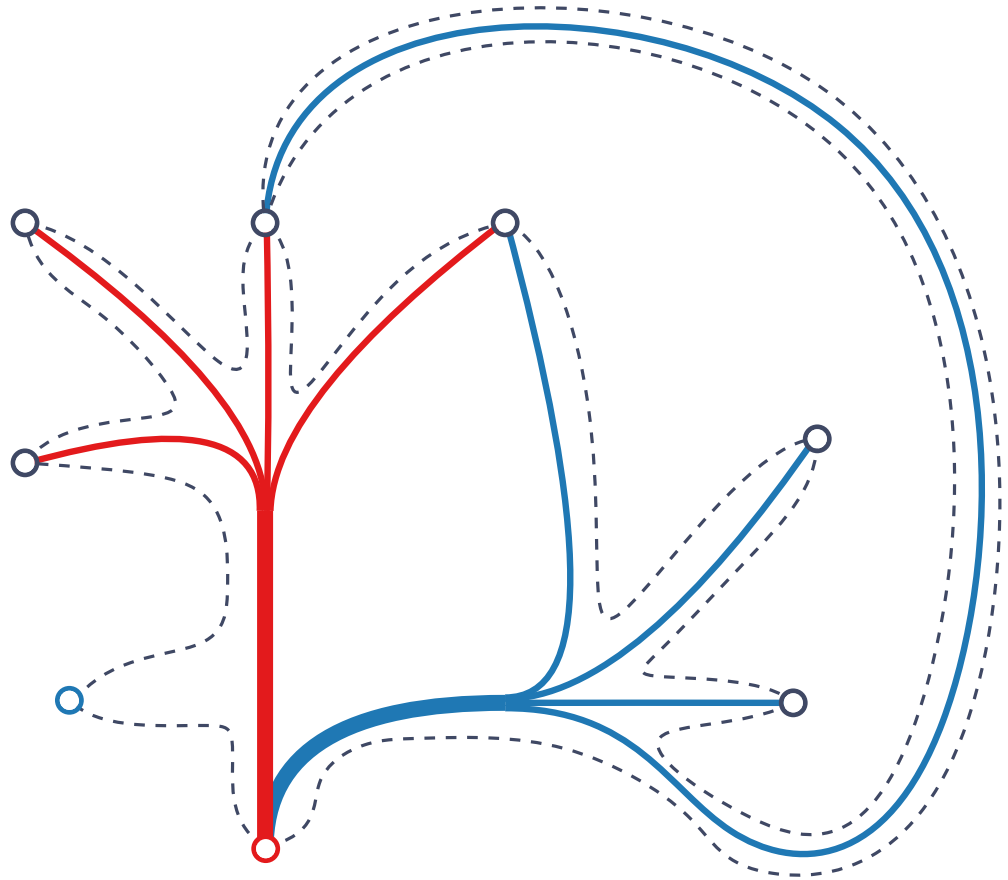
Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



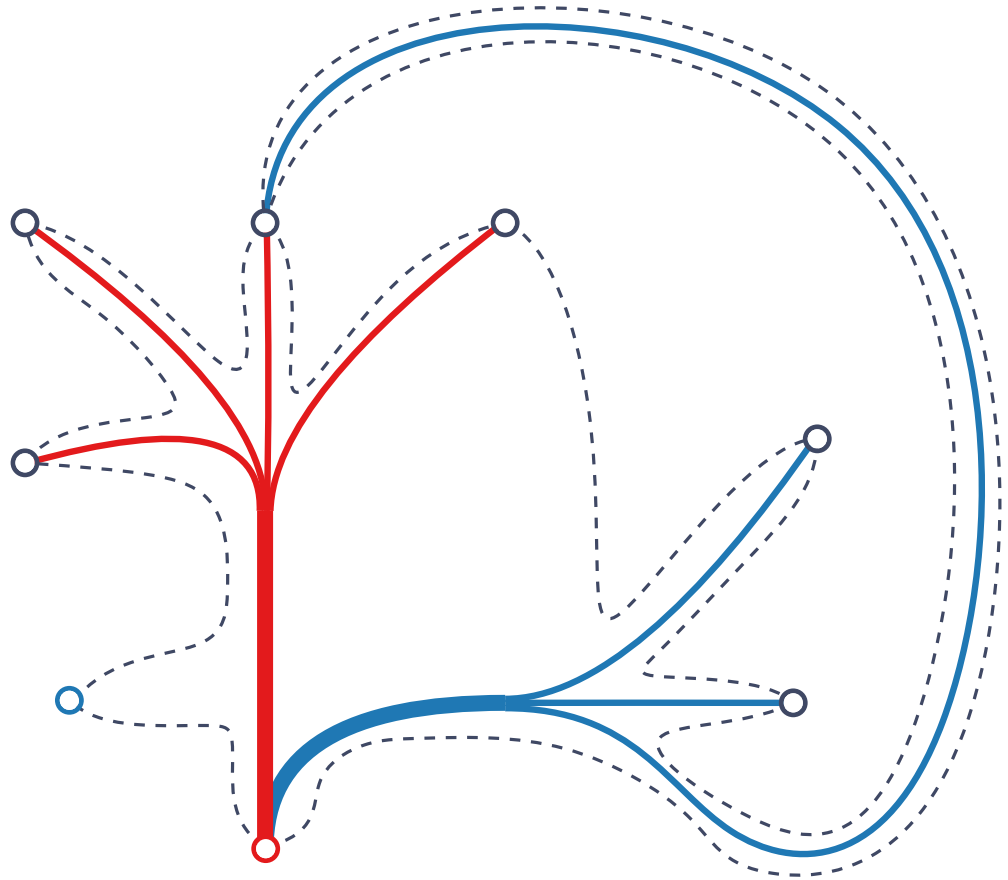
Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



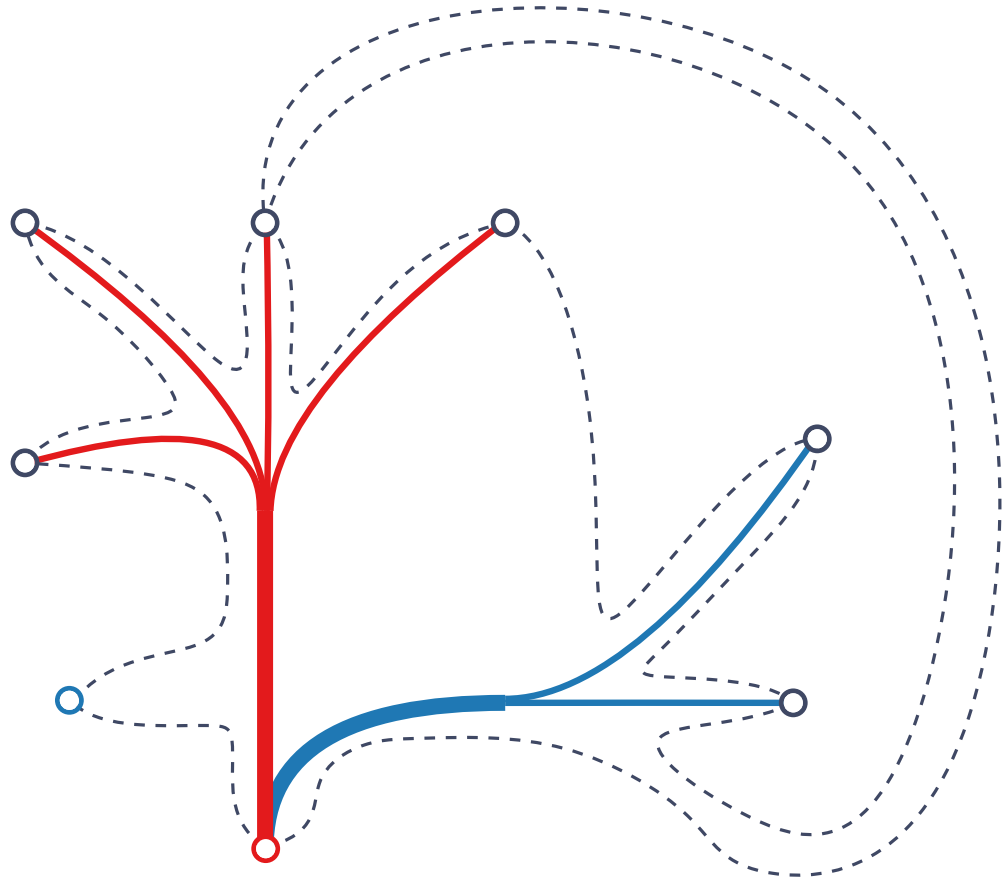
Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



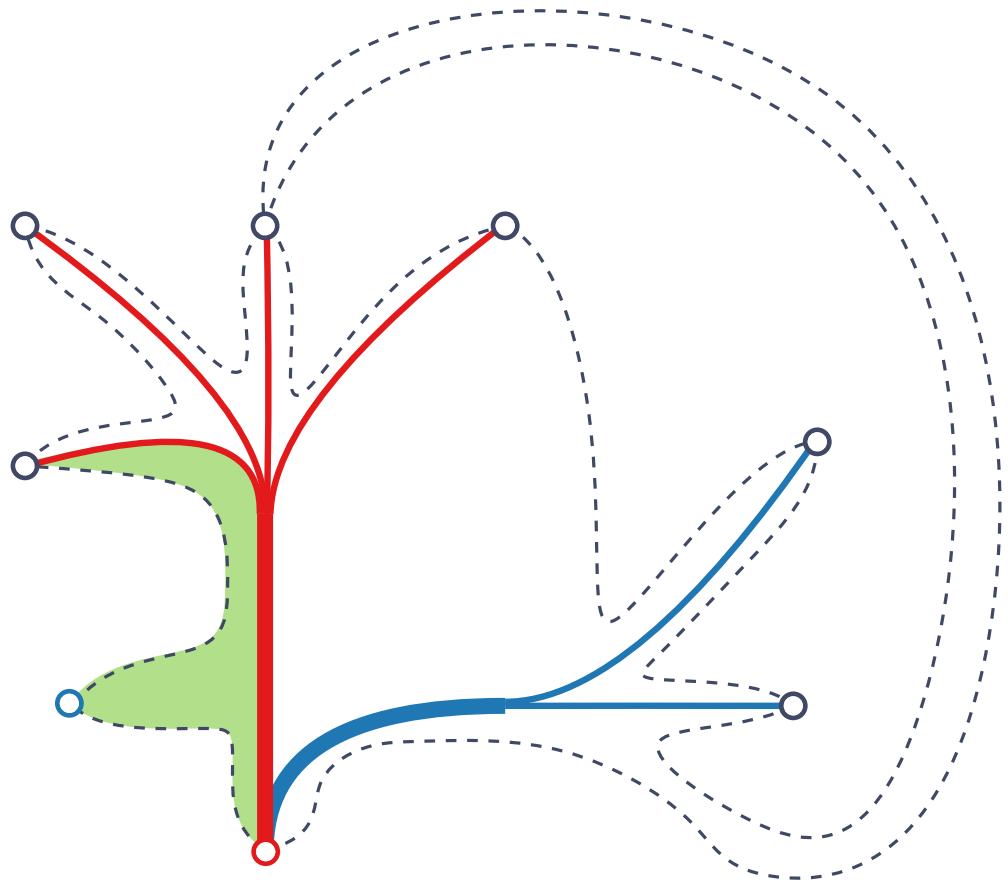
Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



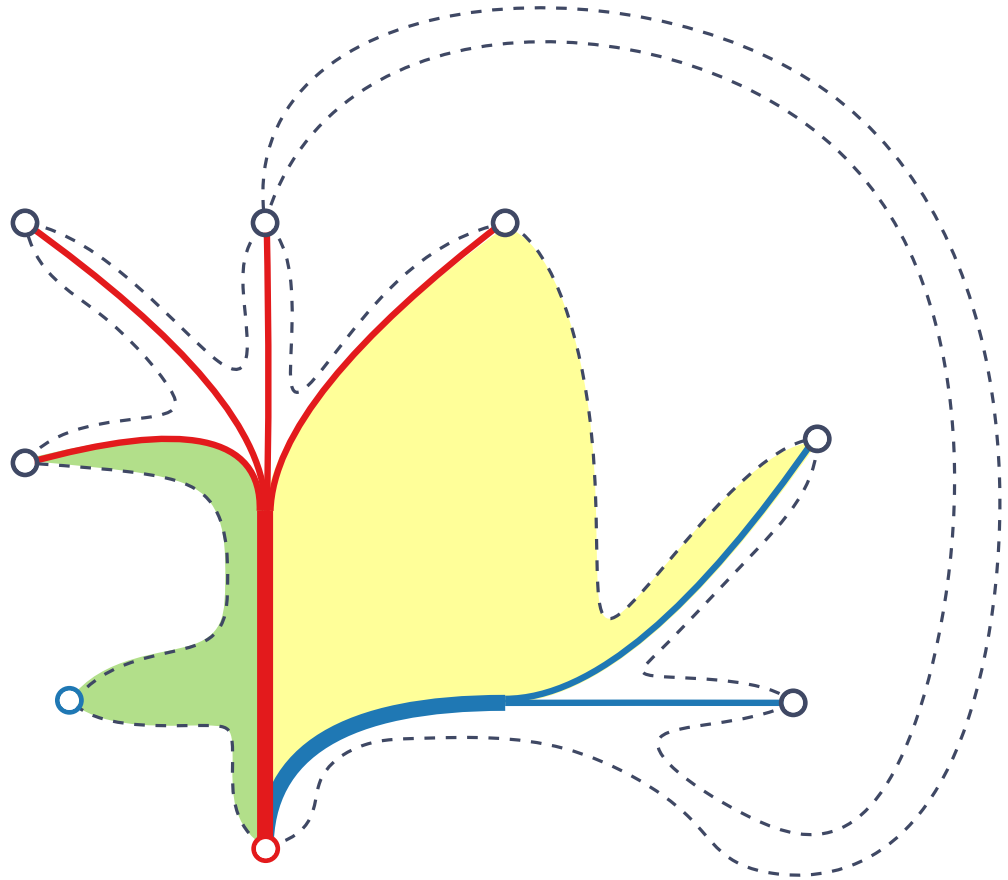
Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



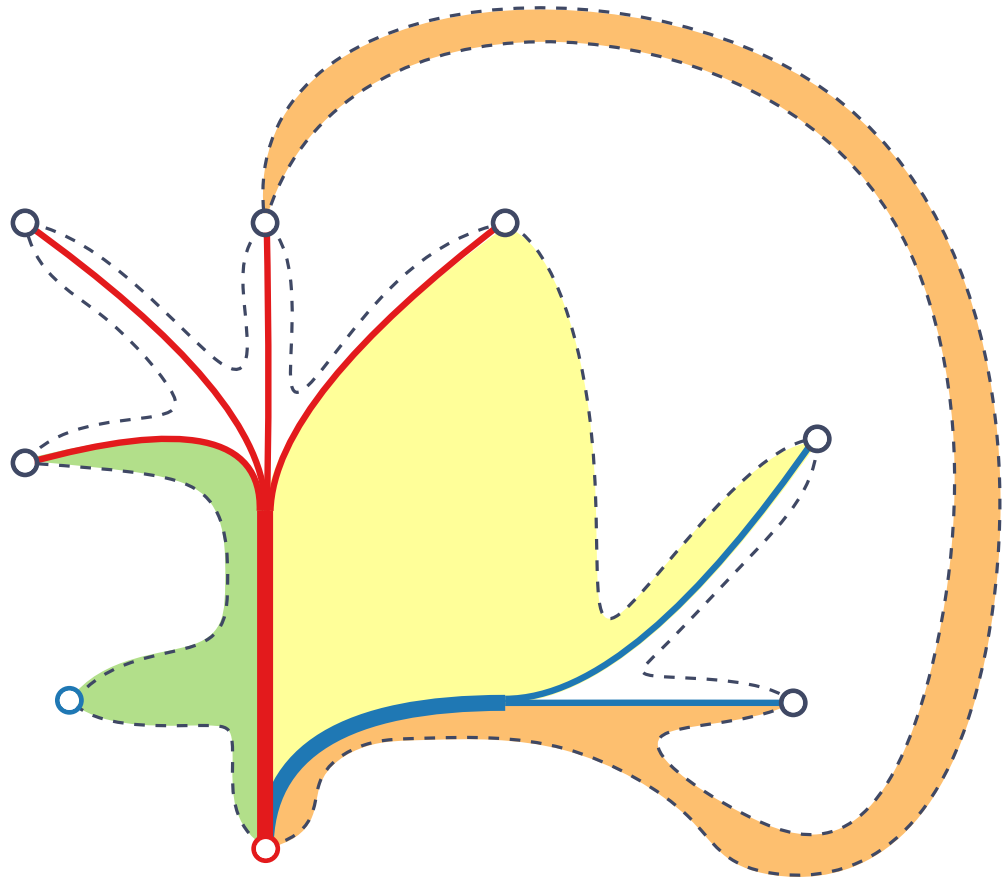
Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



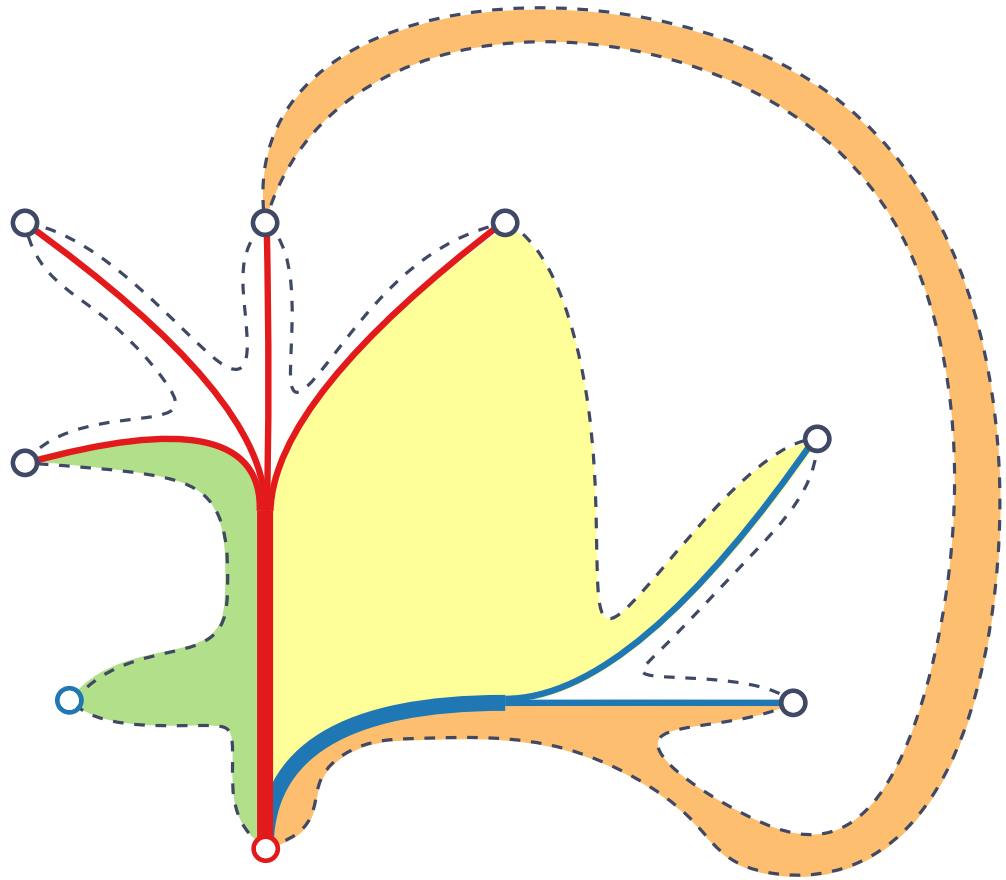
Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



Density: 1-sided

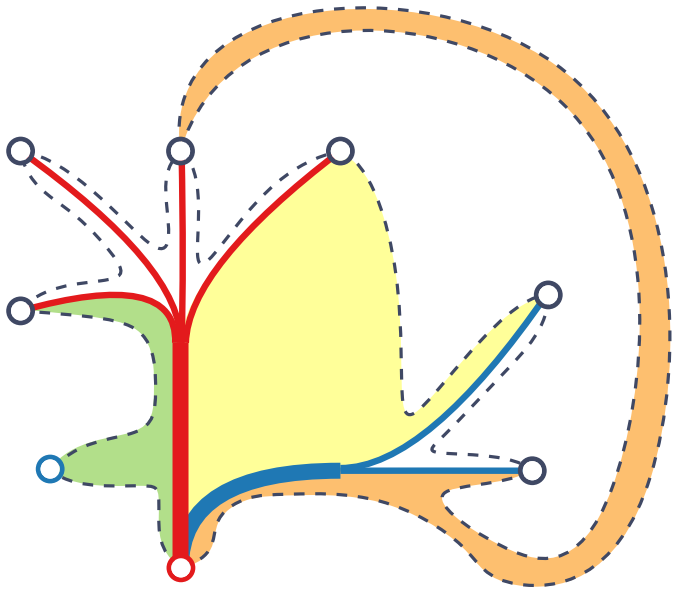
Upper bound: Take maximally dense graph G, make it planar



Remove 2 edges,
create 3 faces

Density: 1-sided

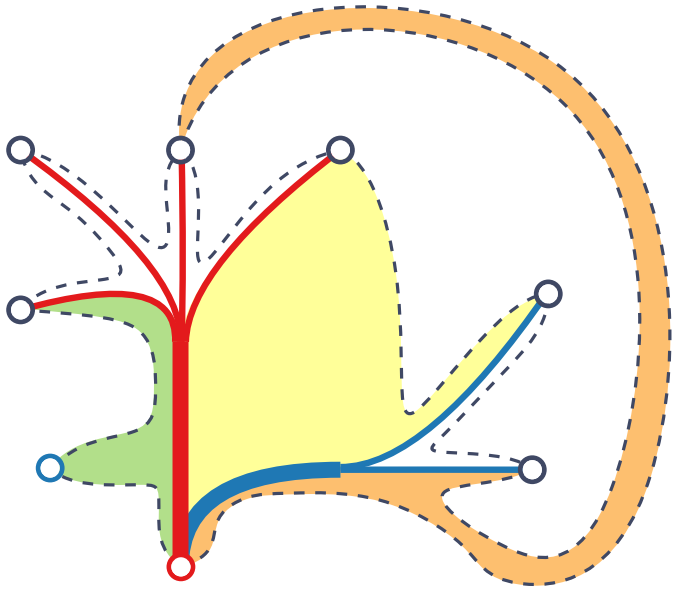
Upper bound: Take maximally dense graph G, make it planar

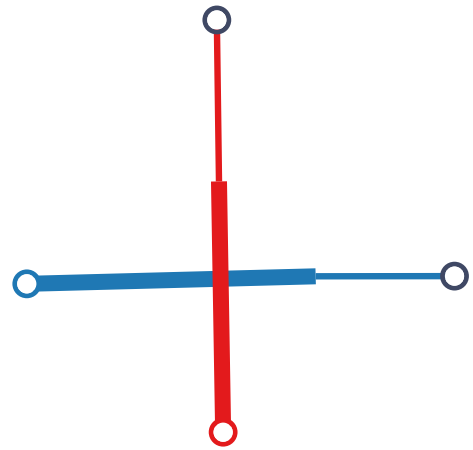


Remove 2 edges,
create 3 faces

Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar

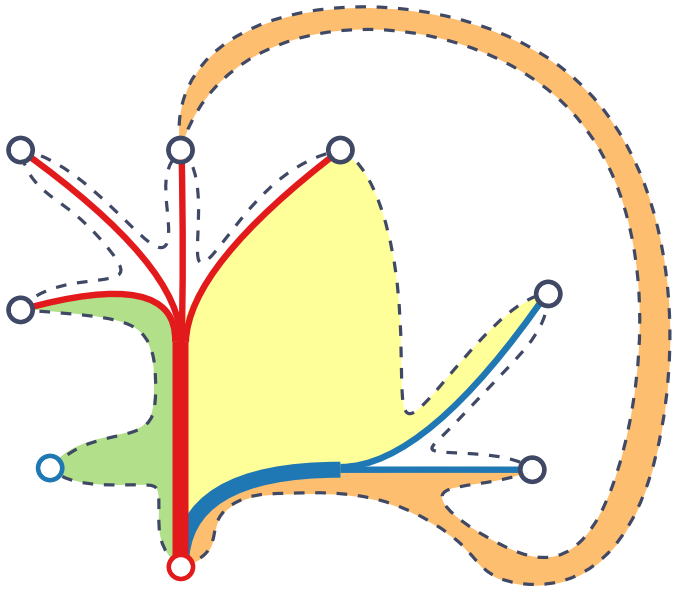


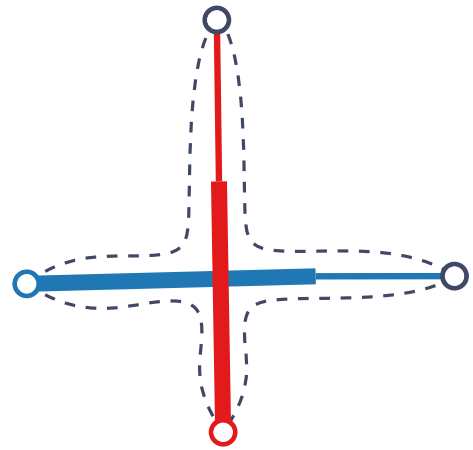


Remove 2 edges,
create 3 faces

Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar

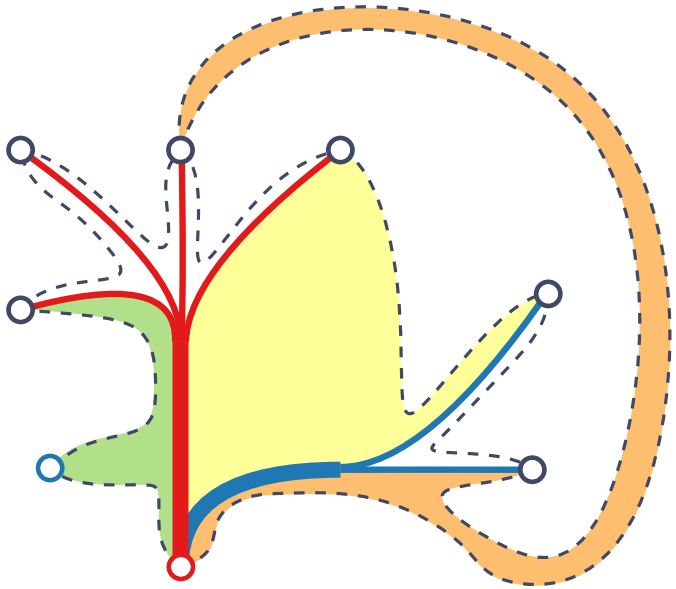


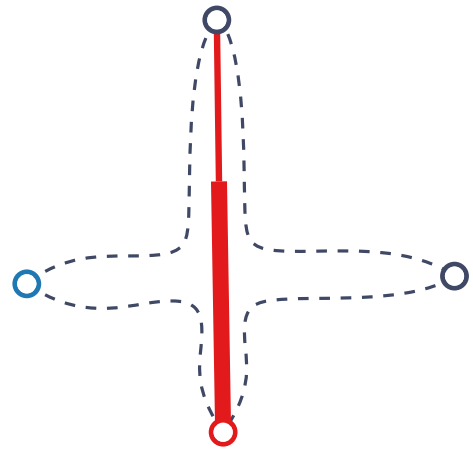


Remove 2 edges,
create 3 faces

Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar

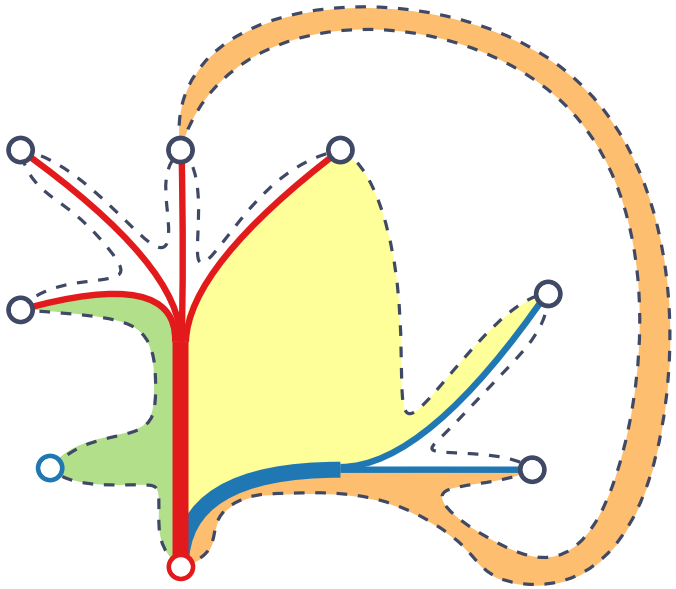


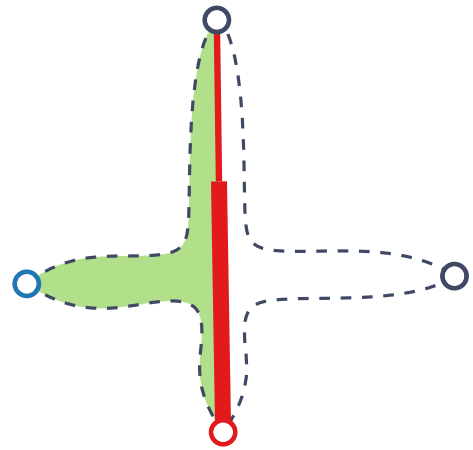


Remove 2 edges,
create 3 faces

Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar

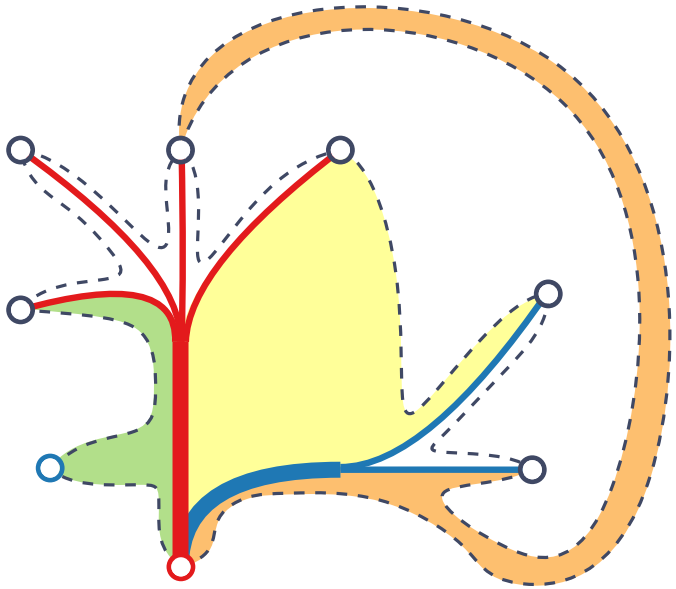


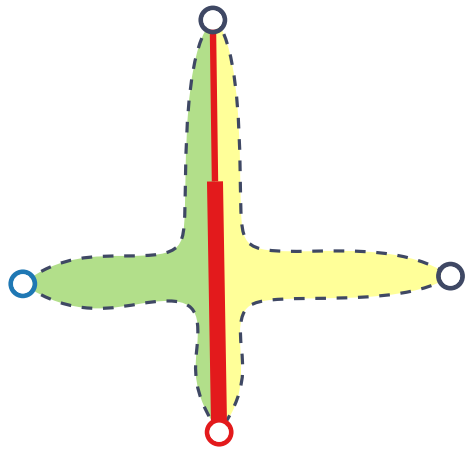


Remove 2 edges,
create 3 faces

Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar

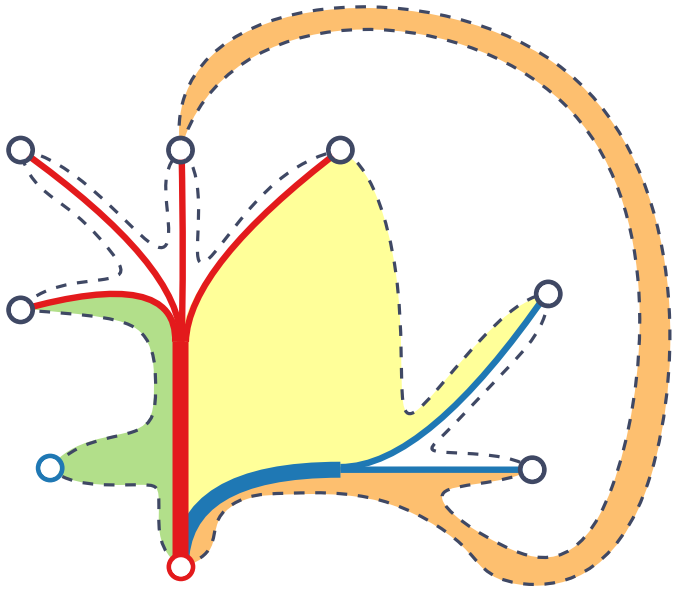




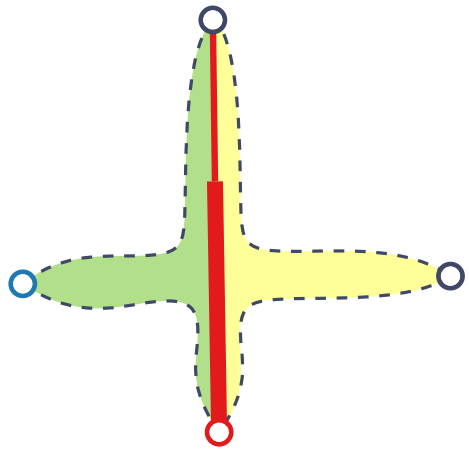
Remove 2 edges,
create 3 faces

Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



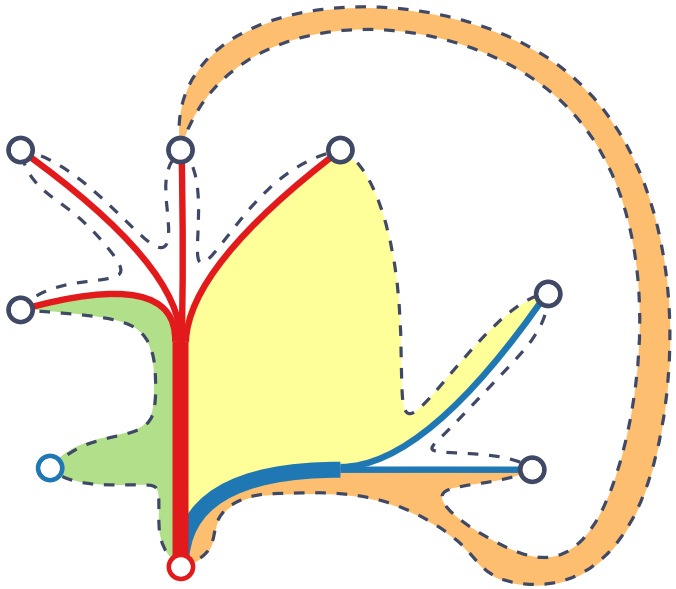
Remove 2 edges,
create 3 faces



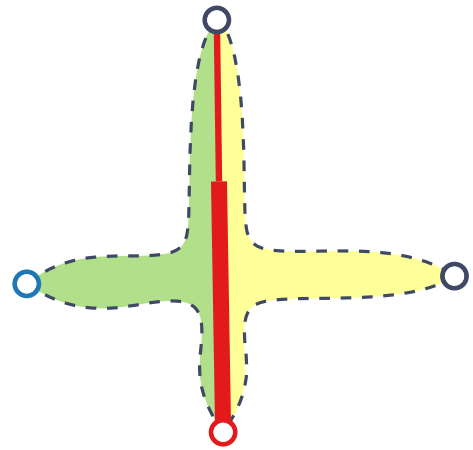
Remove 1 edge,
create 2 faces

Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



Remove 2 edges,
create 3 faces

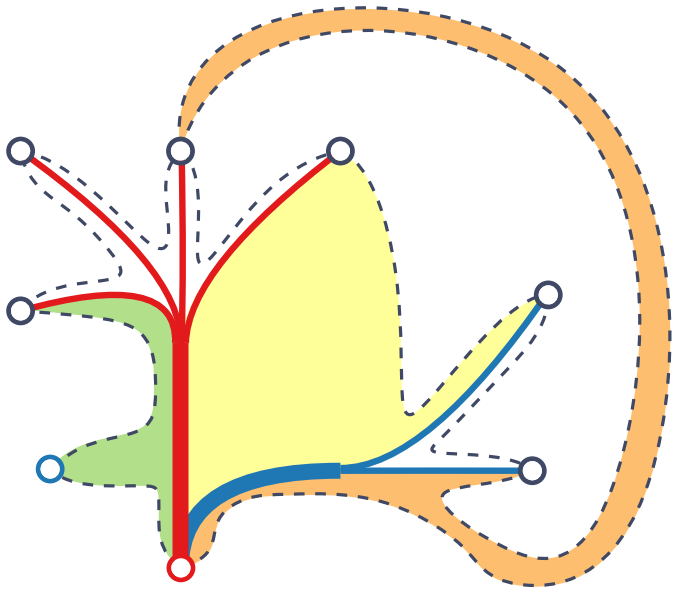


Remove 1 edge,
create 2 faces

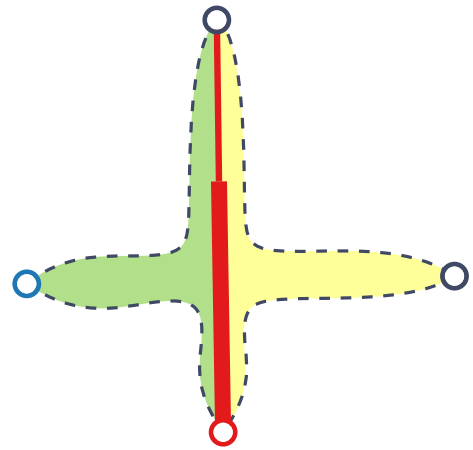
$\Rightarrow$ Planar graph G', $m' \leq 3n - 6$, $f' \leq 2n - 4$

Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



Remove 2 edges,
create 3 faces



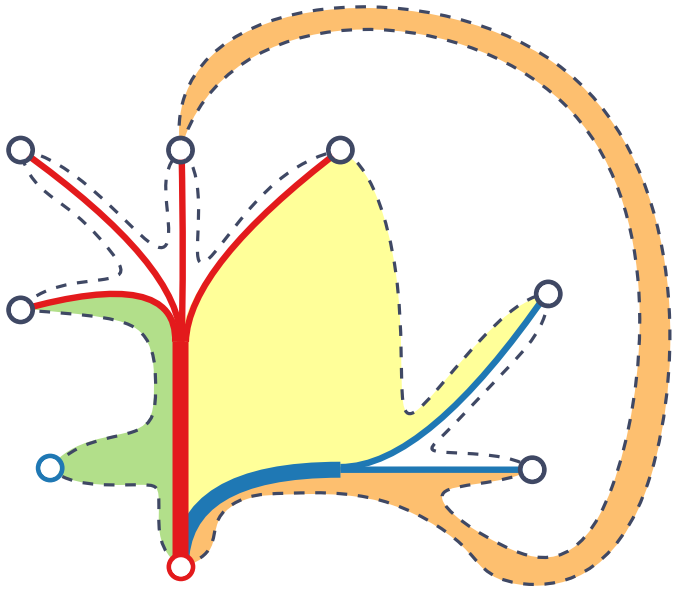
Remove 1 edge,
create 2 faces

$\Rightarrow$ Planar graph G', $m' \leq 3n - 6$, $f' \leq 2n - 4$

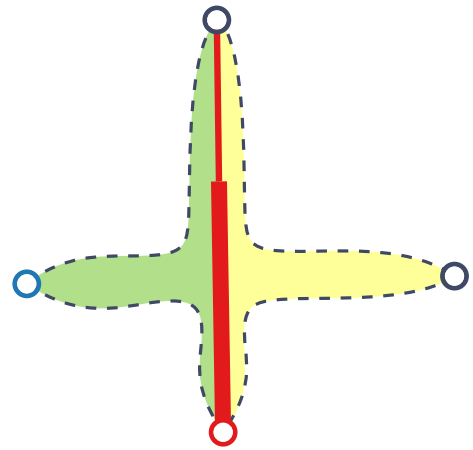
$\Rightarrow m \leq m' + 2f'/3$

Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



Remove 2 edges,
create 3 faces



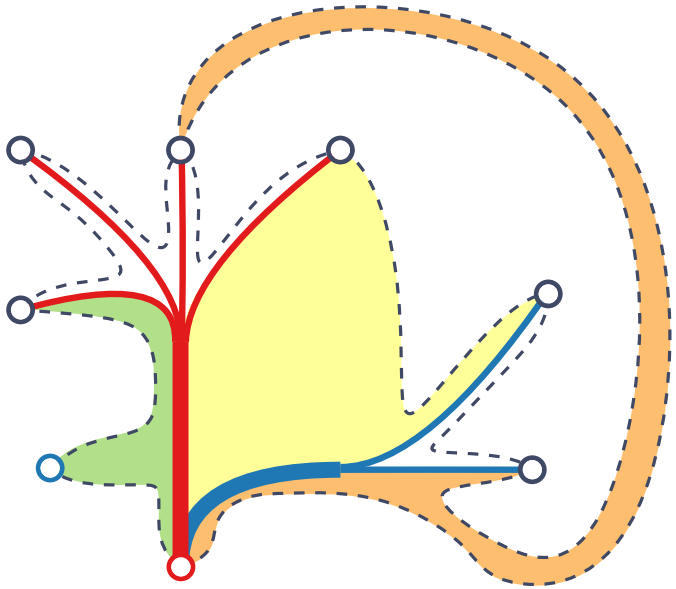
Remove 1 edge,
create 2 faces

$\Rightarrow$ Planar graph G', $m' \leq 3n-6$, $f' \leq 2n-4$

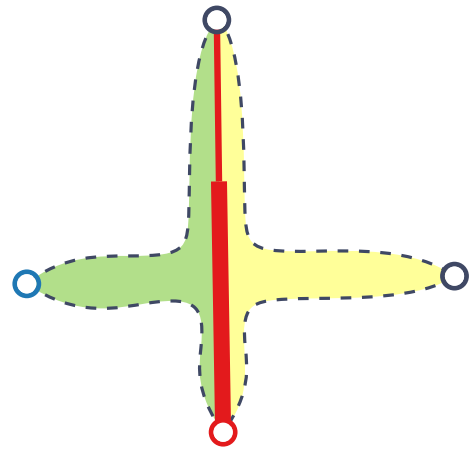
$\Rightarrow m \leq m' + 2f'/3 \leq 3n-6+2\cdot(2n-4)/3$

Density: 1-sided

Upper bound: Take maximally dense graph G, make it planar



Remove 2 edges,
create 3 faces



Remove 1 edge,
create 2 faces

$\Rightarrow$ Planar graph G', $m' \leq 3n-6$, $f' \leq 2n-4$

$\Rightarrow m \leq m' + 2f'/3 \leq 3n-6+2\cdot(2n-4)/3 \leq (13n-26)/3$

Density: 1-sided

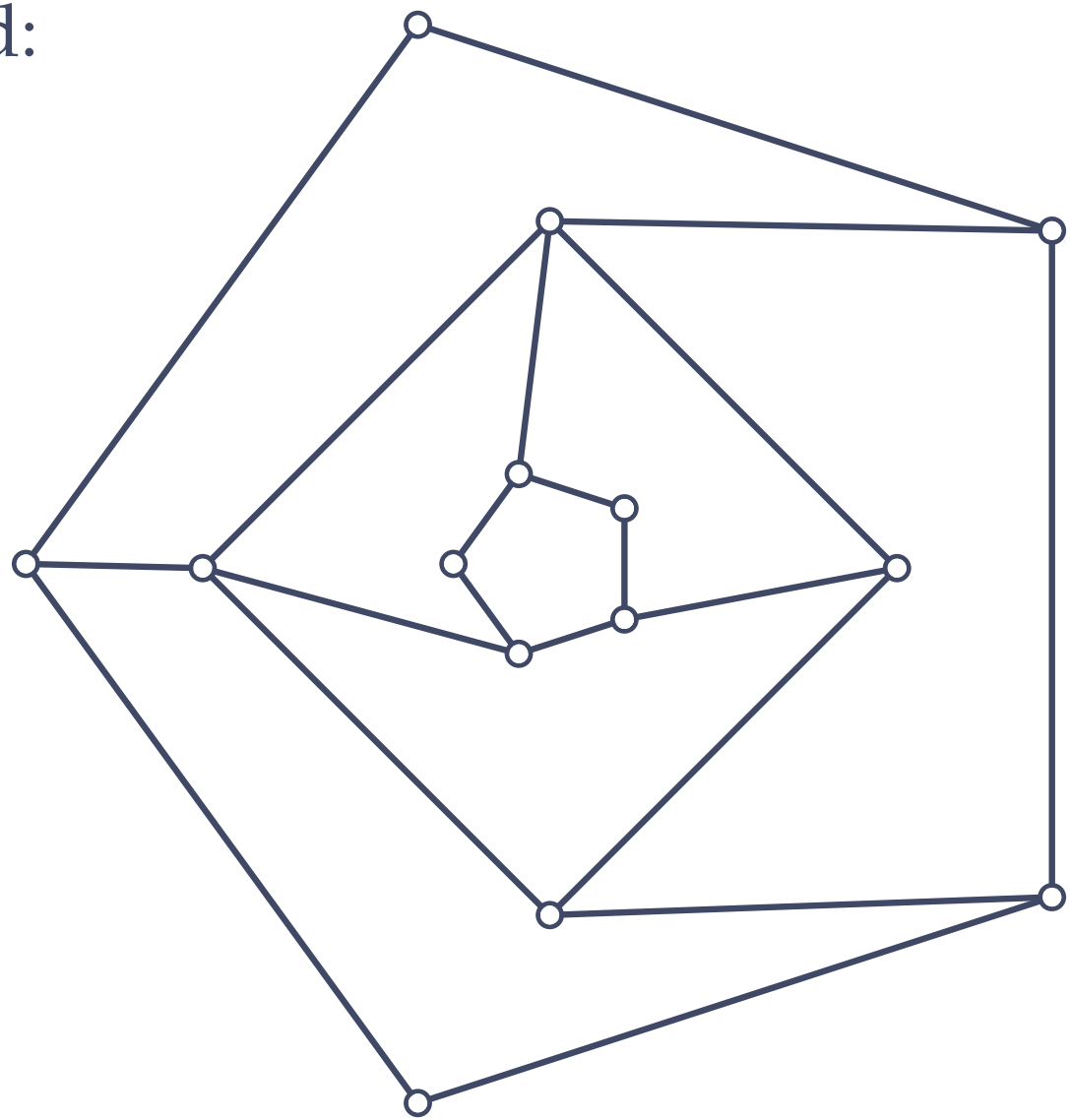
Upper bound: $(13n - 26)/3$

Lower bound:

Density: 1-sided

Upper bound: $(13n - 26)/3$

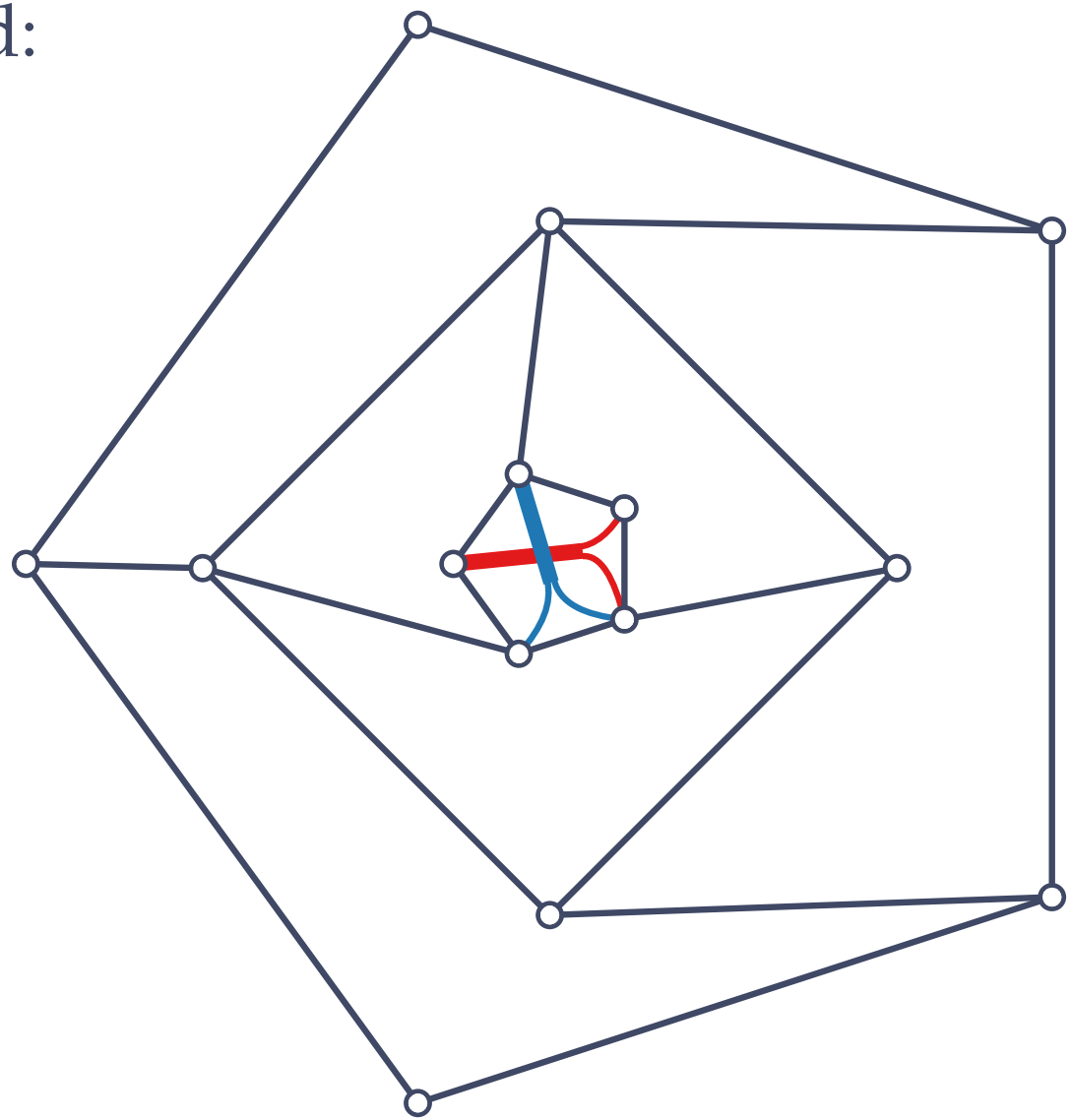
Lower bound:



Density: 1-sided

Upper bound: $(13n - 26)/3$

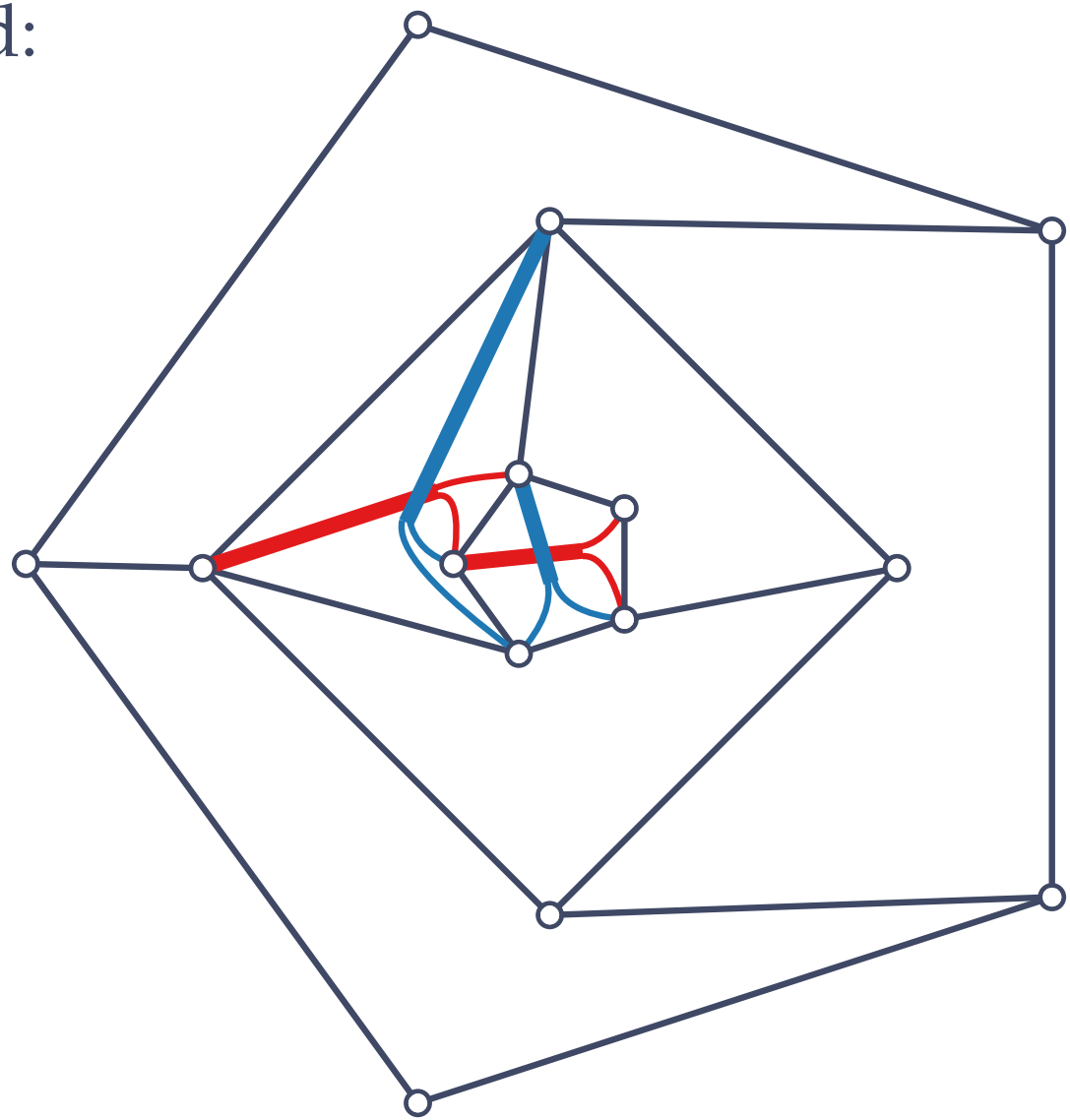
Lower bound:



Density: 1-sided

Upper bound: $(13n - 26)/3$

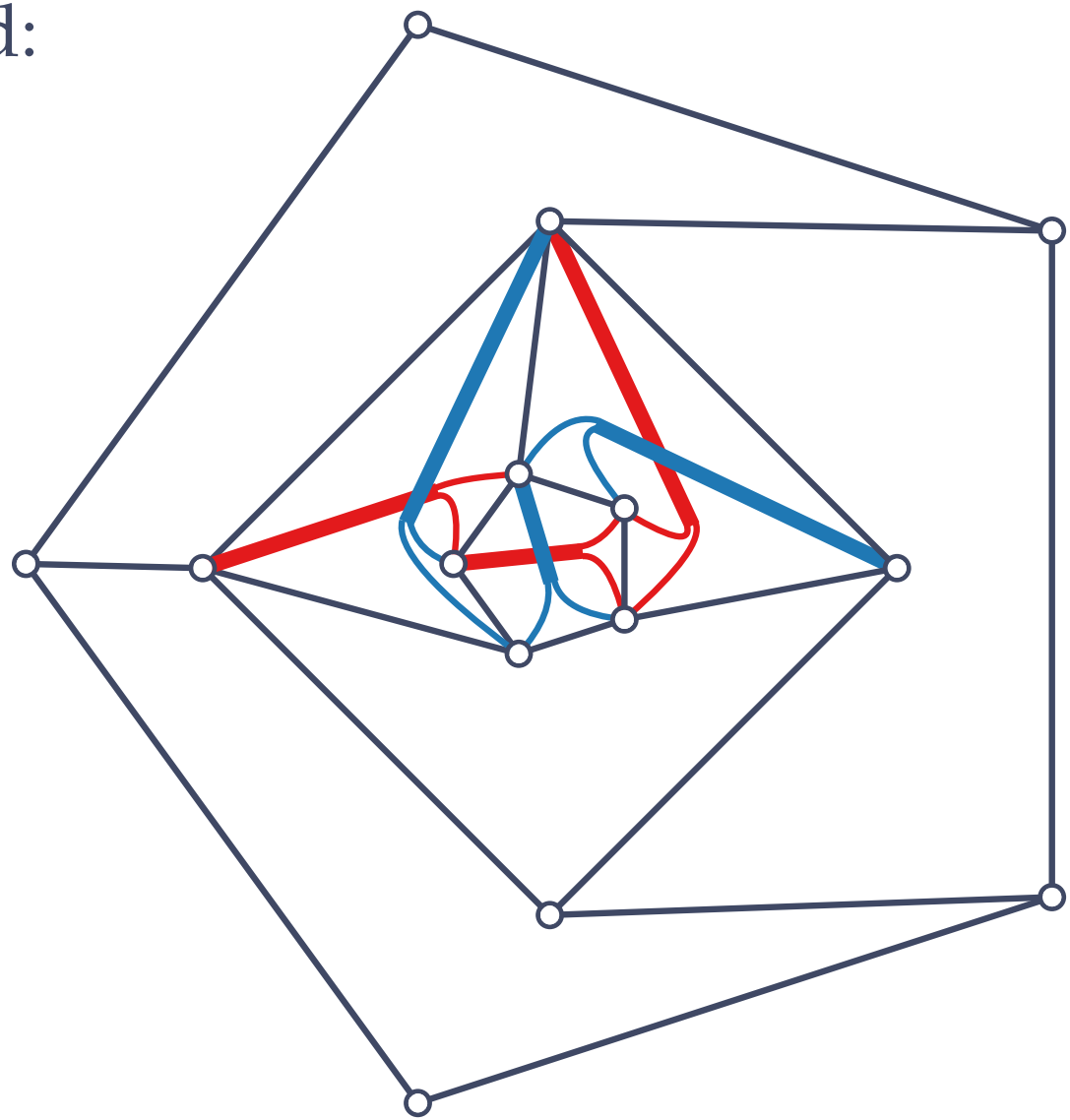
Lower bound:



Density: 1-sided

Upper bound: $(13n - 26)/3$

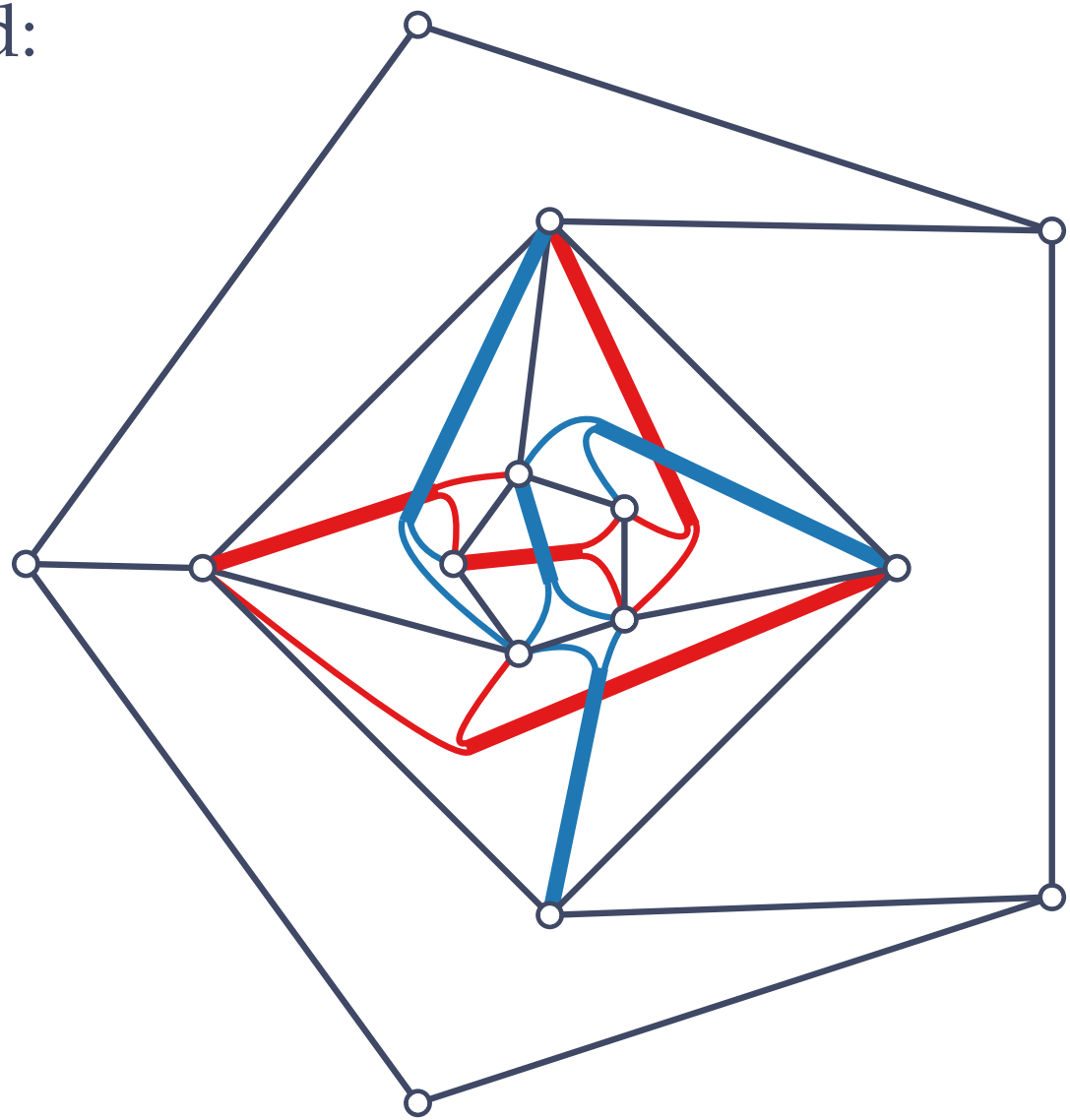
Lower bound:



Density: 1-sided

Upper bound: $(13n - 26)/3$

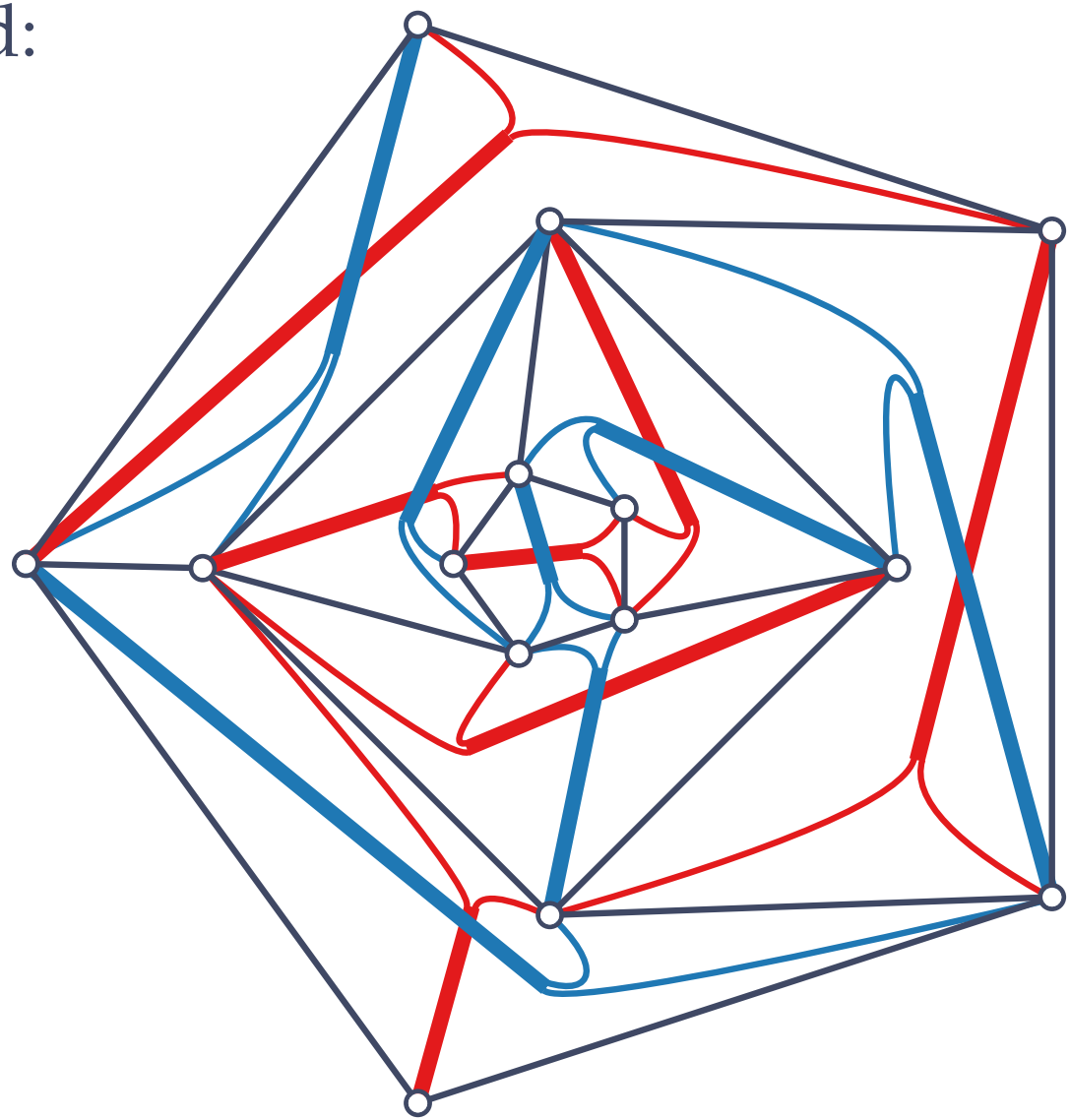
Lower bound:



Density: 1-sided

Upper bound: $(13n - 26)/3$

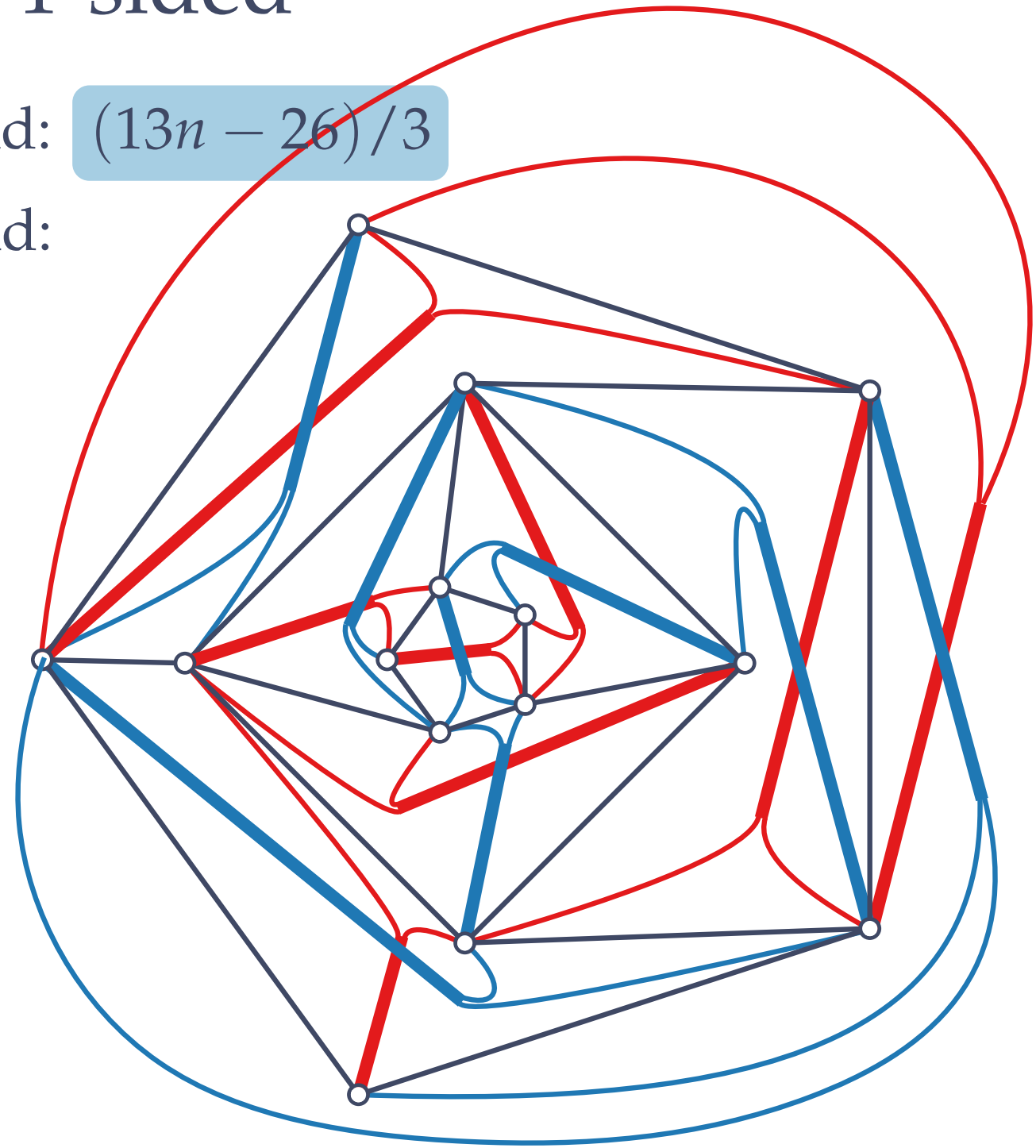
Lower bound:



Density: 1-sided

Upper bound: $(13n - 26)/3$

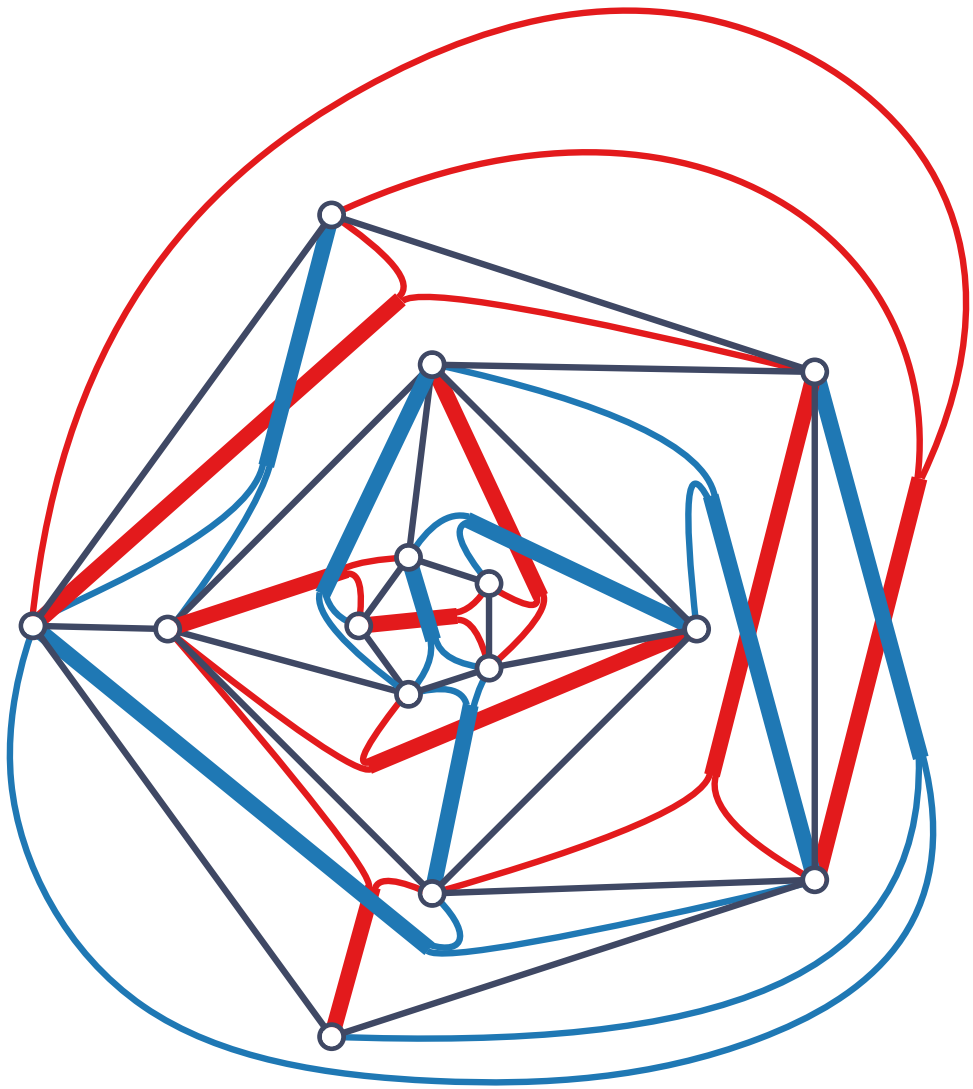
Lower bound:



Density: 1-sided

Upper bound: $(13n - 26)/3$

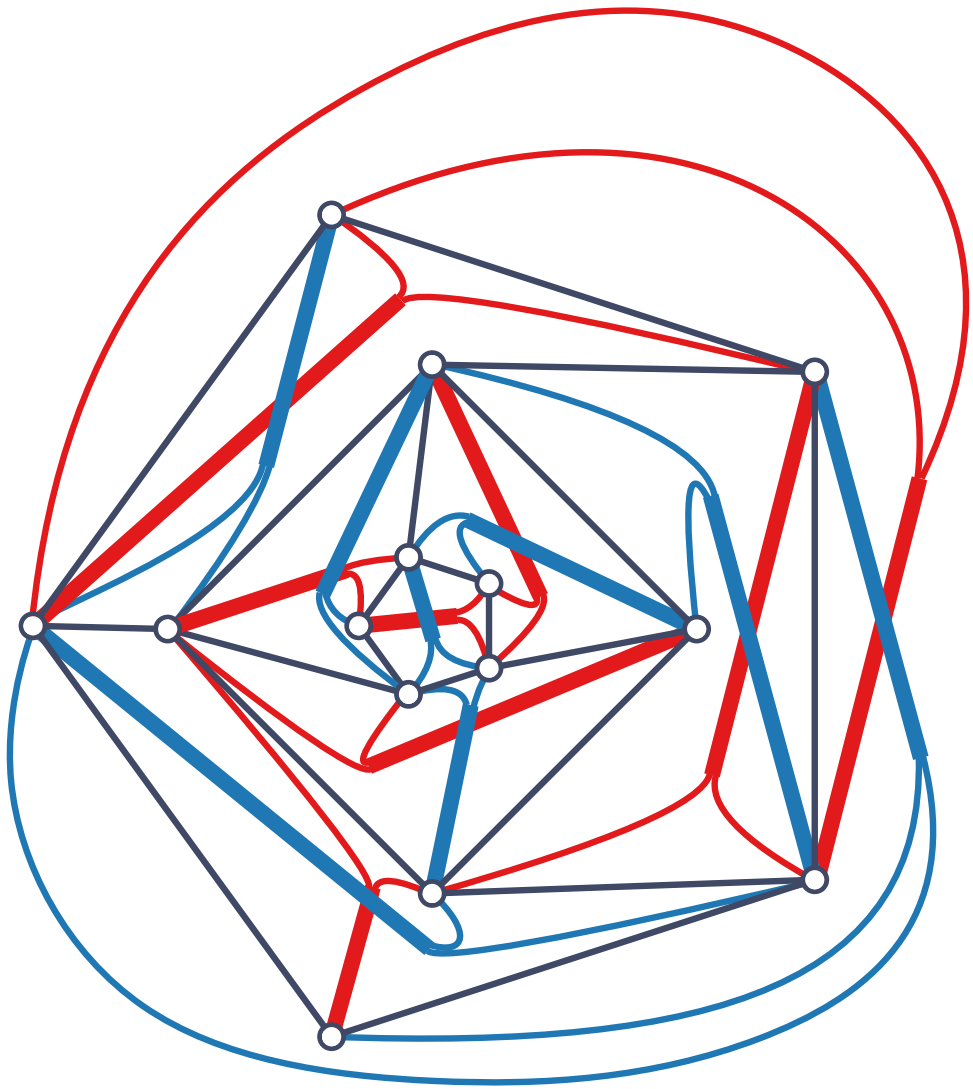
Lower bound: $(5n - 10)/3$



Density: 1-sided

Upper bound: $(13n - 26)/3$

Lower bound: $(5n - 10)/3 + 4 \cdot (2n - 4)/3$

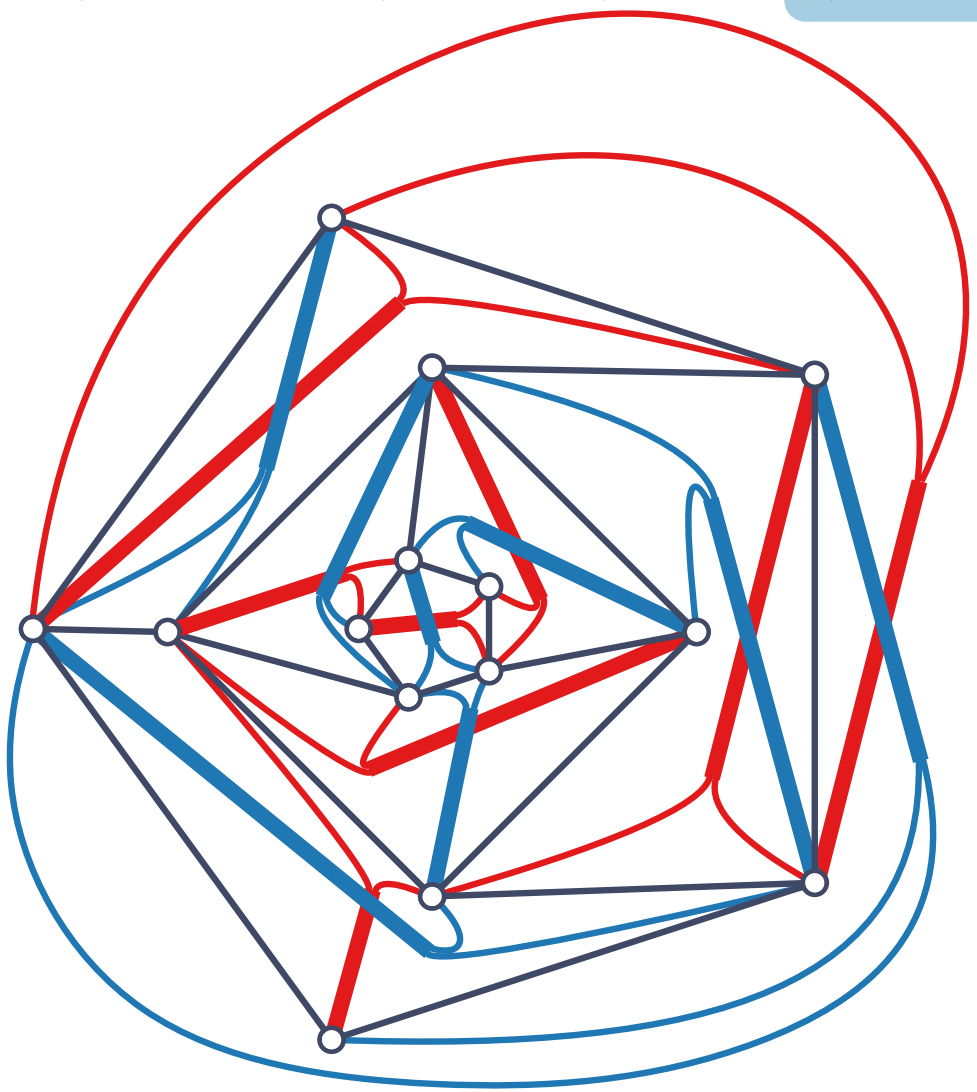


Density: 1-sided

Tight

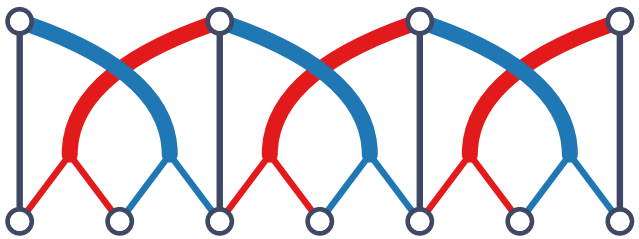
Upper bound: $(13n-26)/3$

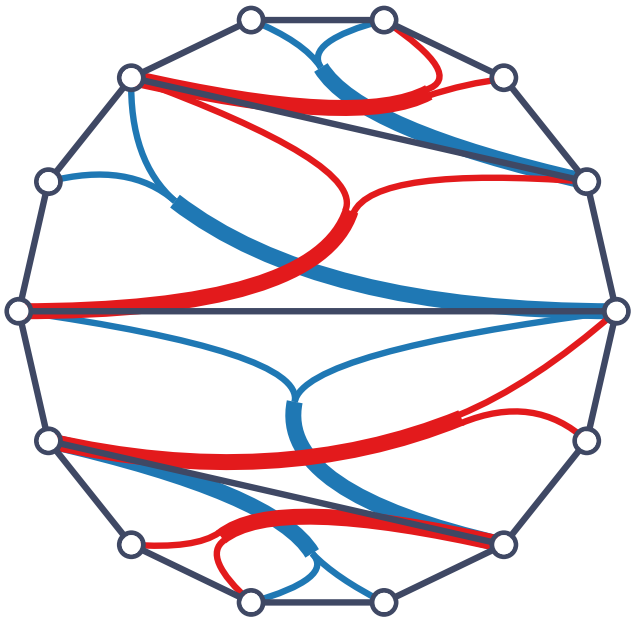
Lower bound: $(5n-10)/3 + 4 \cdot (2n-4)/3 = (13n-26)/3$

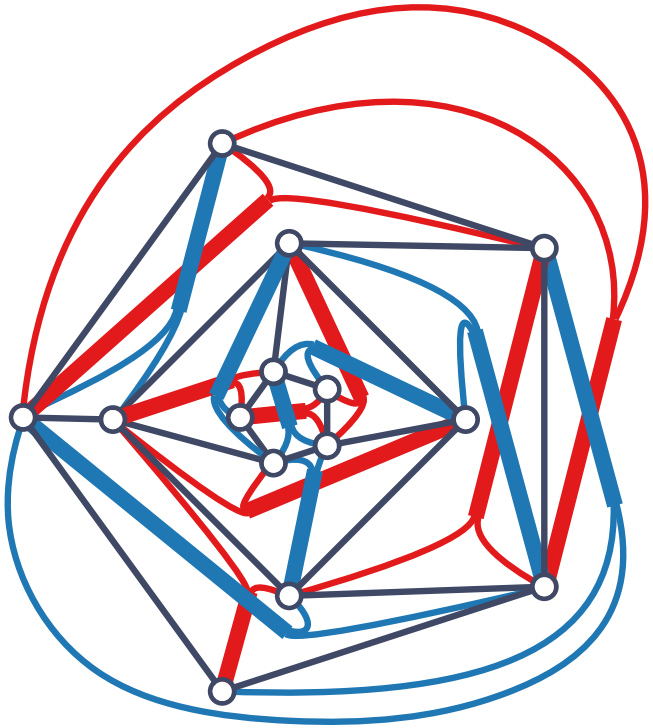


Density: 1-sided

	2-layer		outer		general	
	LB	UB	LB	UB	LB	UB
1-sided	$\frac{5n-7}{3}$	$\frac{5n-7}{3}$	$\frac{8n-13}{3}$	$\frac{8n-13}{3}$	$\frac{13n-26}{3}$	$\frac{13n-26}{3}$







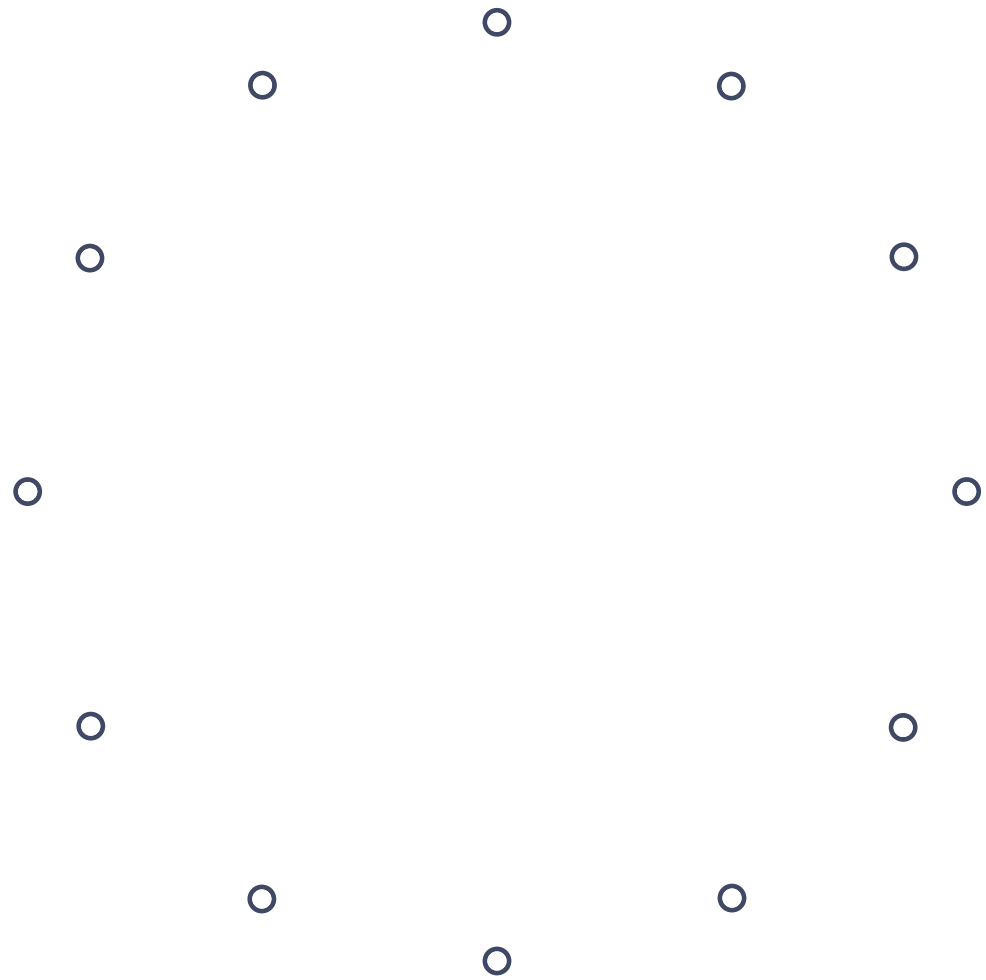
Density: 2-sided

Flower Drawing:

Density: 2-sided

Flower Drawing:

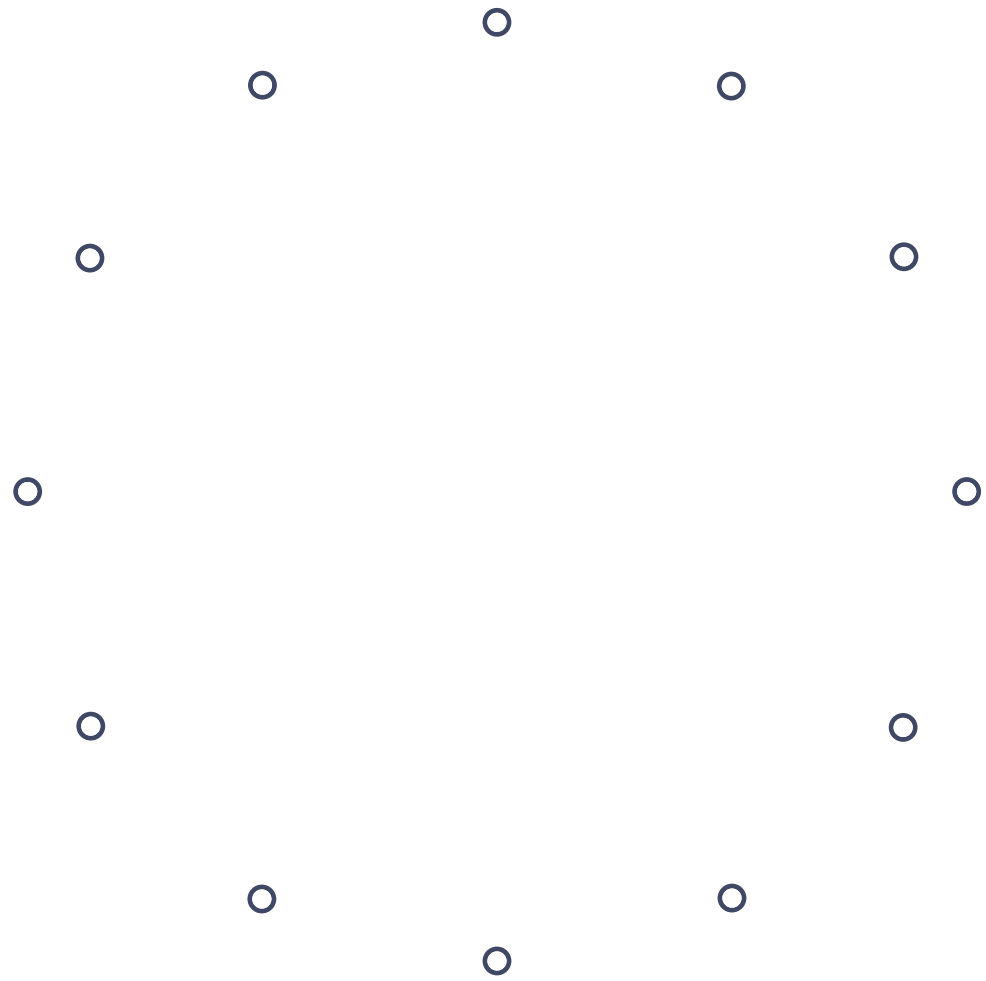
- Vertices on circle



Density: 2-sided

Flower Drawing:

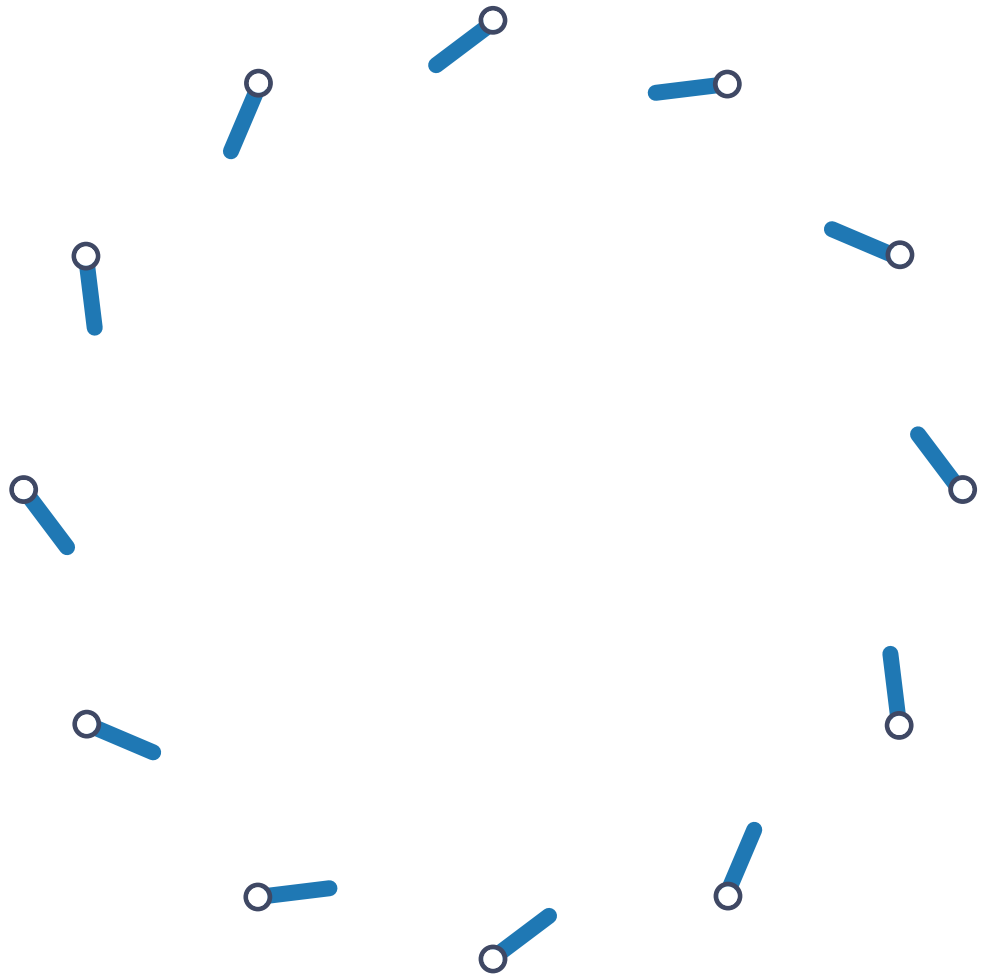
- Vertices on circle
- Every vertex has *left* and *right* bundle



Density: 2-sided

Flower Drawing:

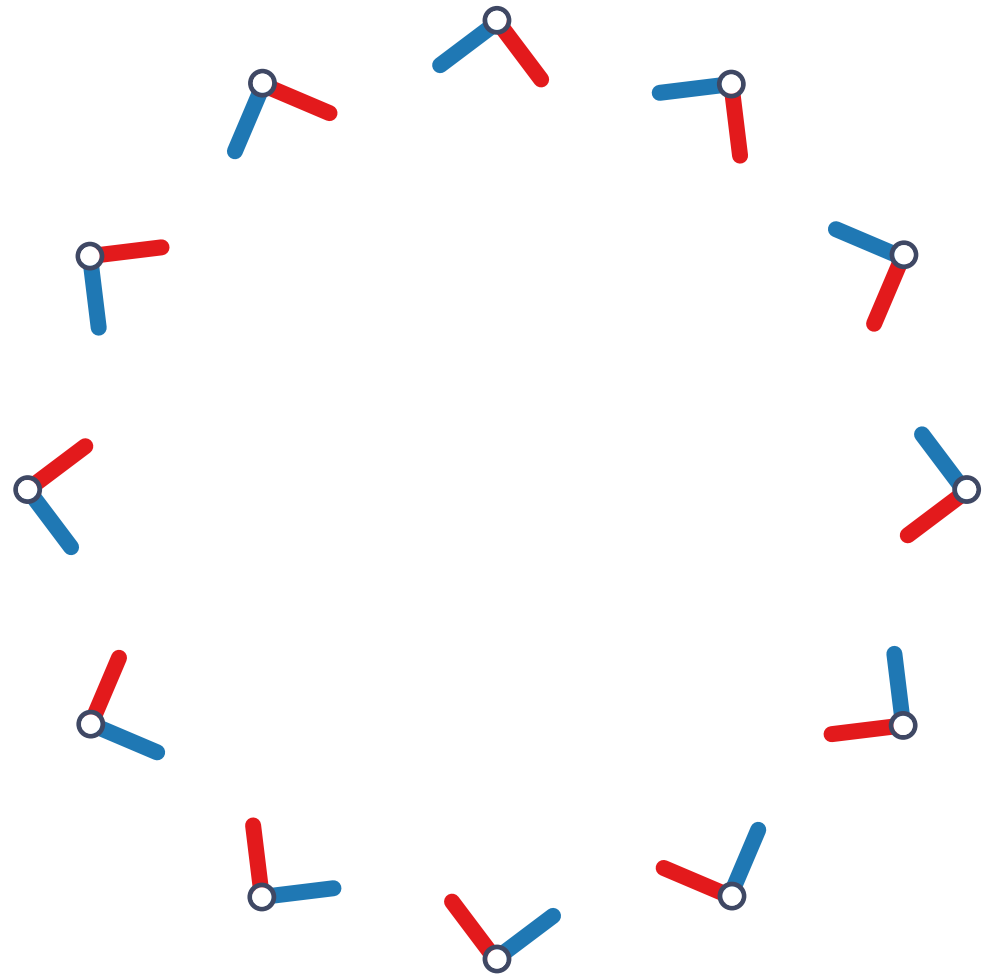
- Vertices on circle
- Every vertex has *left* and *right* bundle



Density: 2-sided

Flower Drawing:

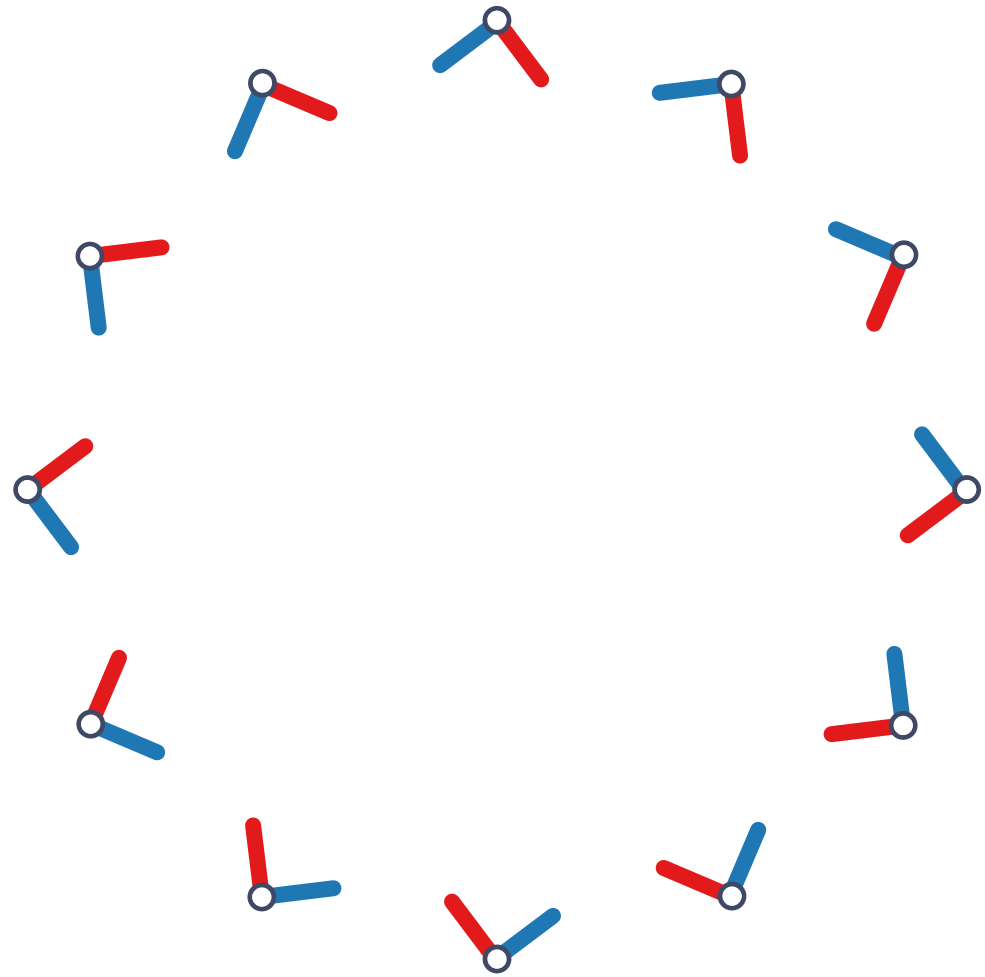
- Vertices on circle
- Every vertex has *left* and *right* bundle



Density: 2-sided

Flower Drawing:

- Vertices on circle
- Every vertex has *left* and *right* bundle
- Consecutive bundles cross



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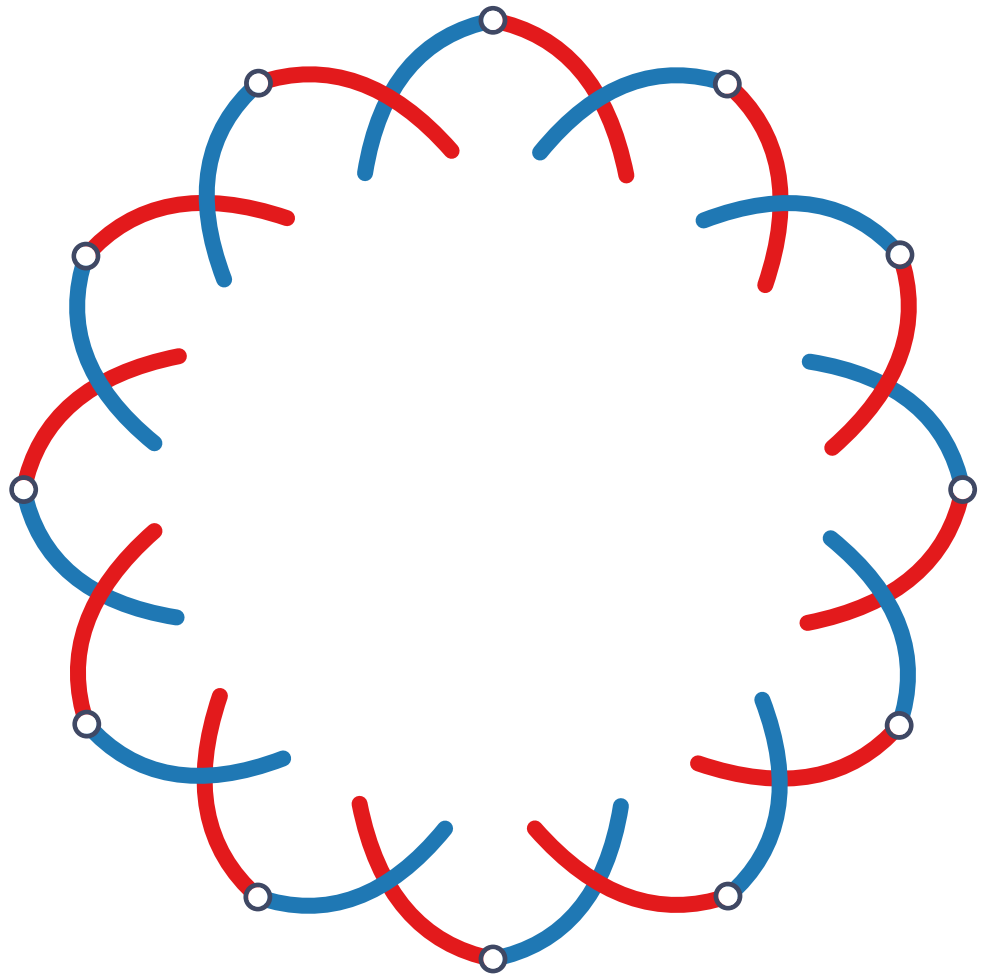

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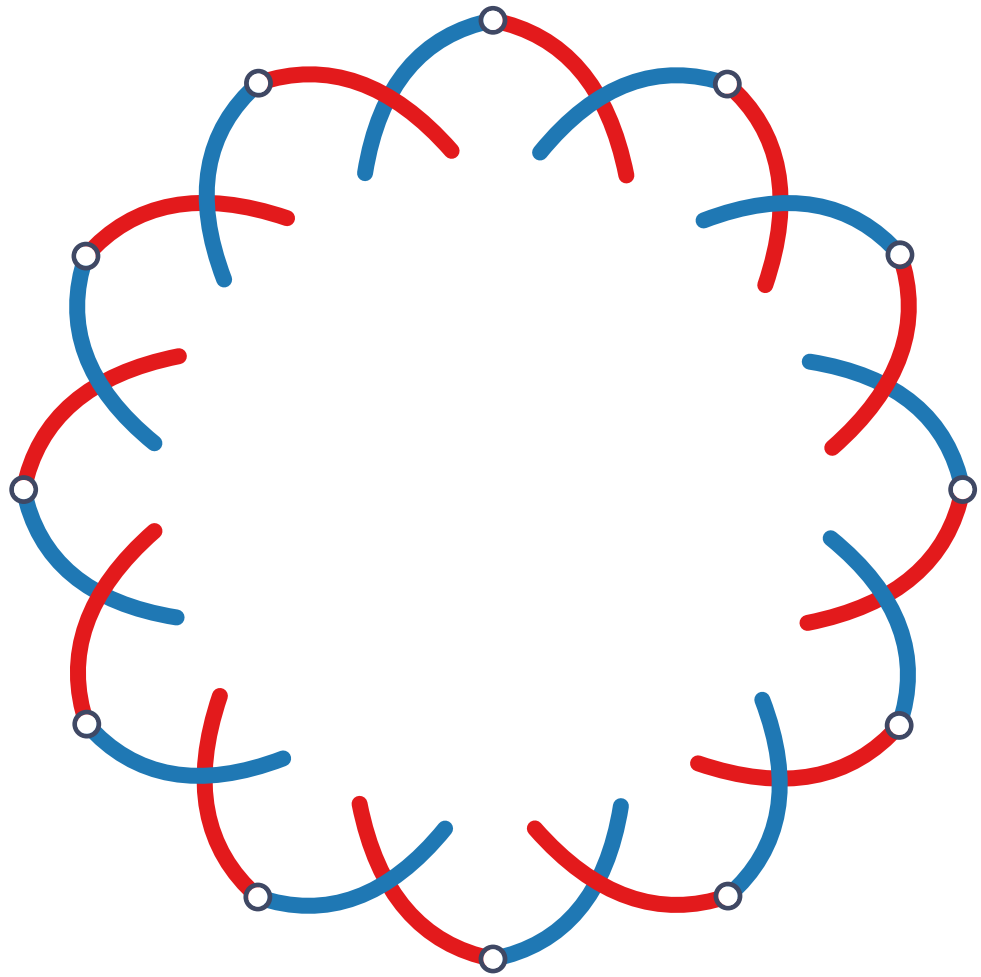
Water Lily Drawing:

- Flower Drawing



Density: 2-sided

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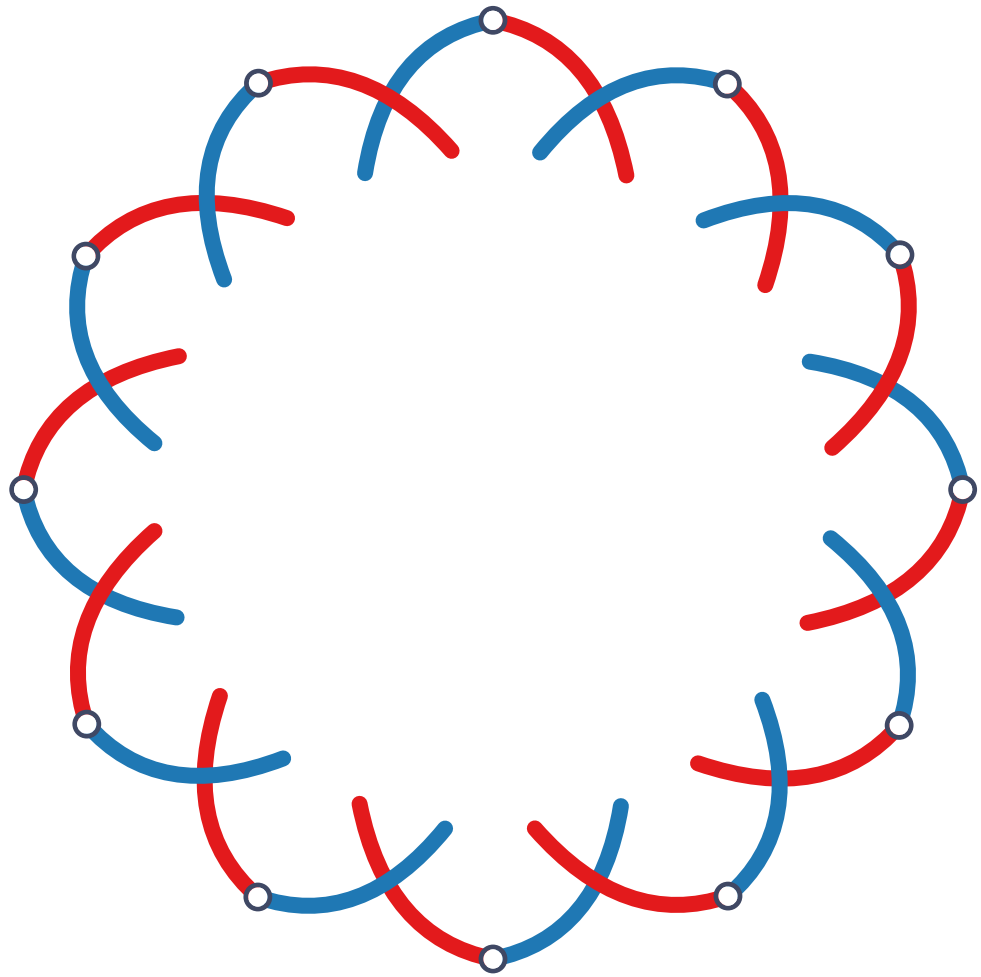
Water Lily Drawing:

- Flower Drawing
- Terminals partitioned into 3 sets



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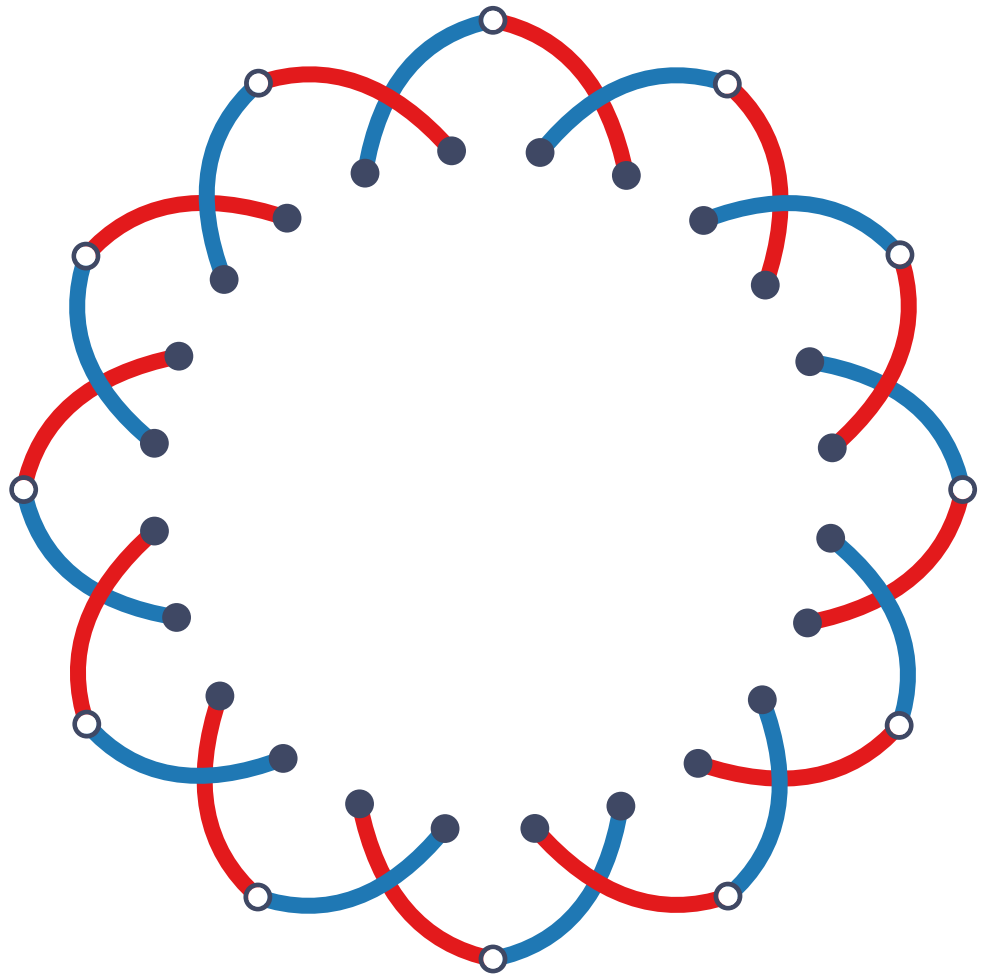

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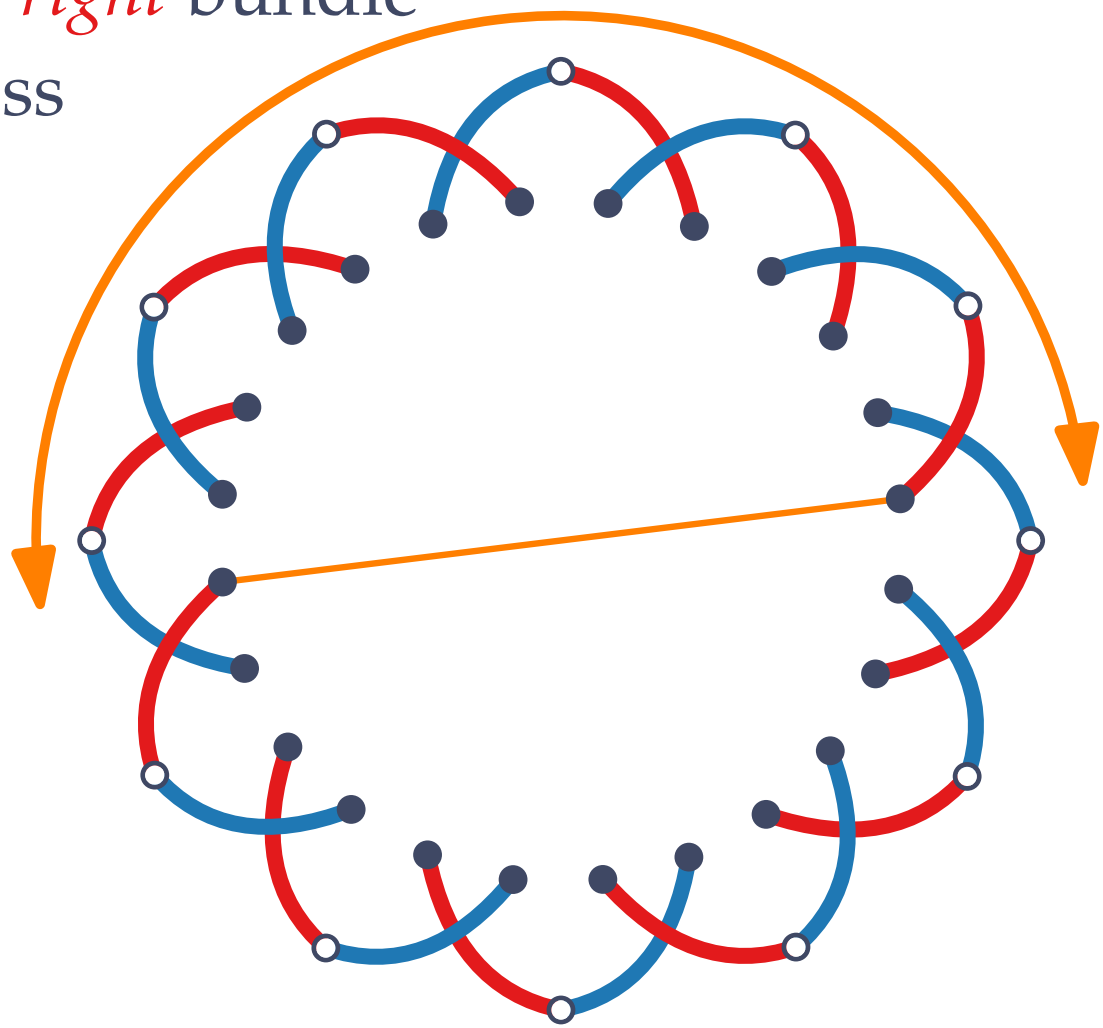

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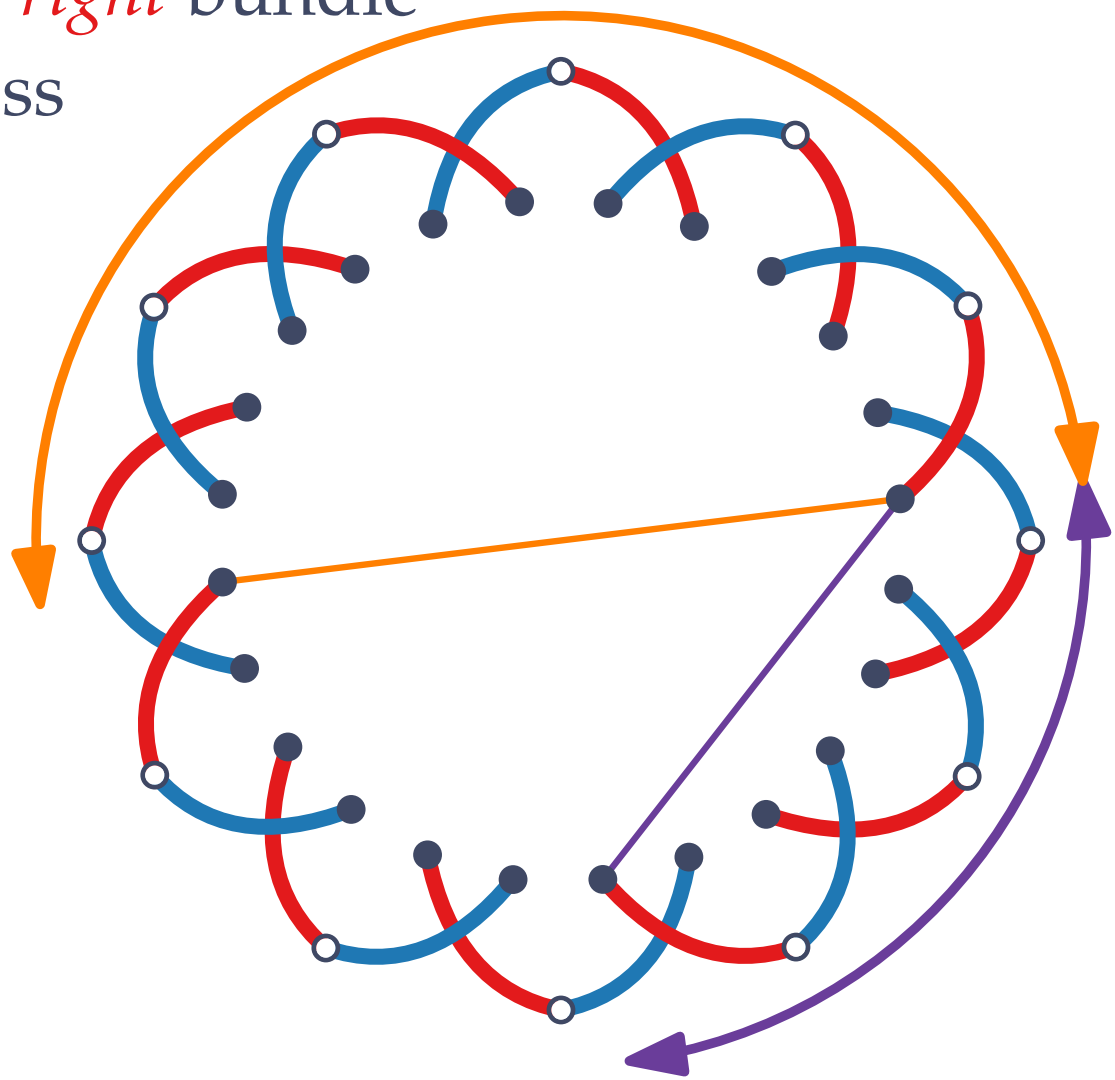

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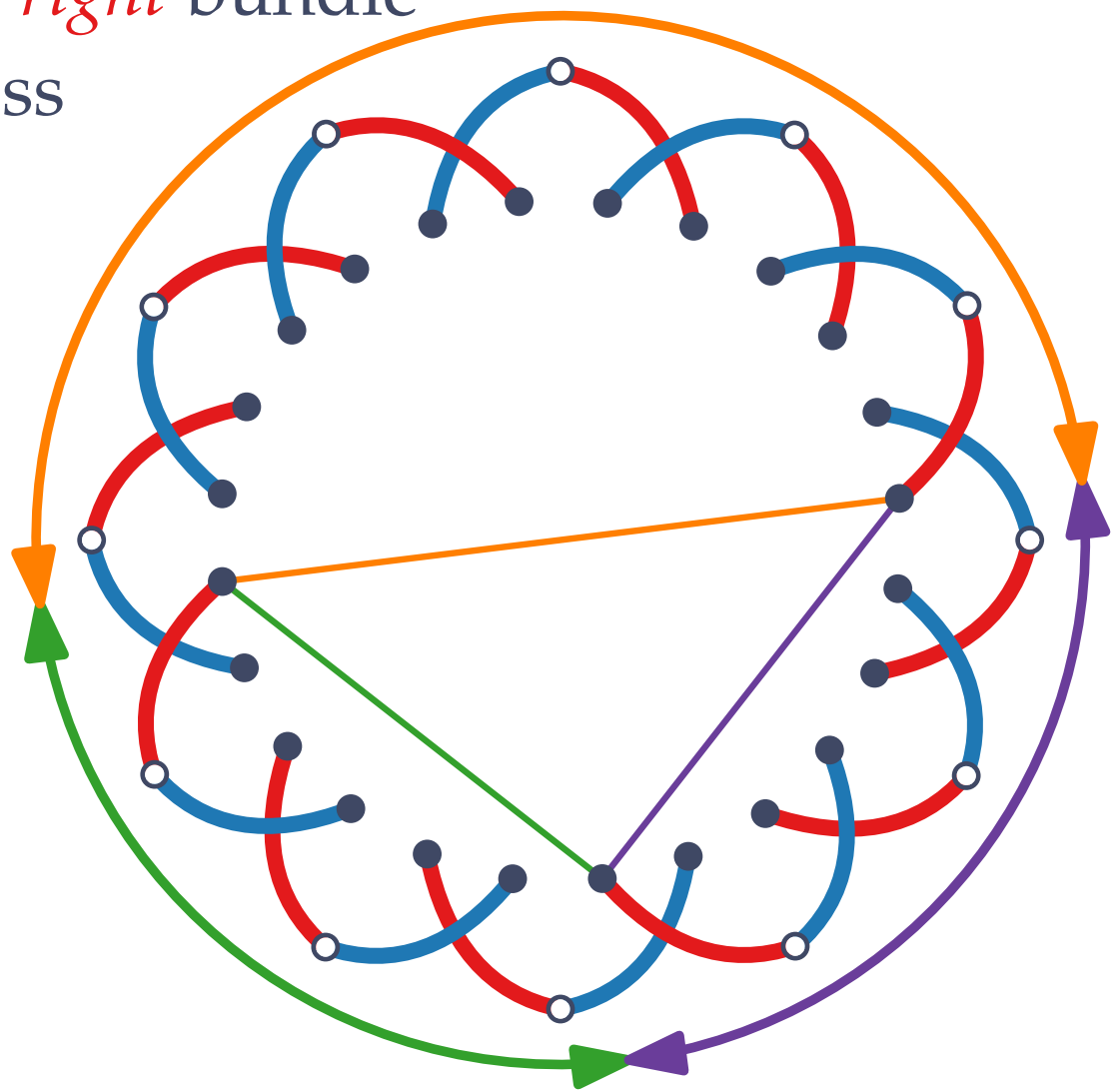

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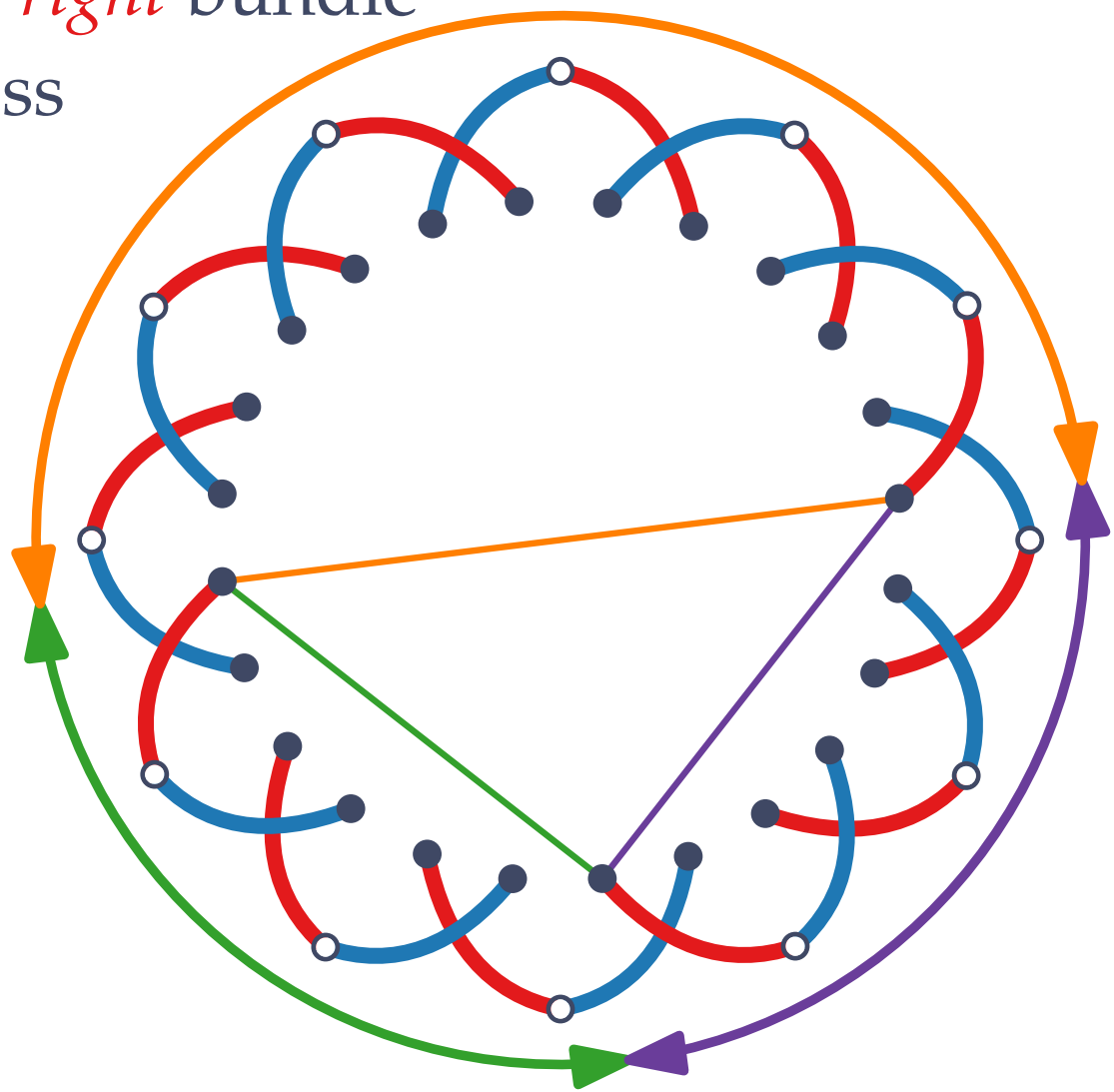
Water Lily Drawing:

- Flower Drawing
- Terminals partitioned into 3 sets
- Cycle through terminals



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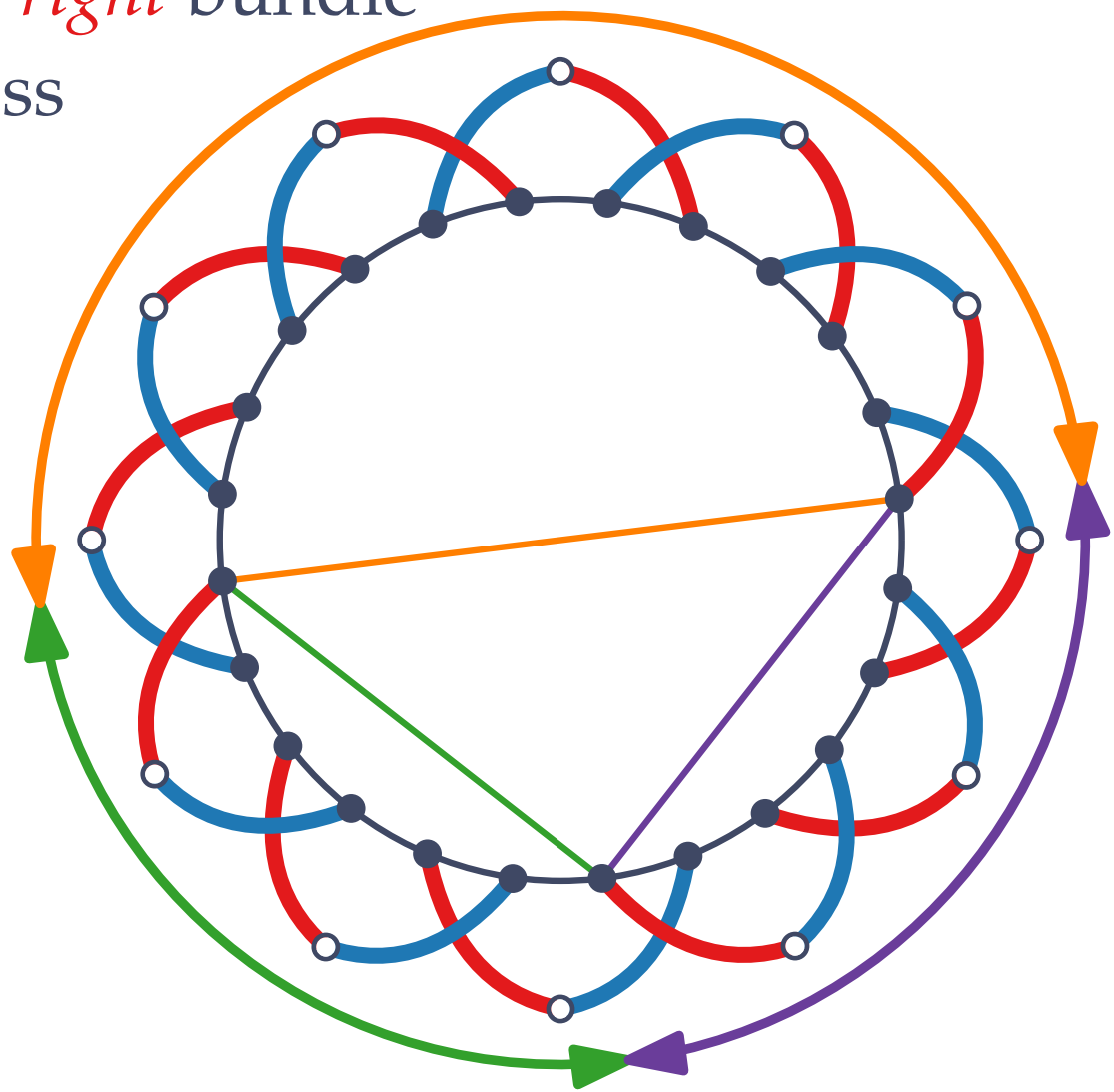

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Water Lily Drawing:

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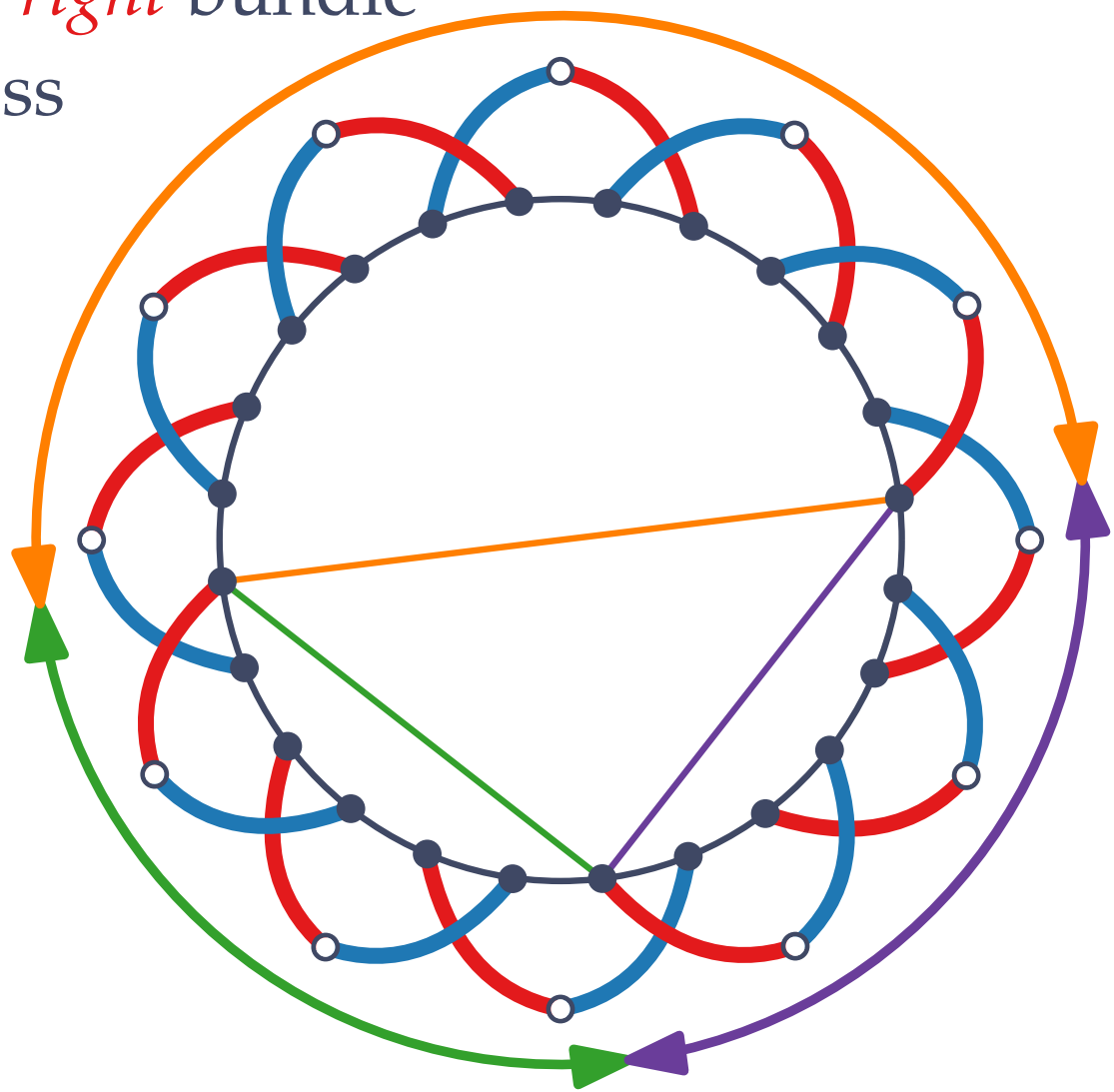
- Cycle through terminals
- Zigzag pattern in each partition



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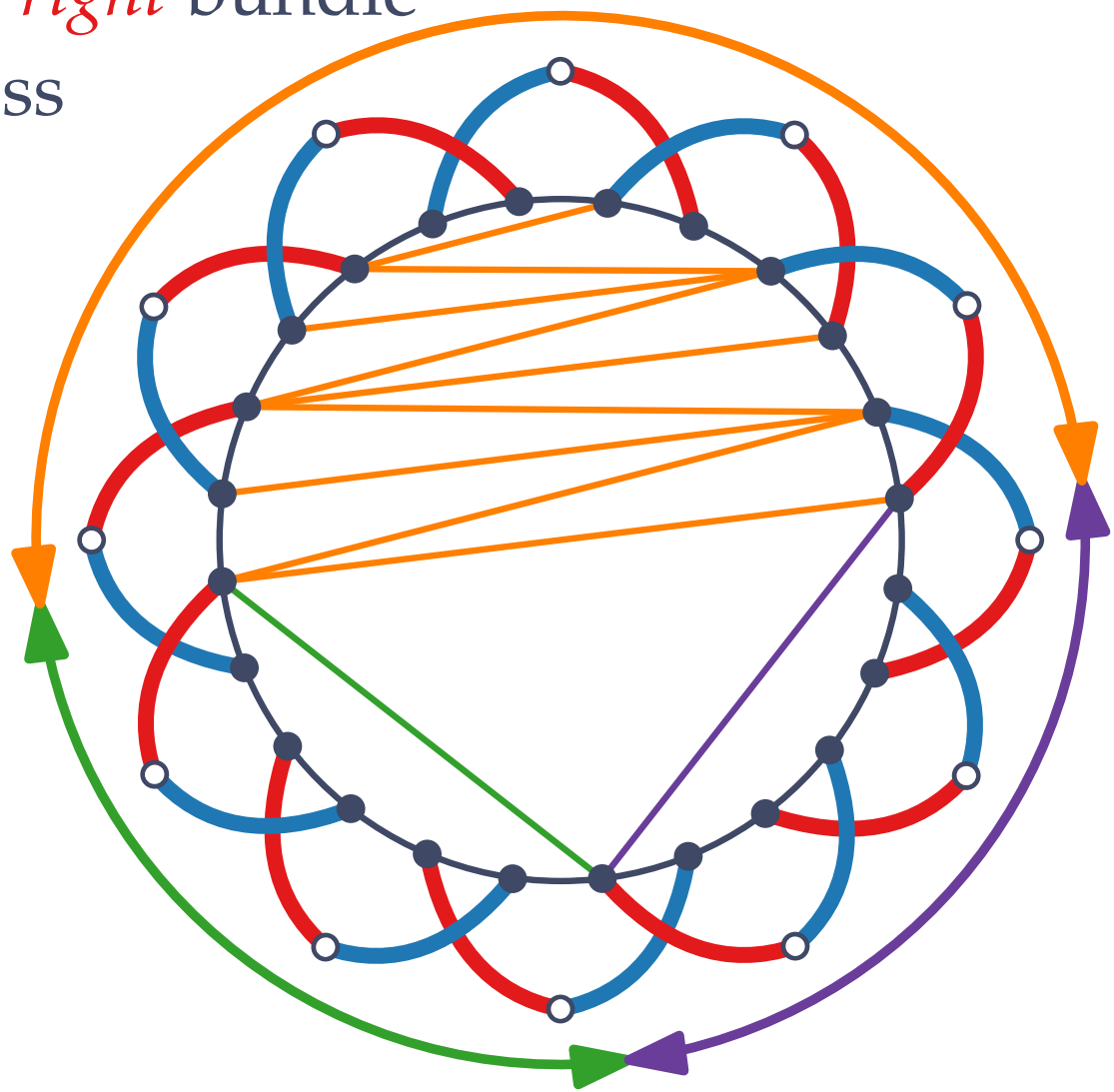

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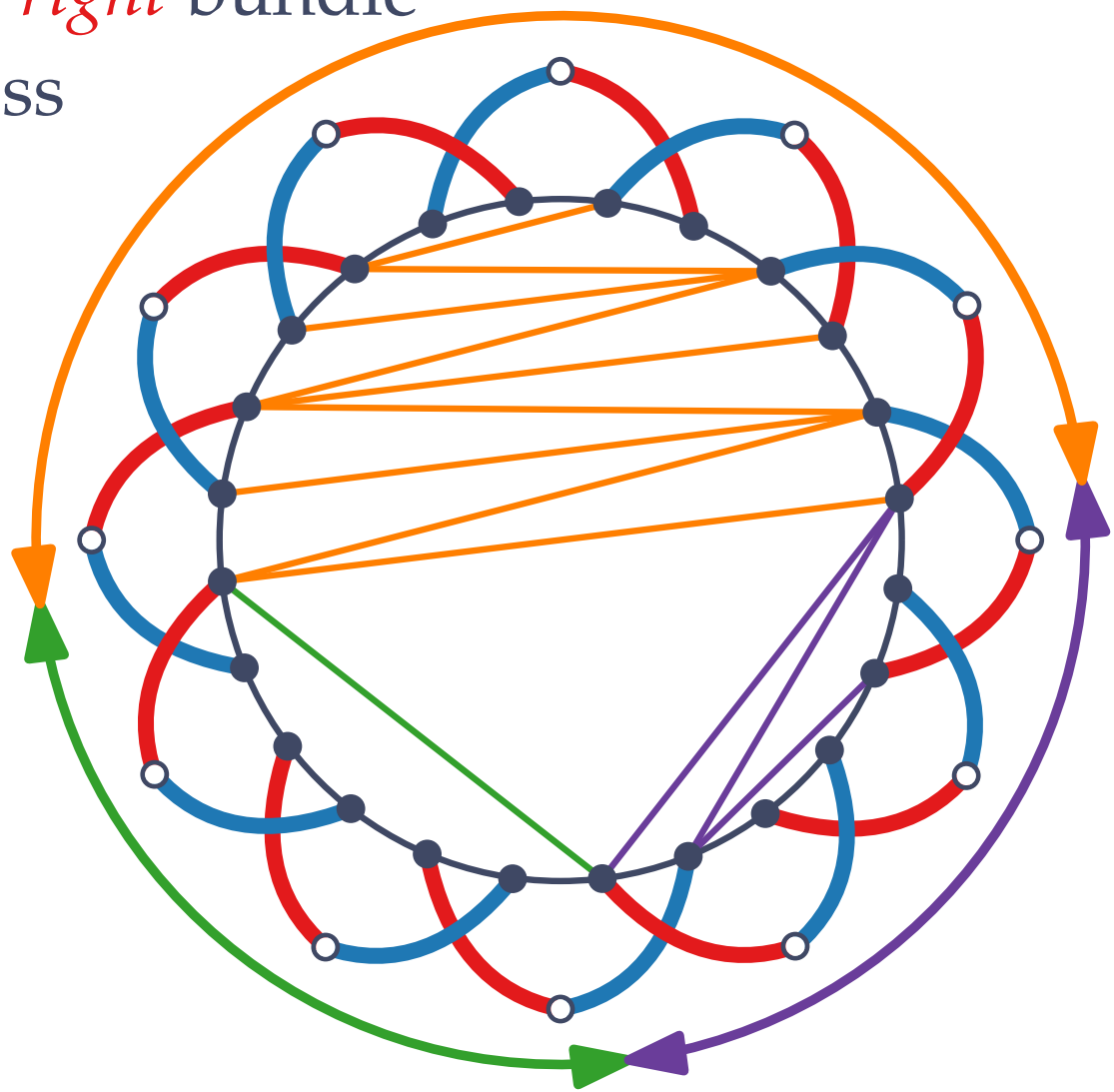

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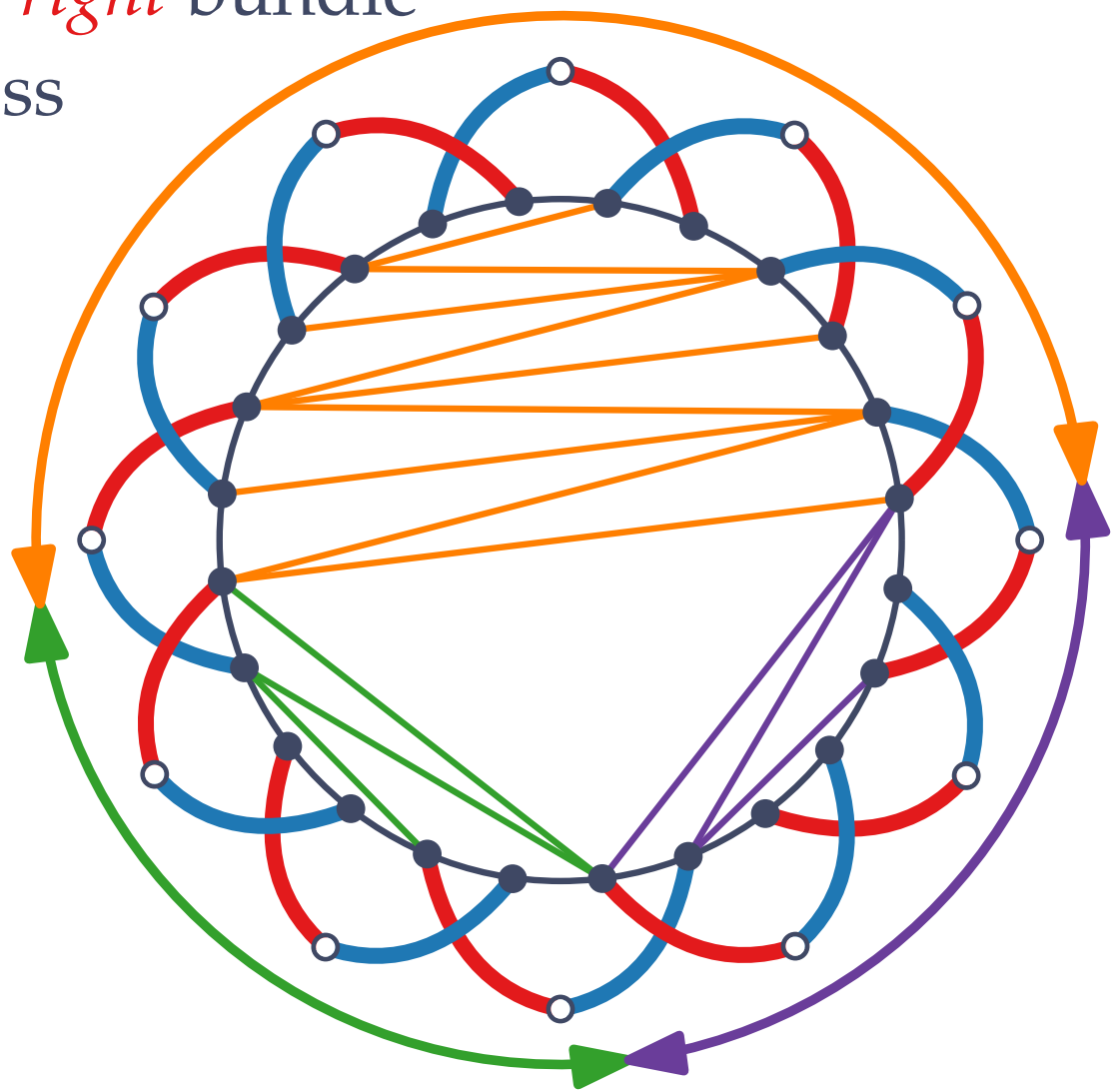

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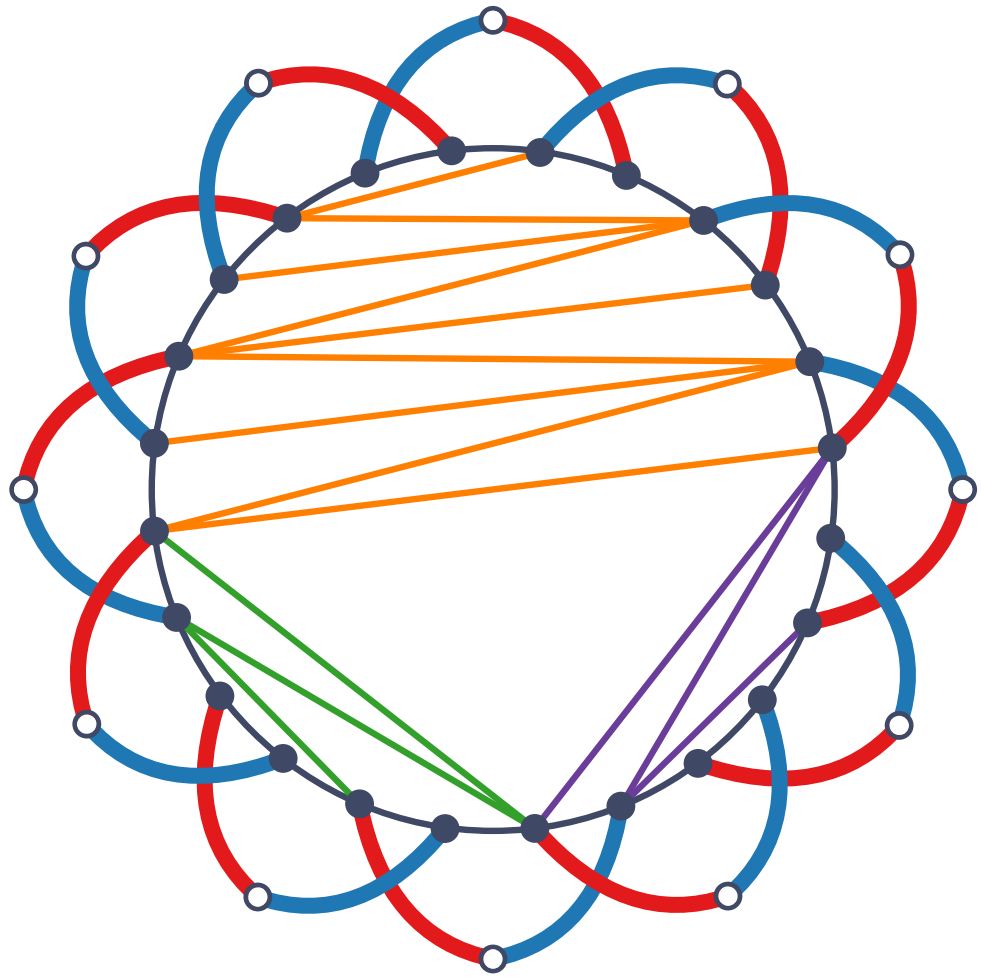
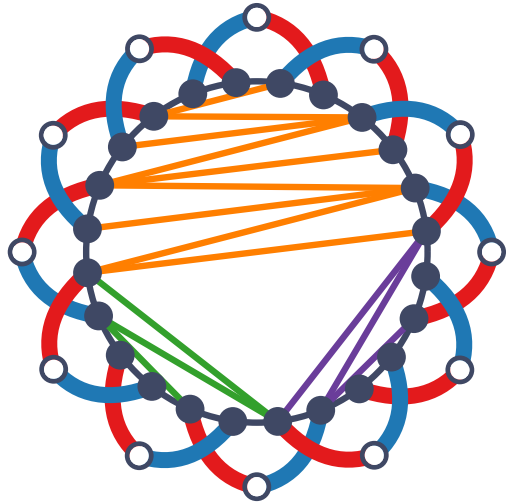


Density: 2-sided

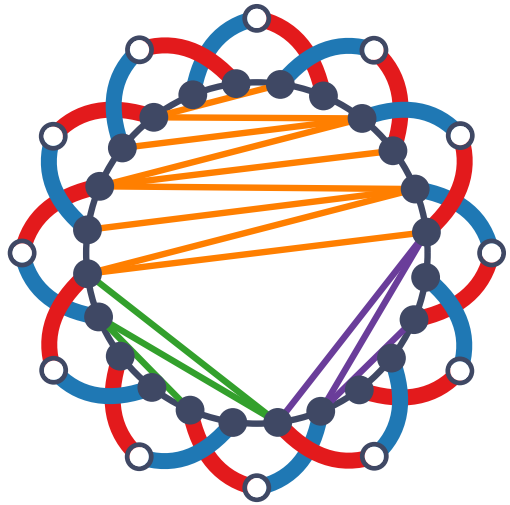
Water Lily Drawings have $4n - 9$ edges



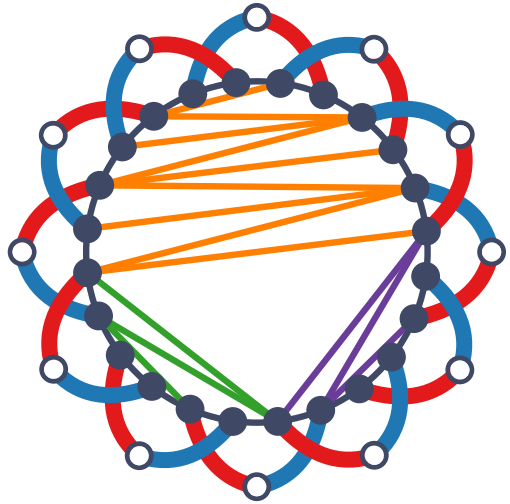
Density: 2-sided

Water Lily Drawings have $4n - 9$ edges

$\Rightarrow$ LB for outer-2-sided: $4n - 9$



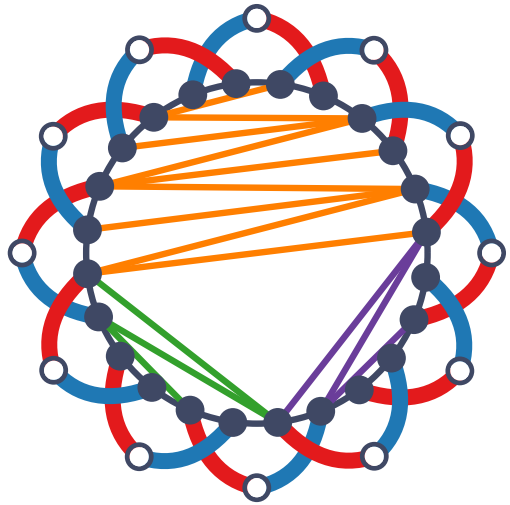
Density: 2-sided



Water Lily Drawings have $4n - 9$ edges
$\Rightarrow$ LB for outer-2-sided: $4n - 9$

Merge 2 Water Lilys
$\Rightarrow$ LB for 2-sided: $6n - 18$

Density: 2-sided



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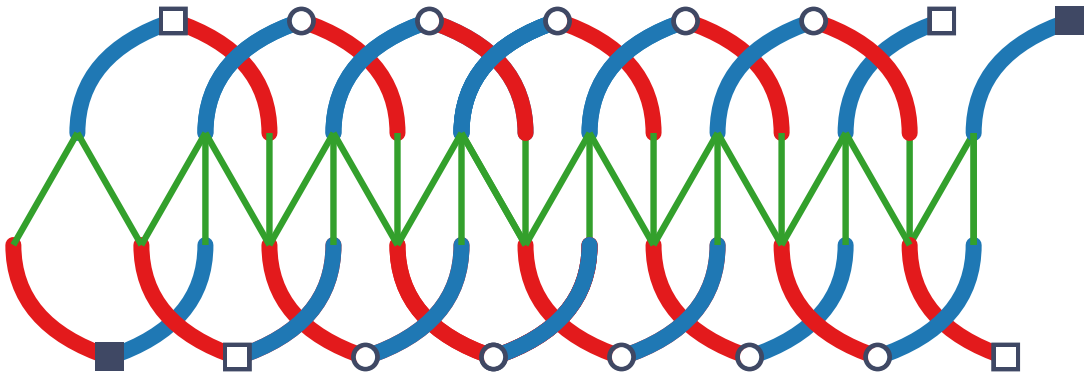
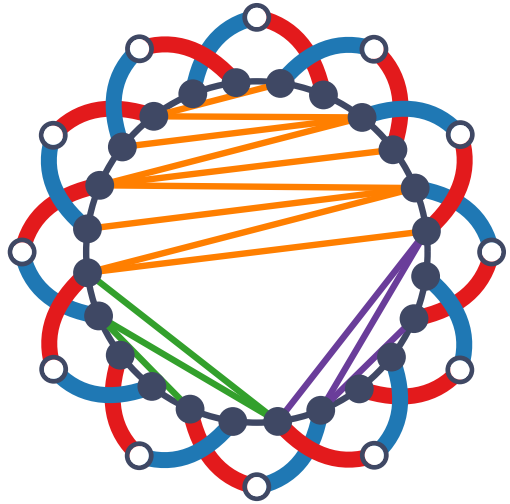
Merge 2 Water Lilys
$\Rightarrow$ LB for 2-sided: $6n - 18$



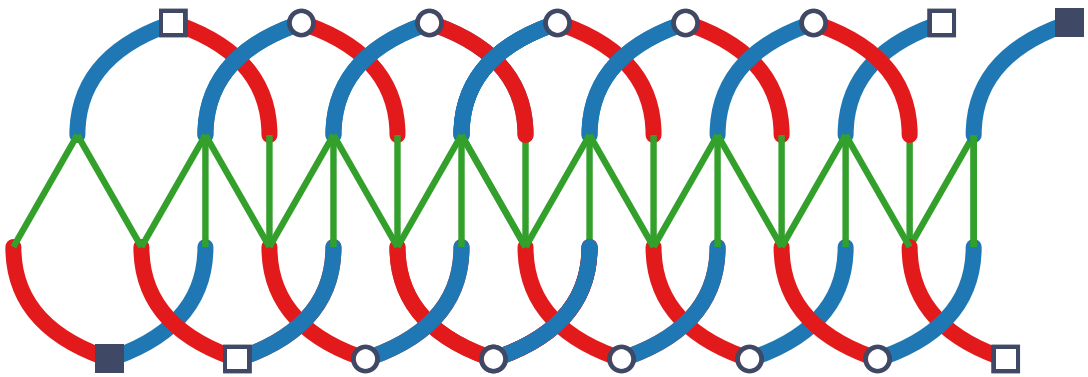
Density: 2-sided



Water Lily Drawings have $4n - 9$ edges
$\Rightarrow$ LB for outer-2-sided: $4n - 9$

Merge 2 Water Lilys
$\Rightarrow$ LB for 2-sided: $6n - 18$

$\Rightarrow$ LB for 2-layer 2-sided: $2n - 4$



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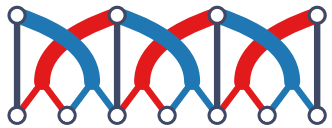
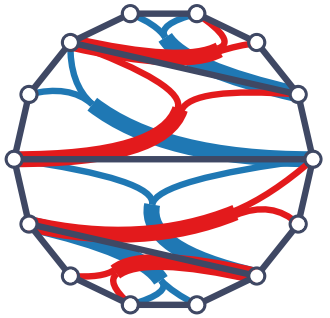
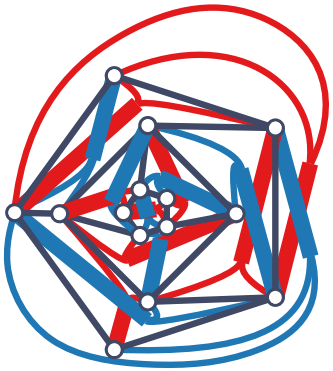
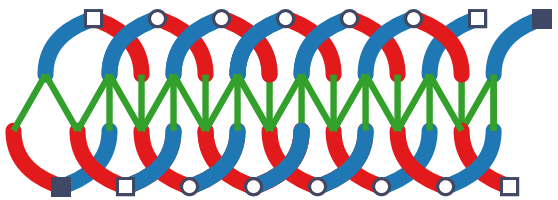
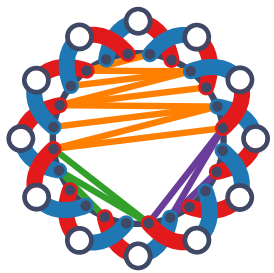






	2-layer		outer		general	
	LB	UB	LB	UB	LB	UB
1-sided	$\frac{5n-7}{3}$	$\frac{5n-7}{3}$	$\frac{8n-13}{3}$	$\frac{8n-13}{3}$	$\frac{13n-26}{3}$	$\frac{13n-26}{3}$
2-sided	$2n-4$		$4n-9$		$6n-18$	





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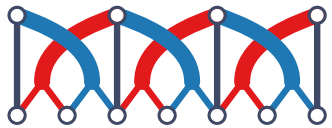

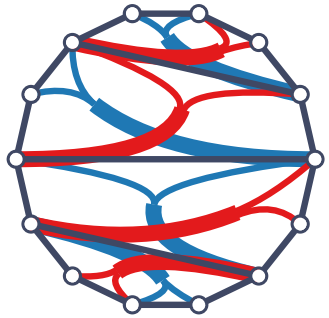

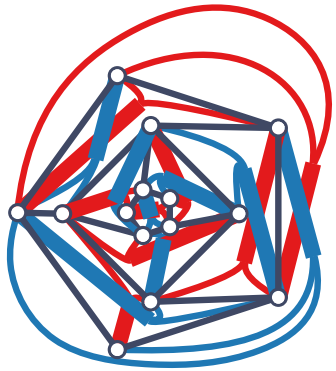

	2-layer		outer		general	
	LB	UB	LB	UB	LB	UB
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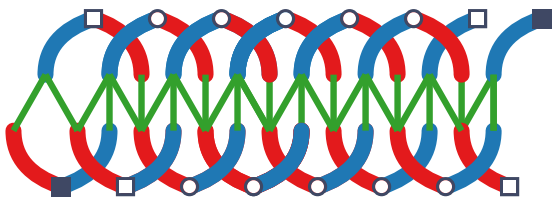
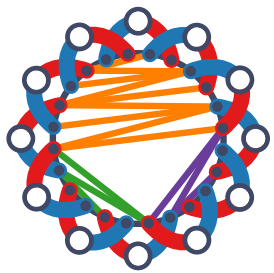
2-sided	$2n-4$	$3n-7$	$4n-9$		$6n-18$	





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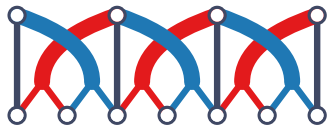

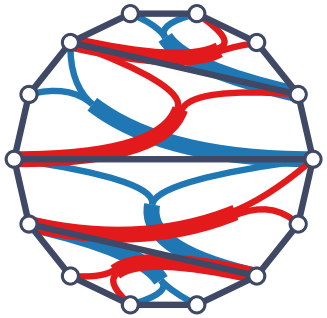

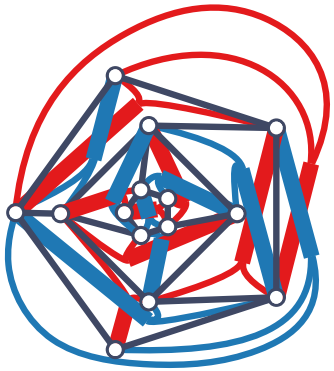

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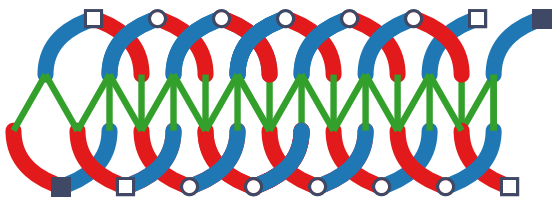
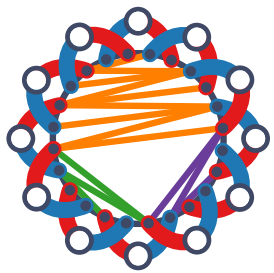
2-sided	$2n-4$	$3n-7$	$4n-9$	$4n-9$	$6n-18$	





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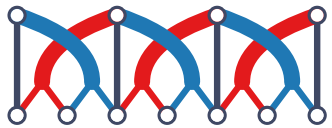

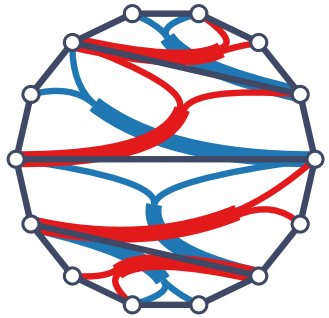

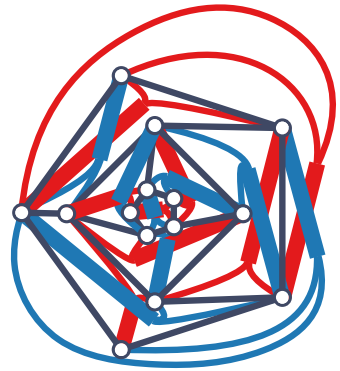

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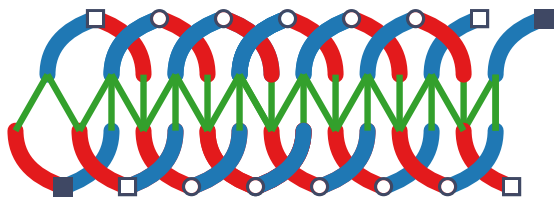
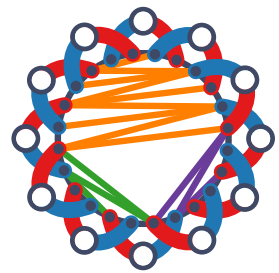
2-sided	$2n-4$	$3n-7$	$4n-9$	$4n-9$	$6n-18$	$8.6n-15.6$





Relationships

Relationships

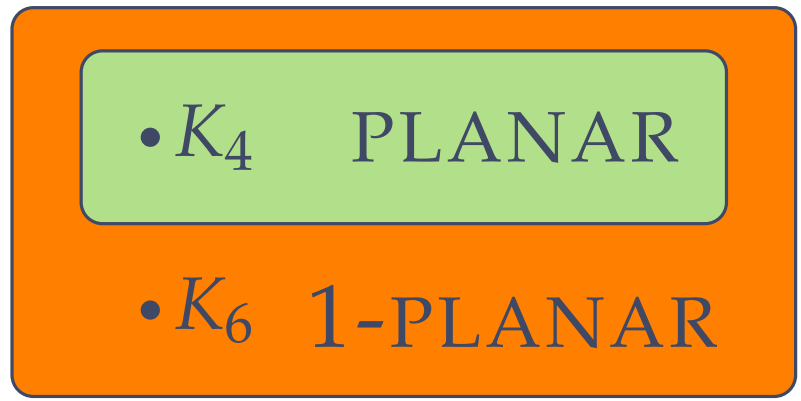
PLANAR

Relationships

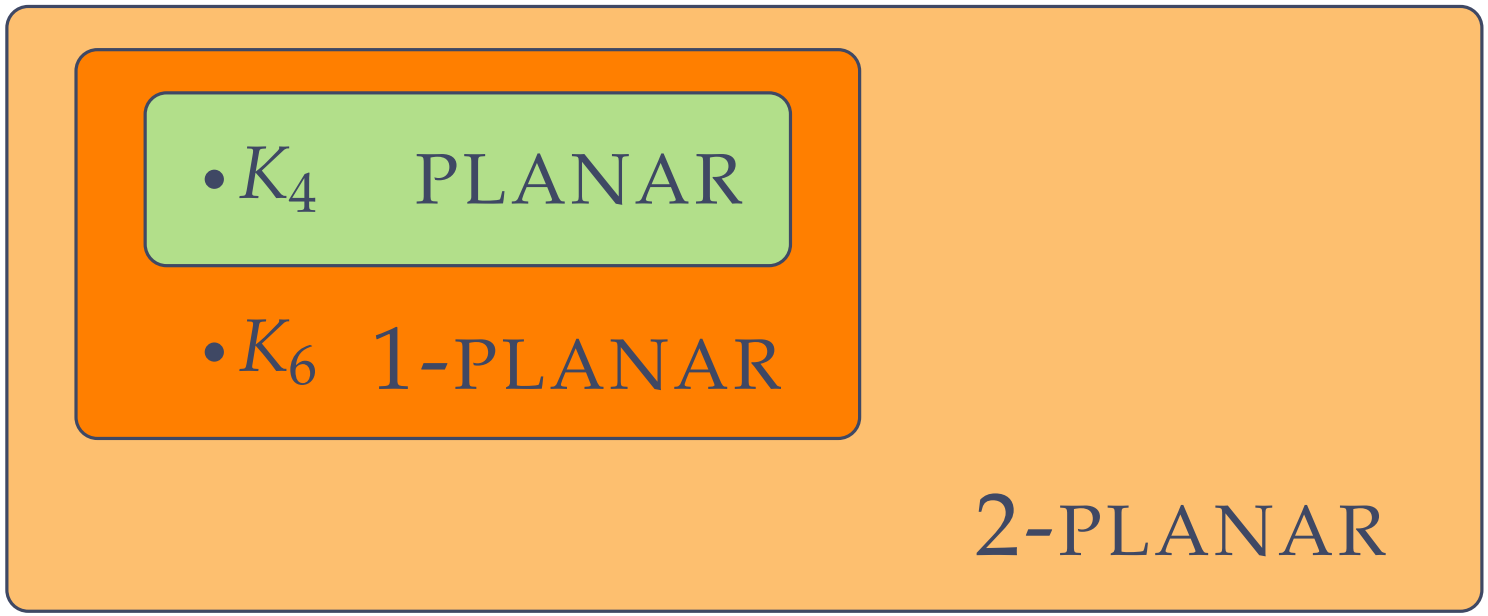
- K_4 PLANAR

Relationships

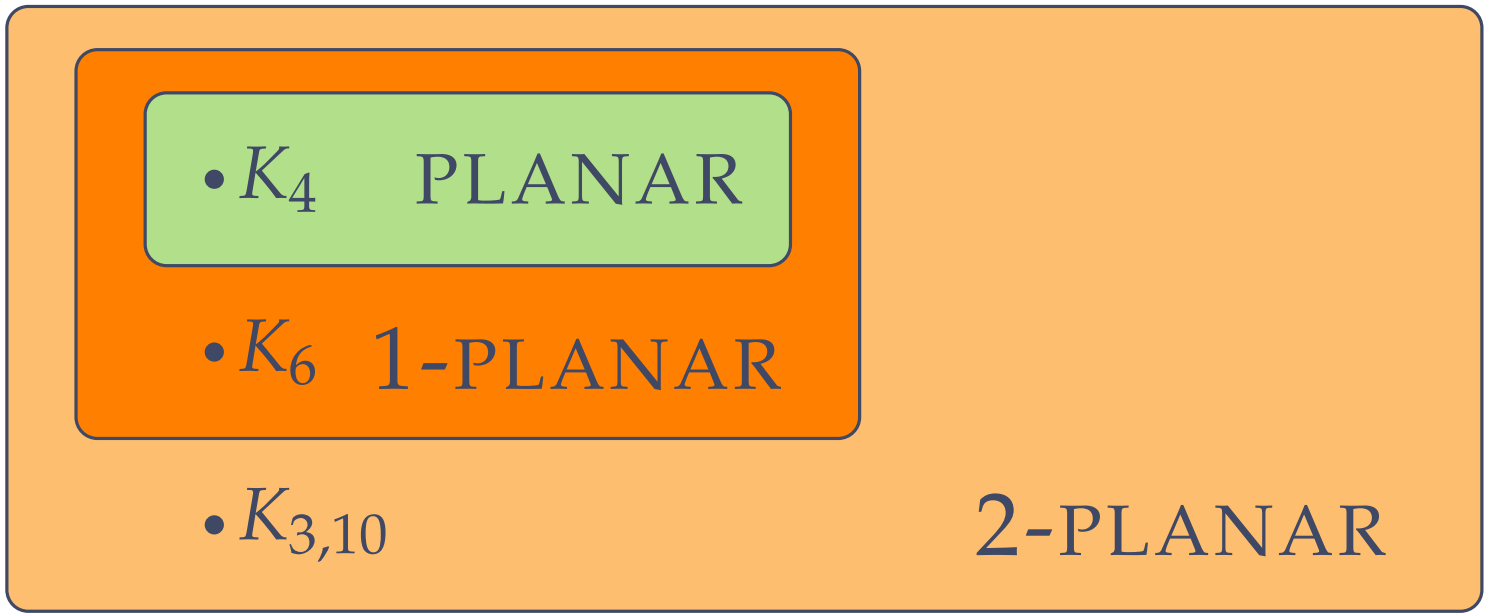
Relationships



Relationships



Relationships



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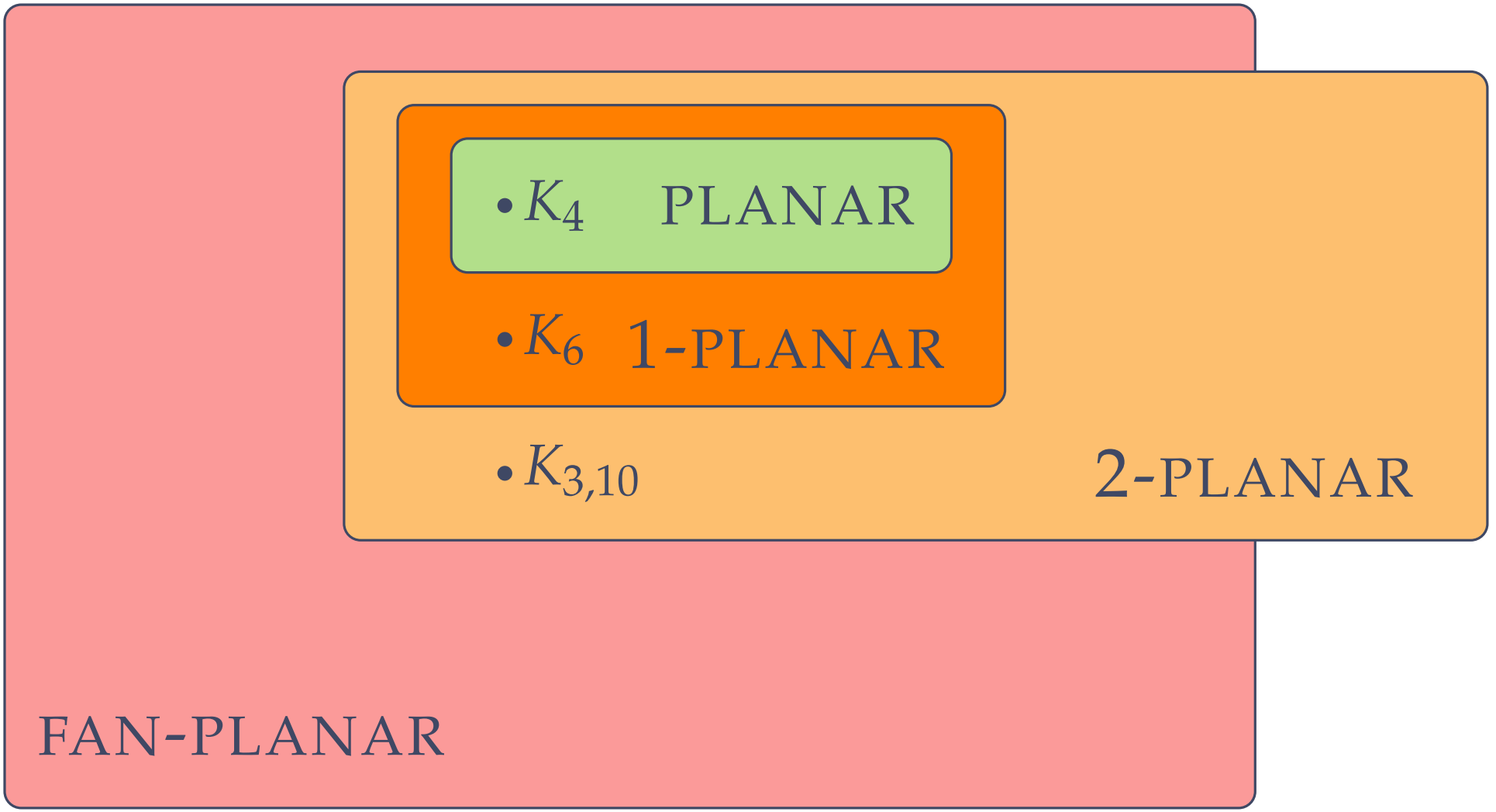

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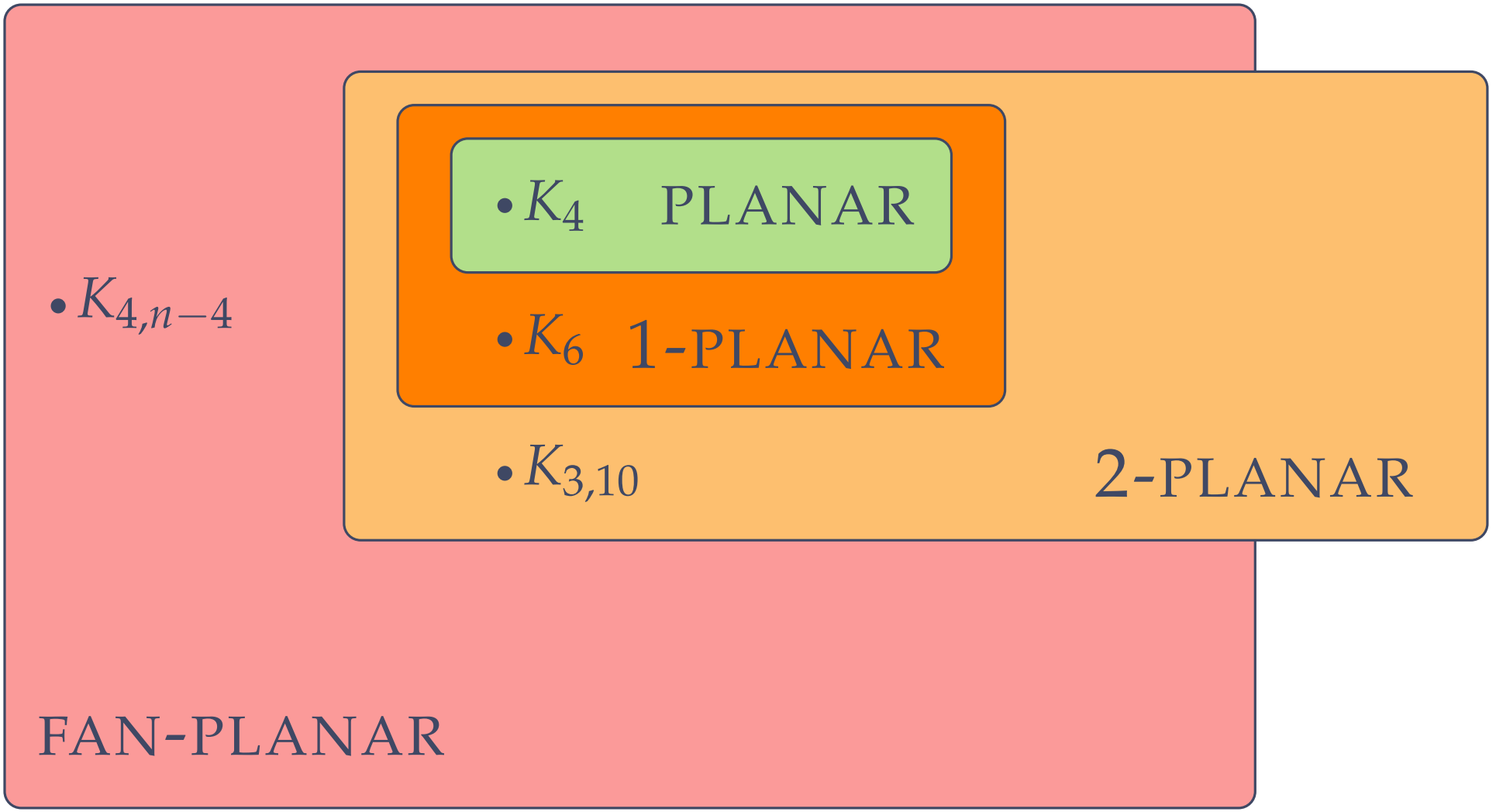

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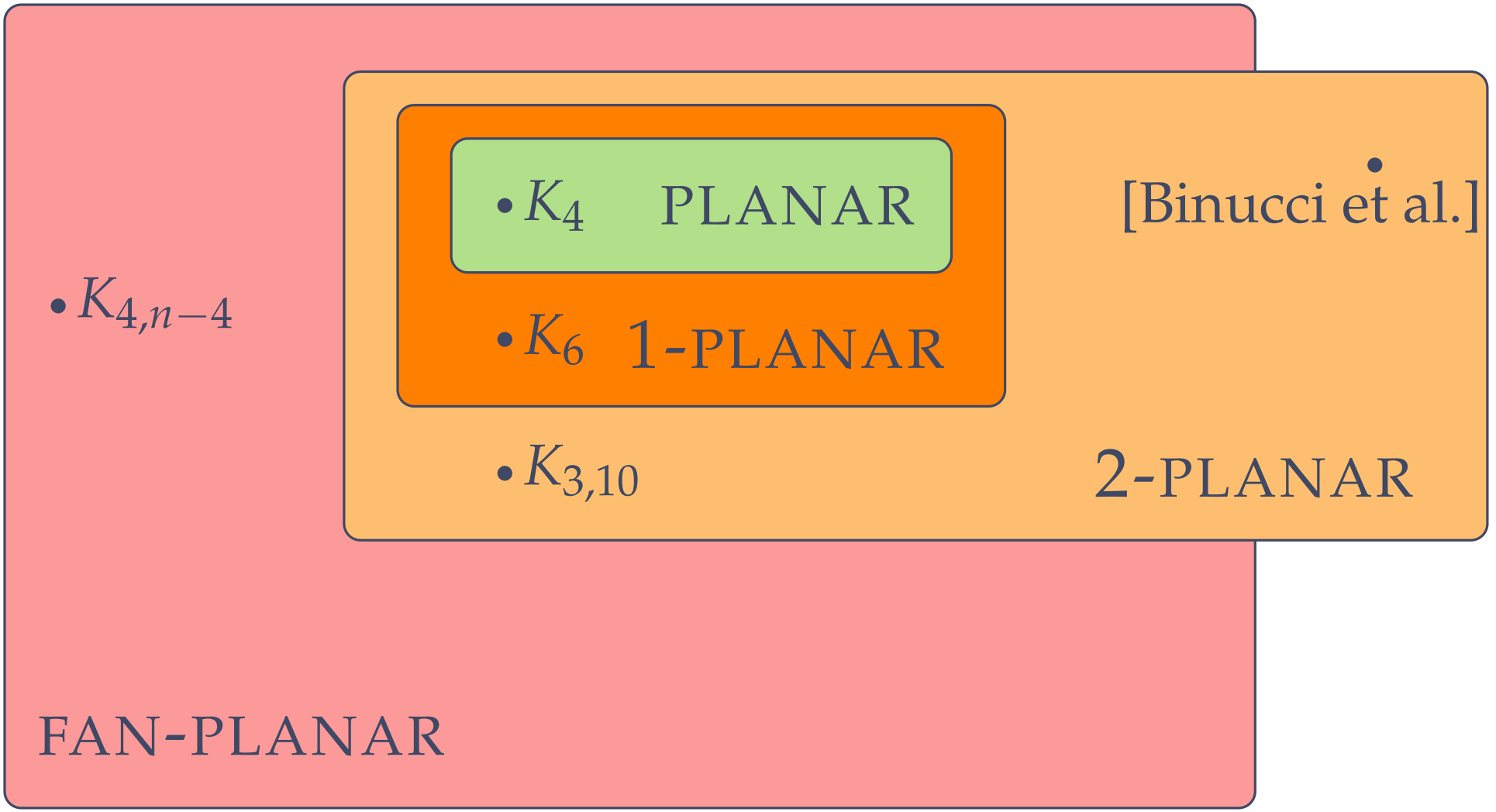

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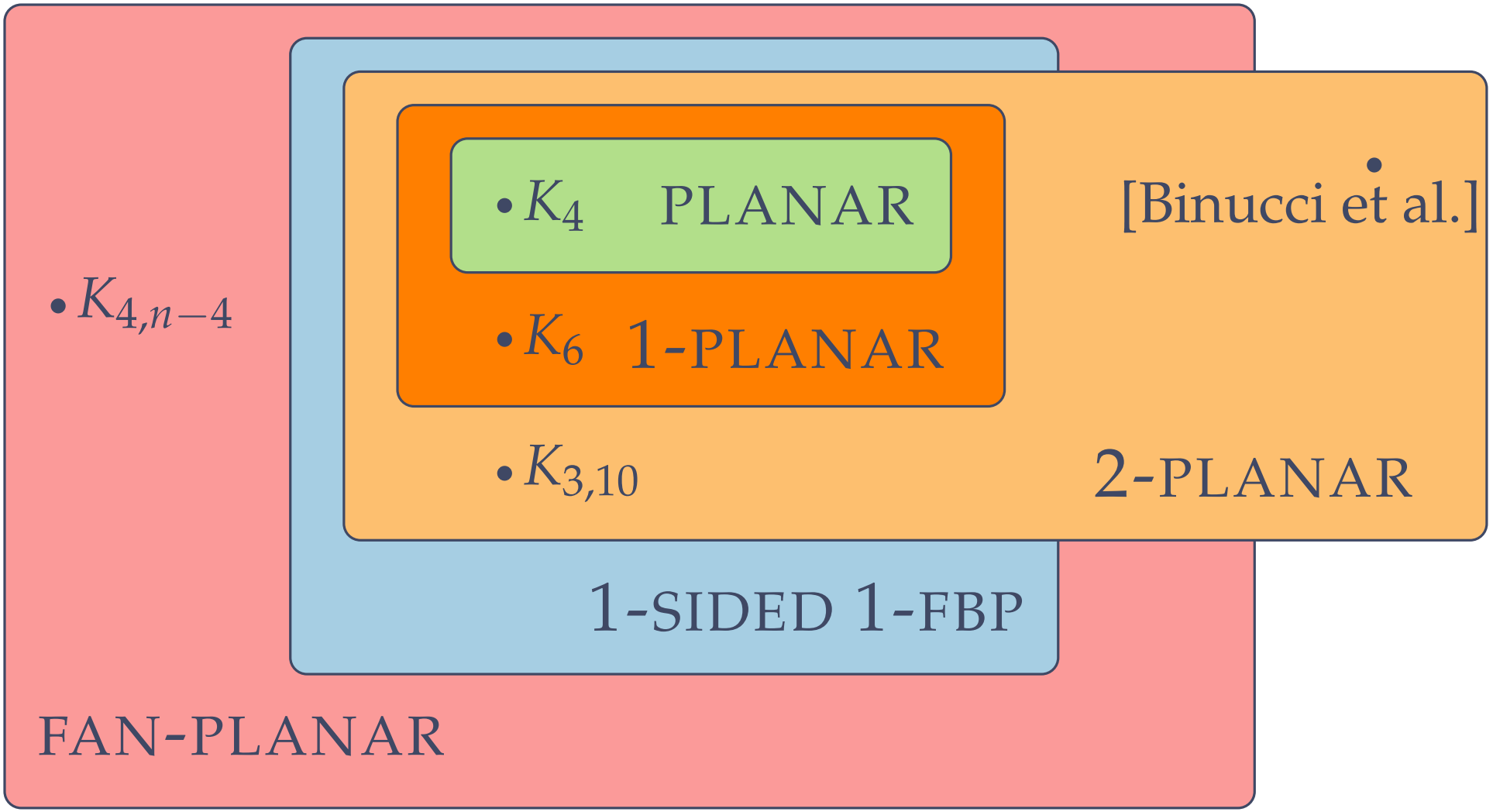
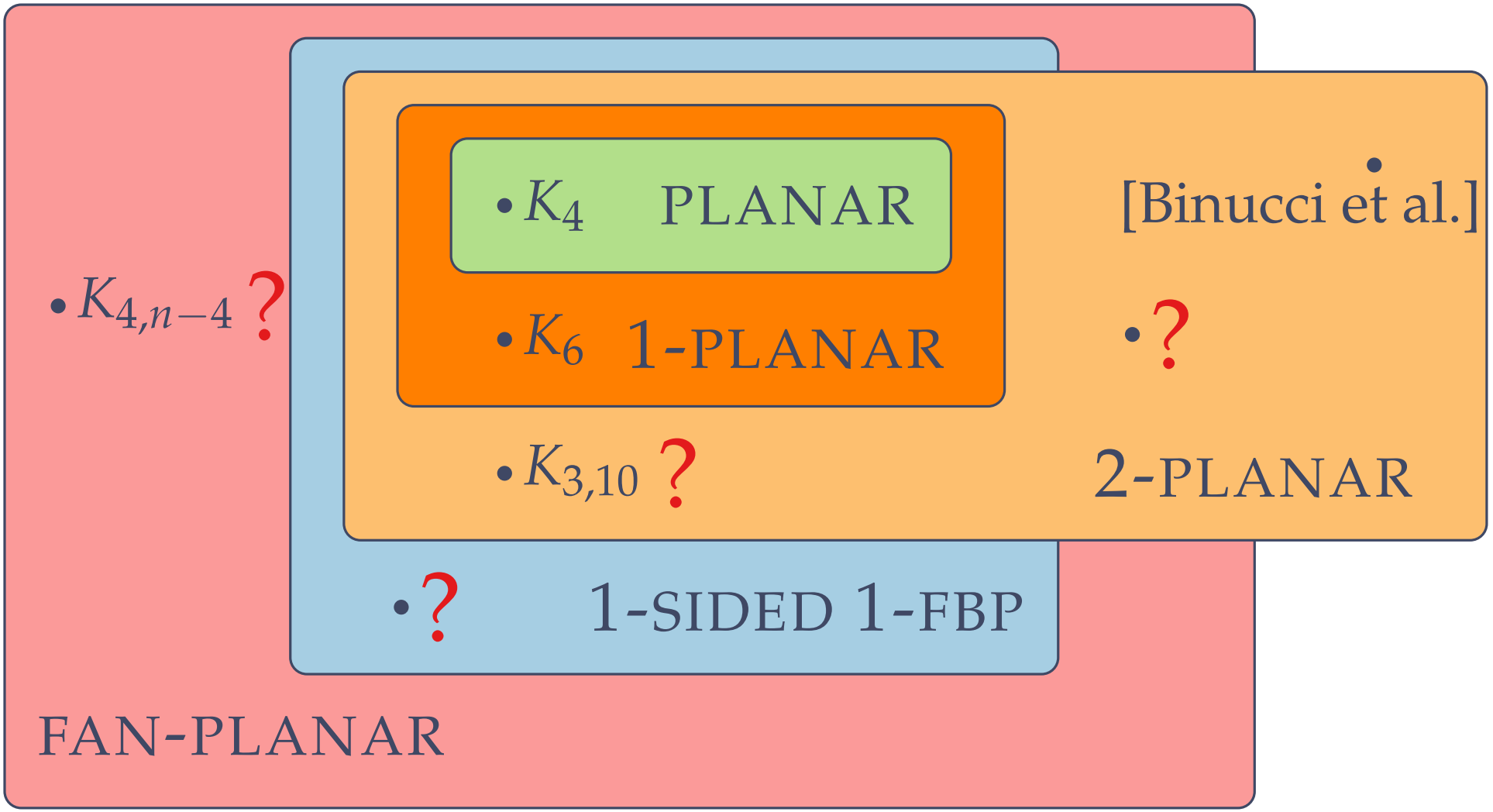
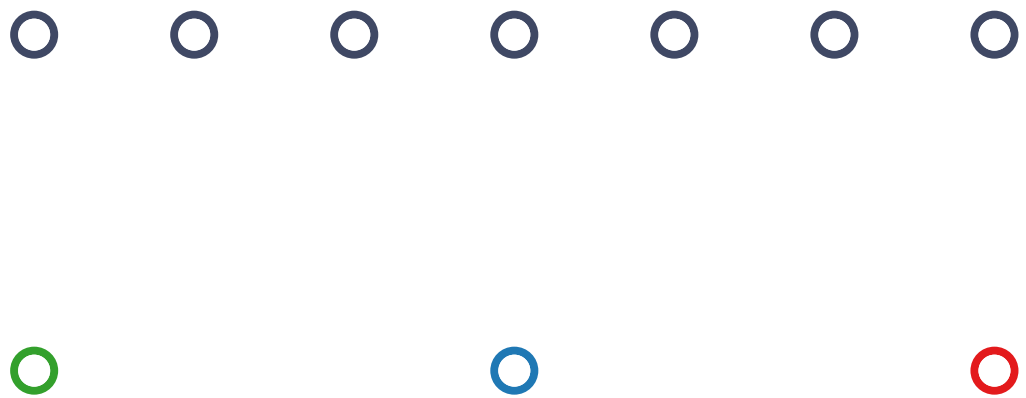


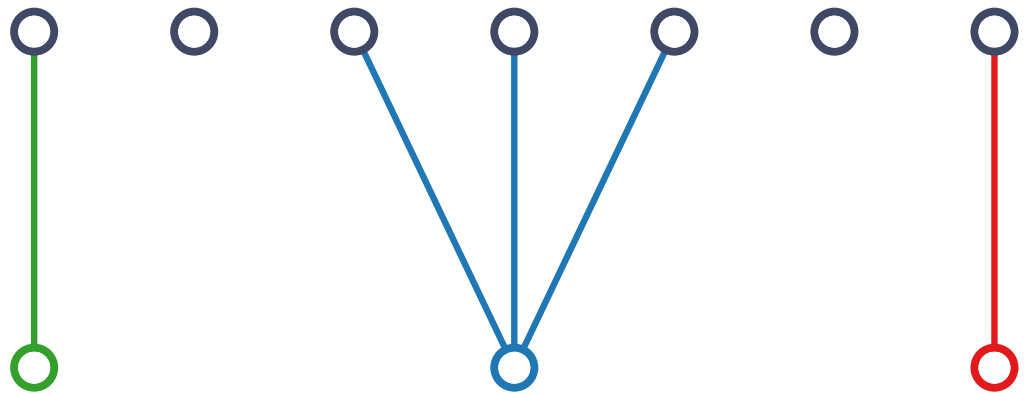
Relationships



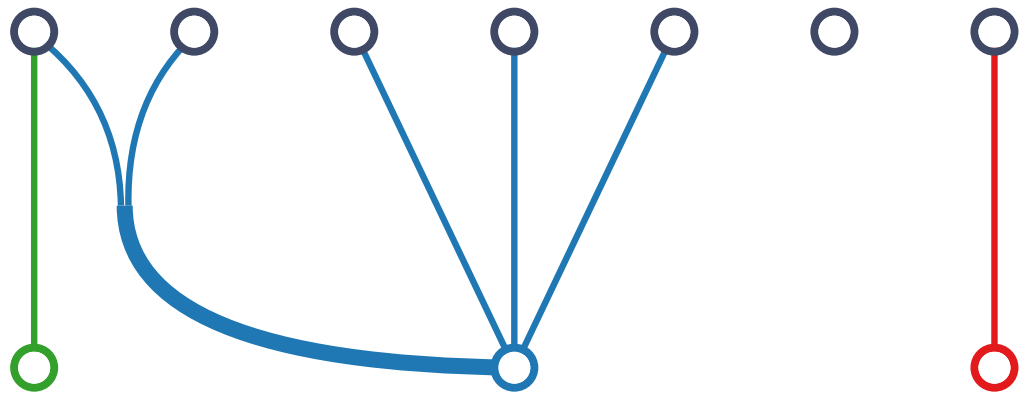
$K_{3,14}$



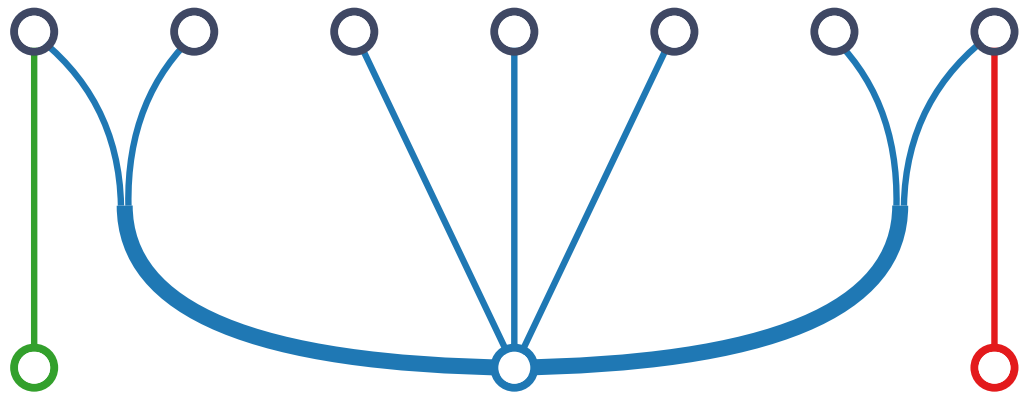
$K_{3,14}$



$K_{3,14}$



$K_{3,14}$



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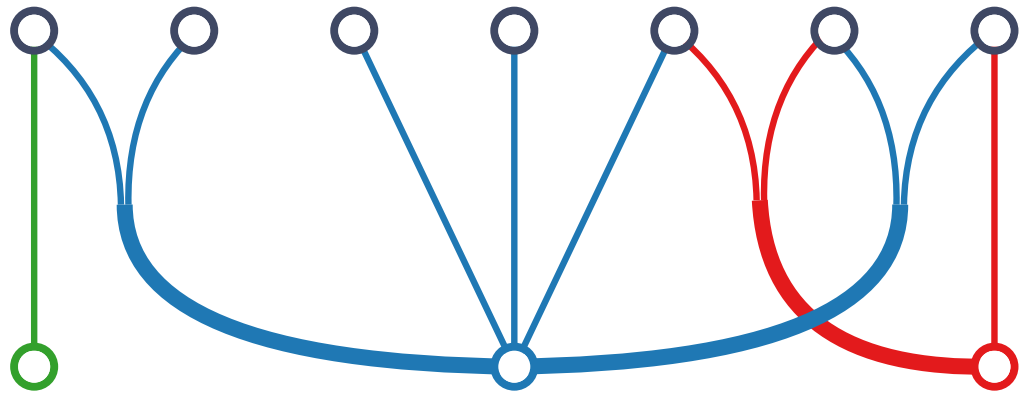
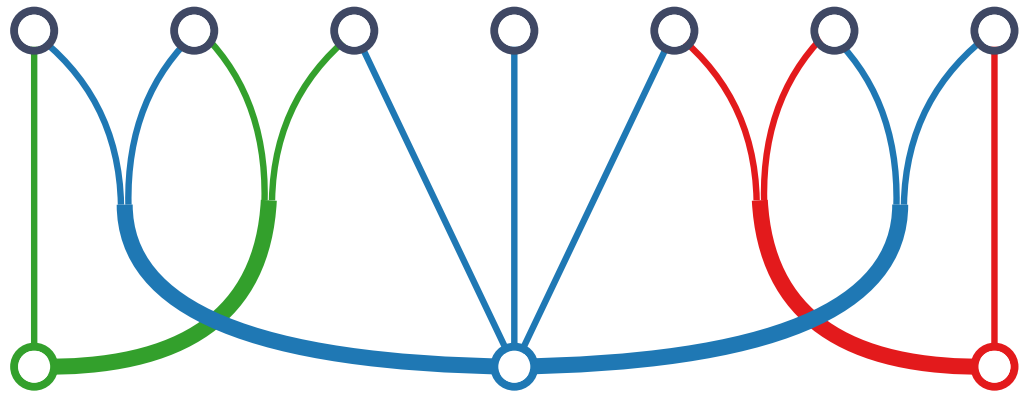
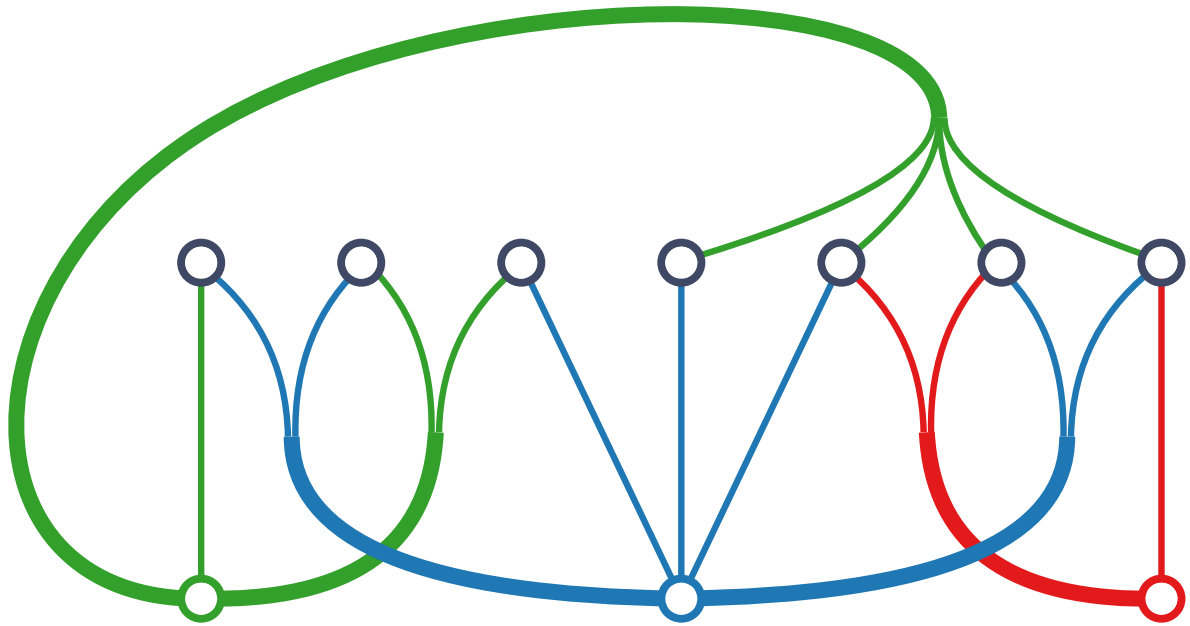


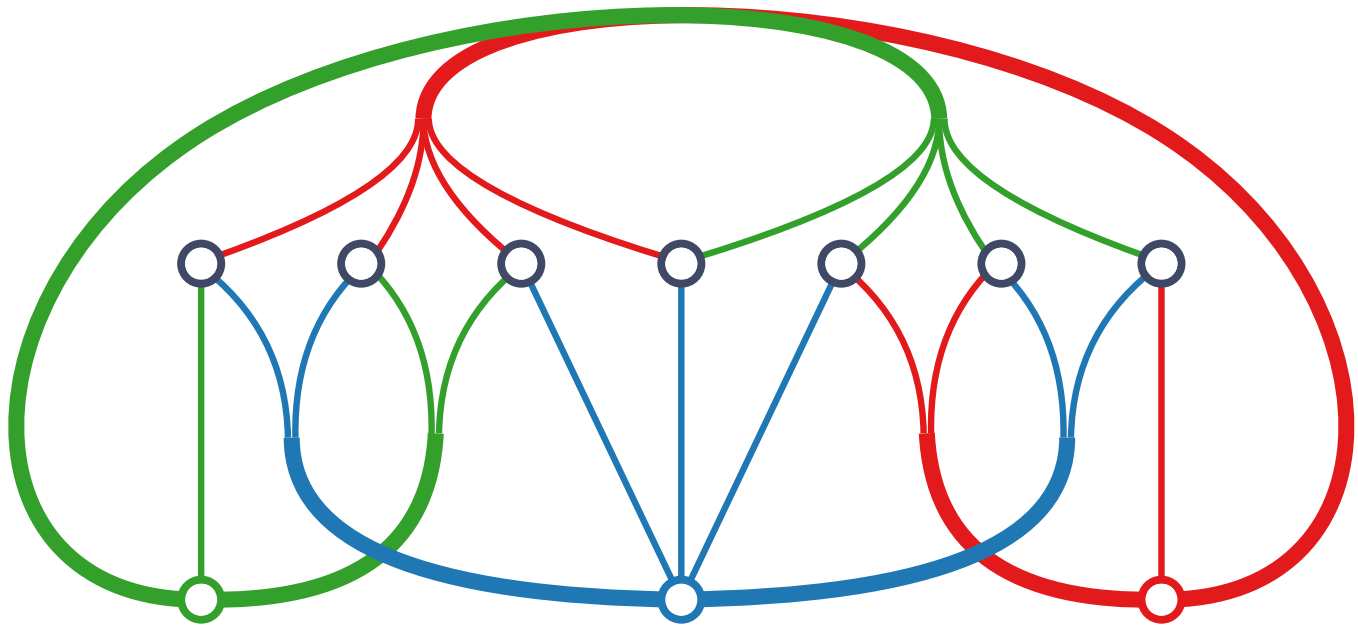
$K_{3,14}$



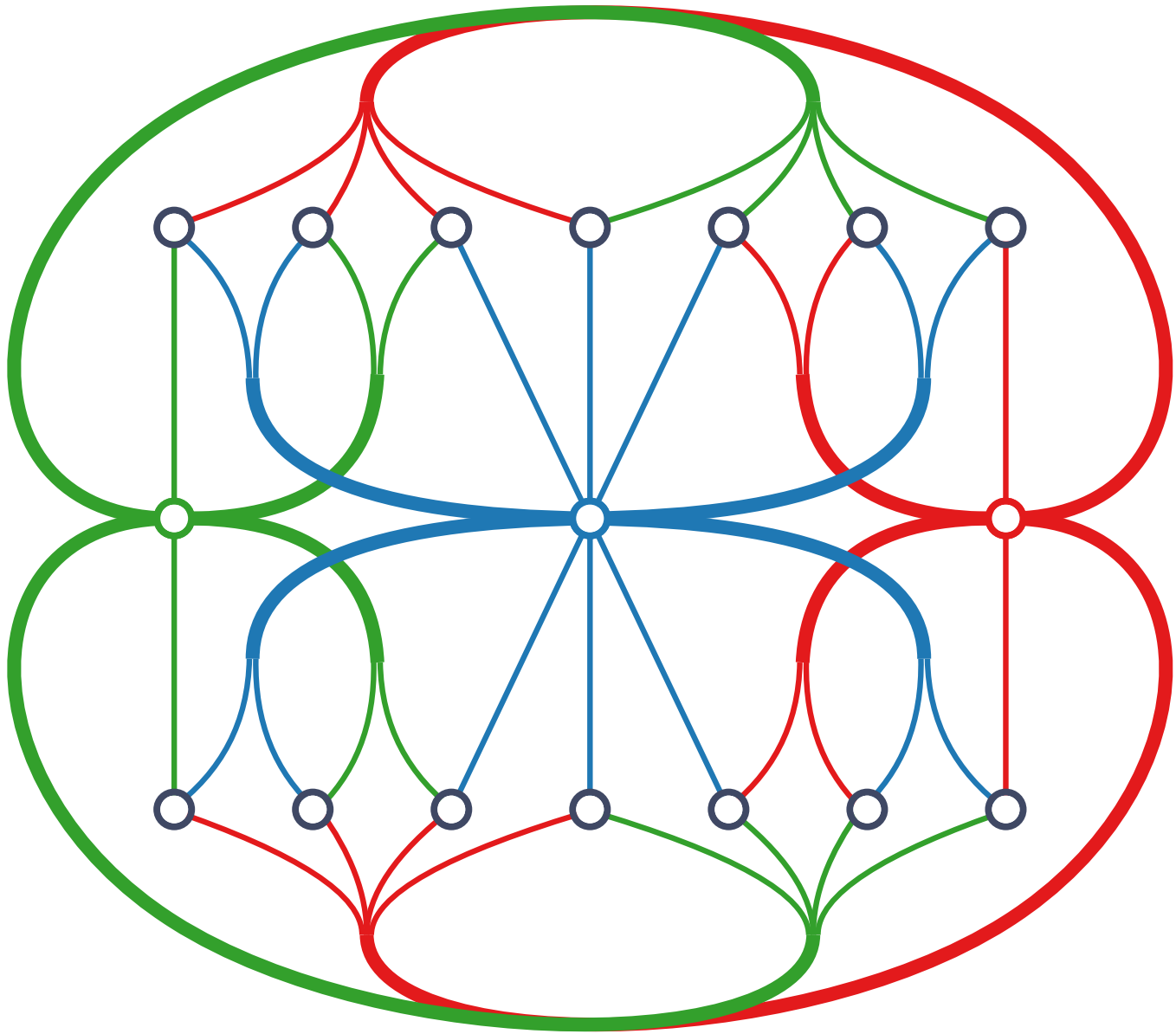
$K_{3,14}$



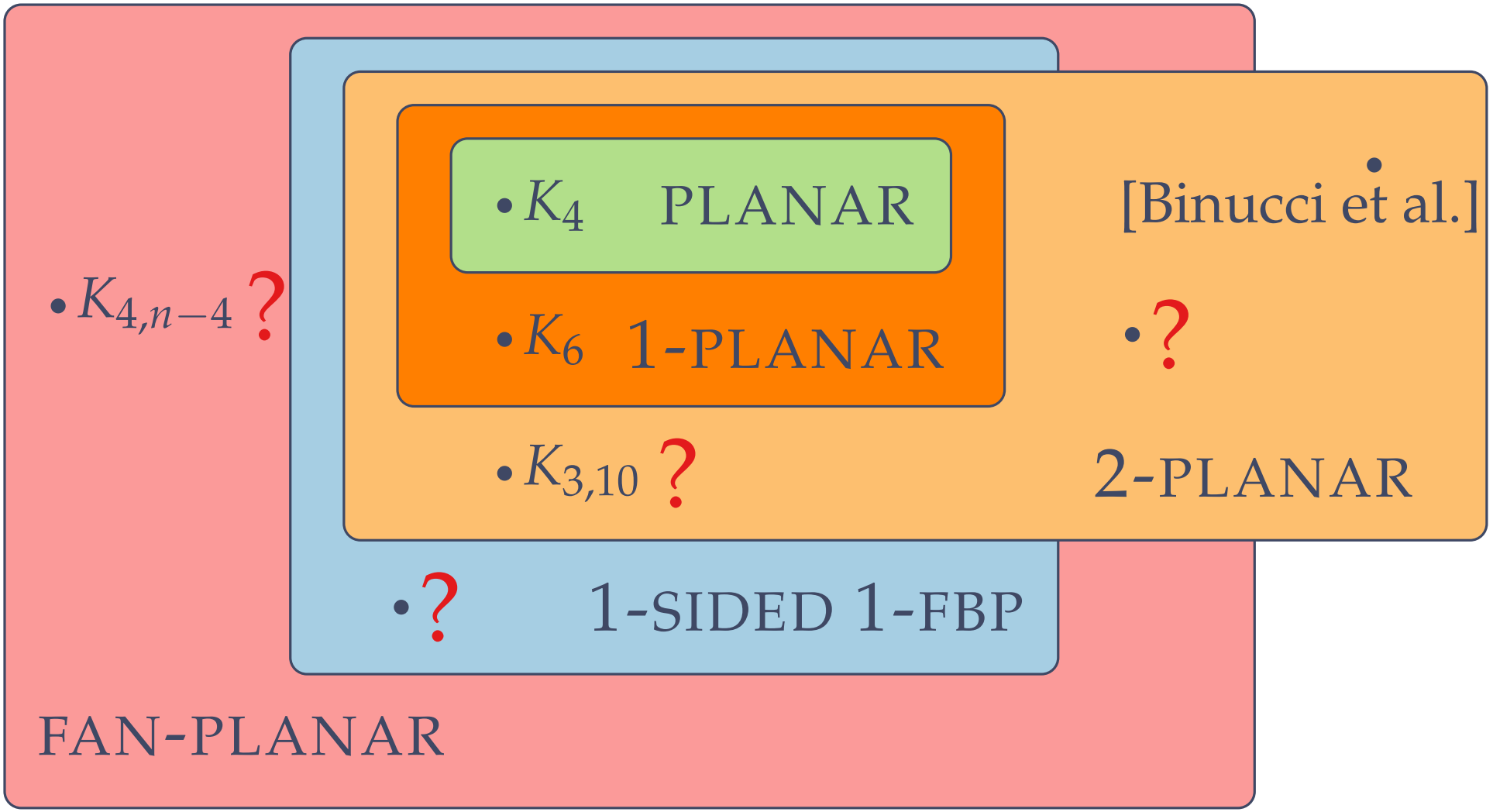
$K_{3,14}$



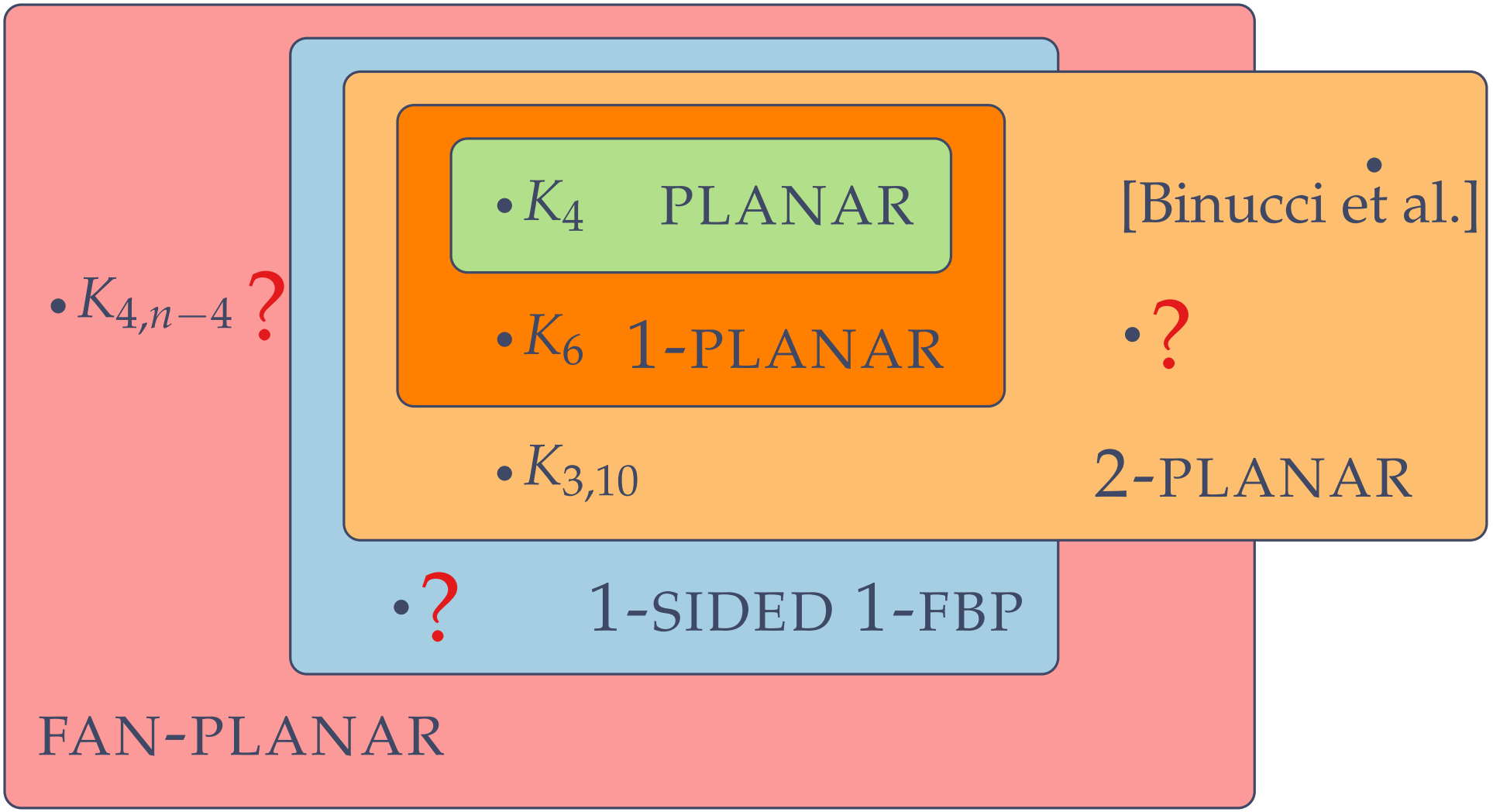
$K_{3,14}$



Relationships



Relationships



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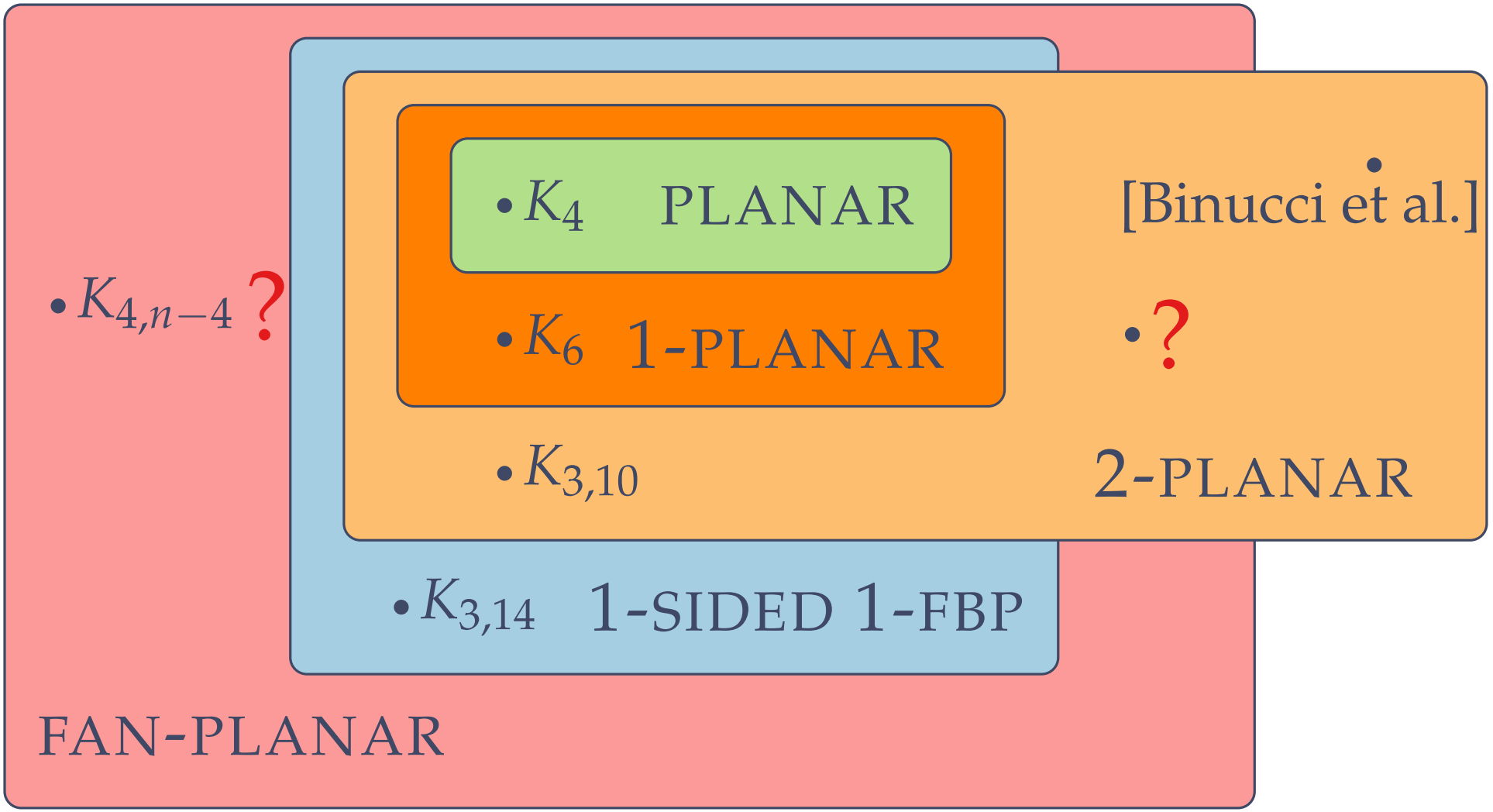

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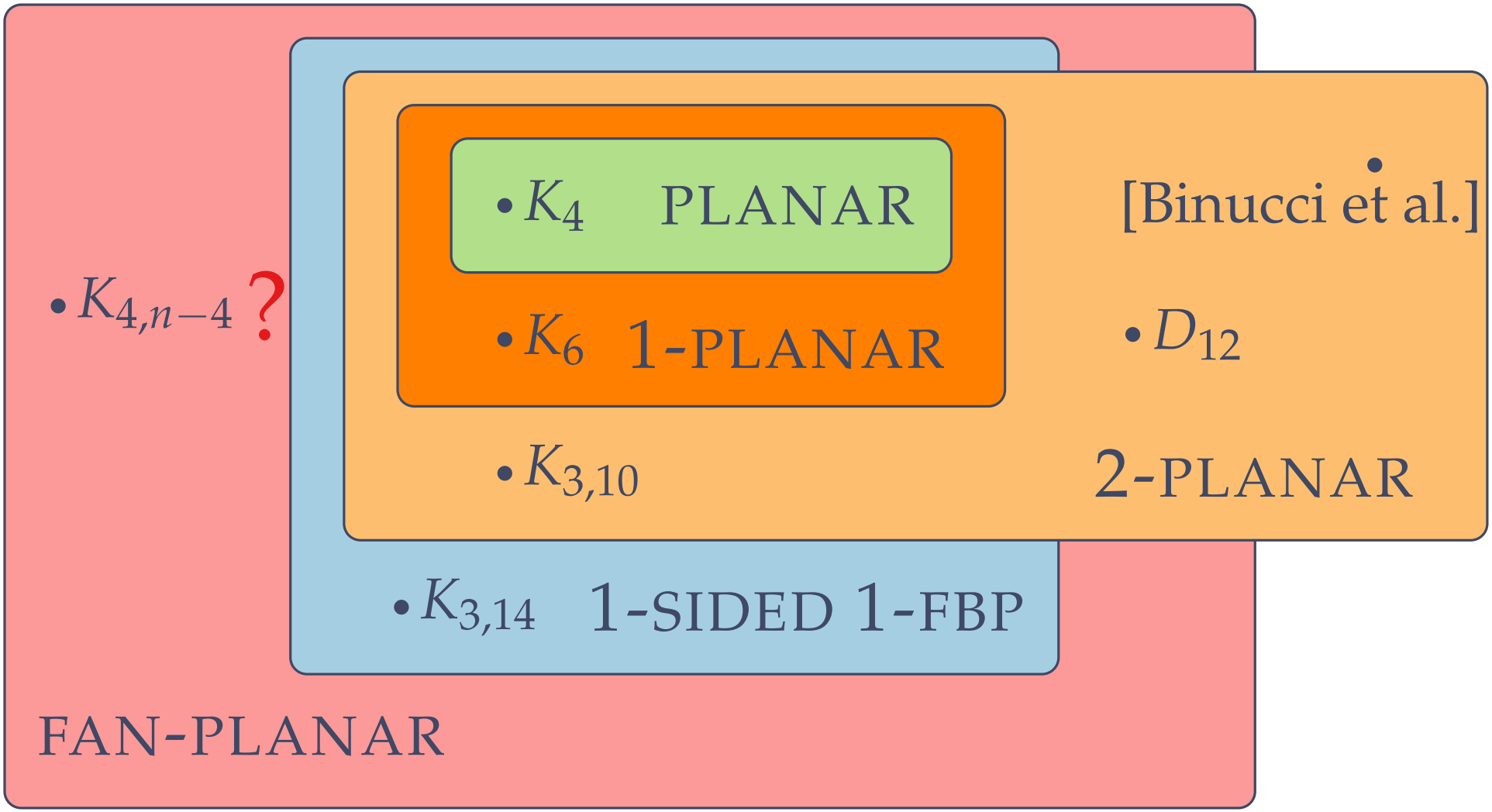

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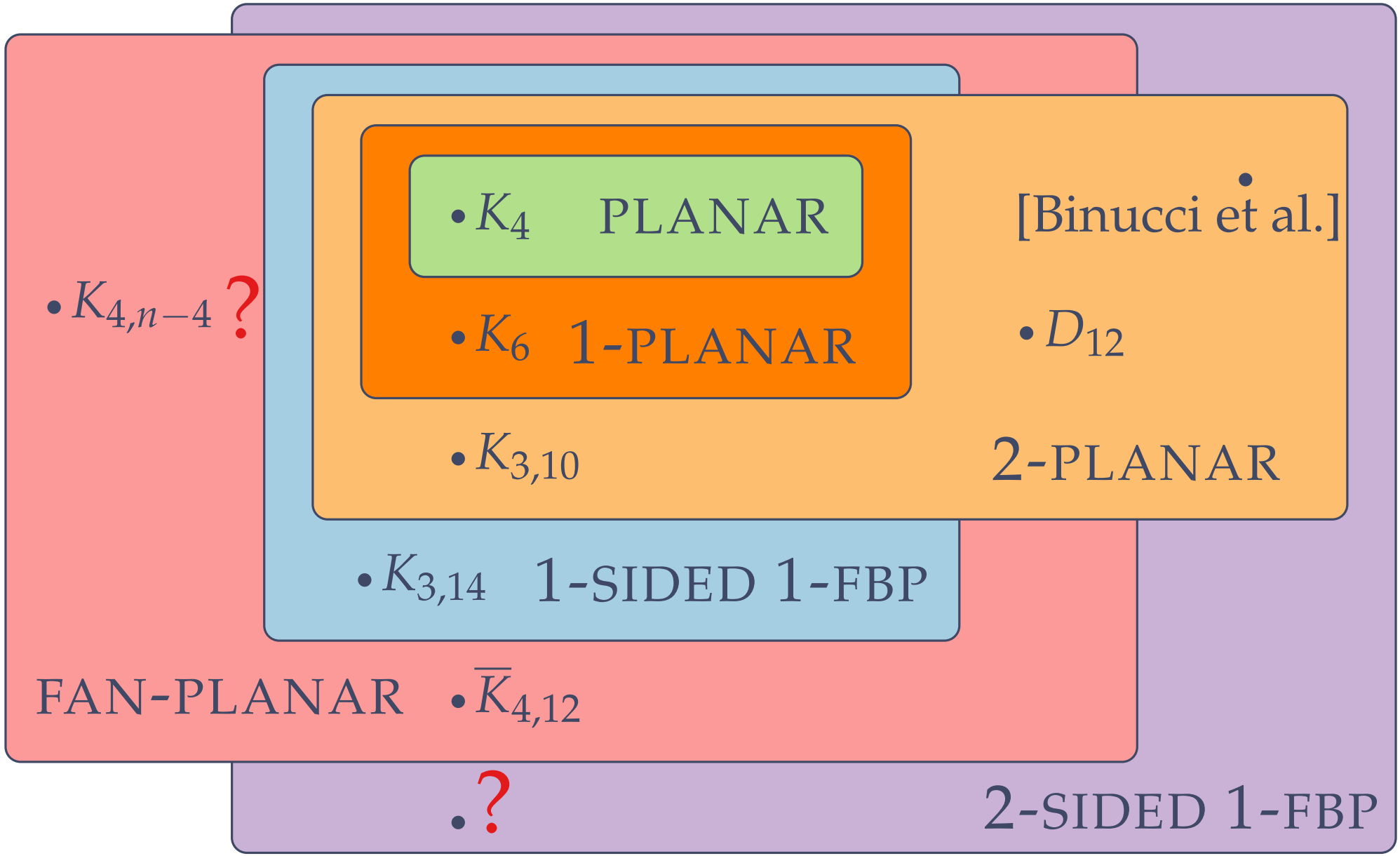
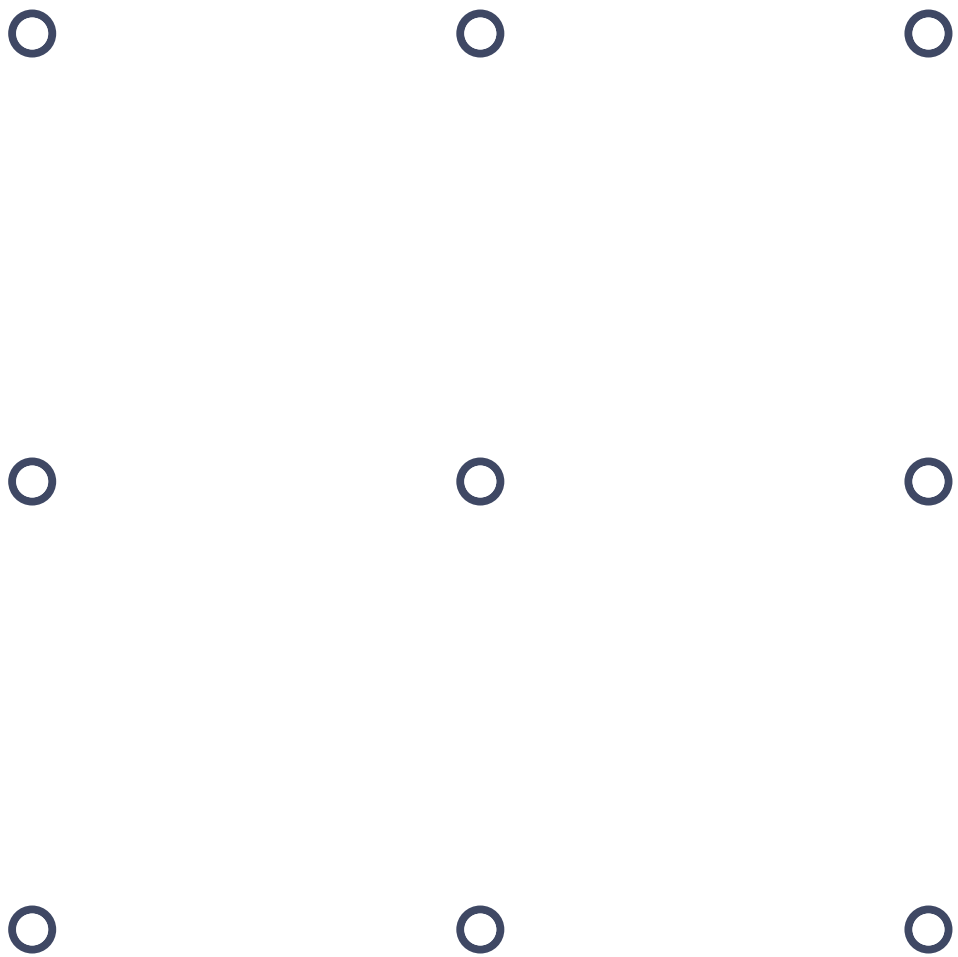
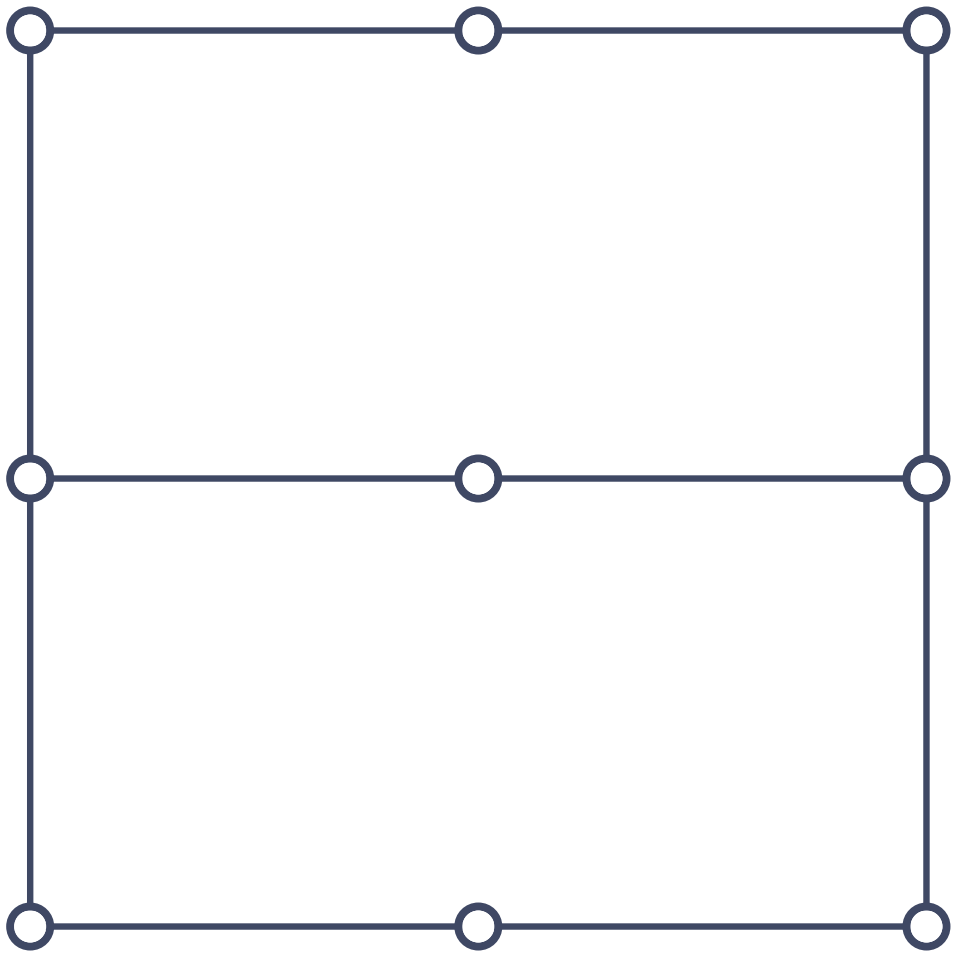


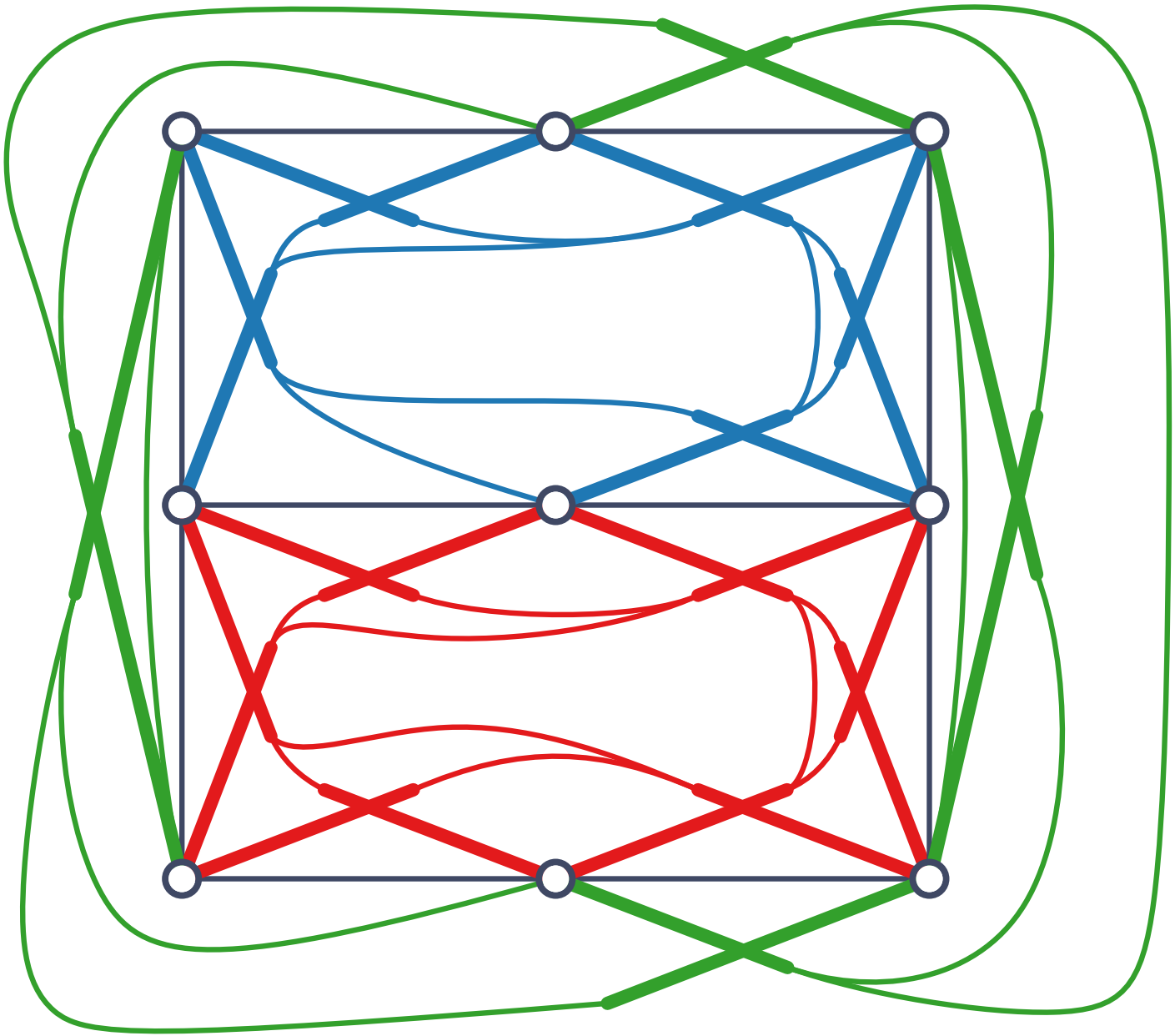
K_9



K_9



K_9



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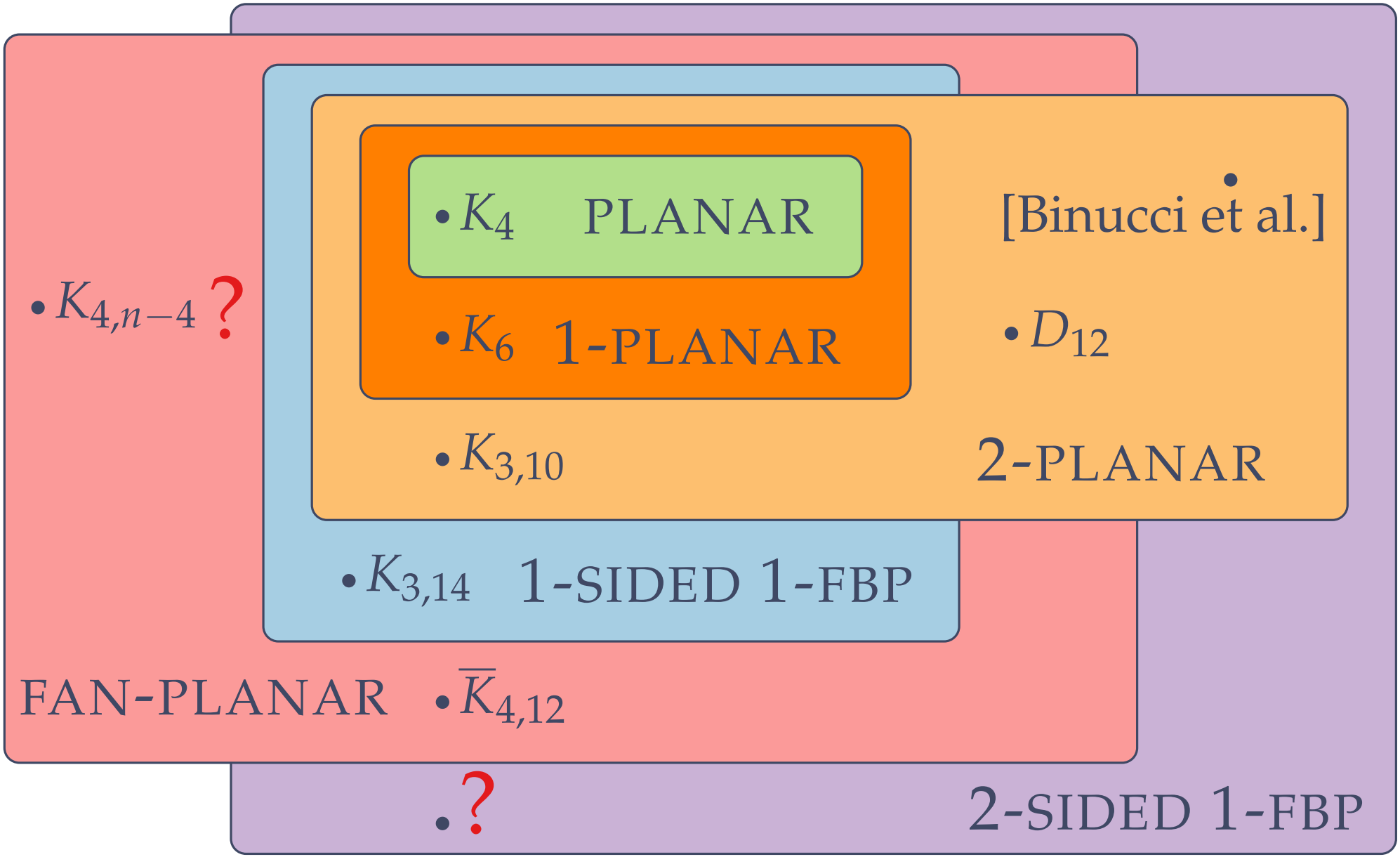

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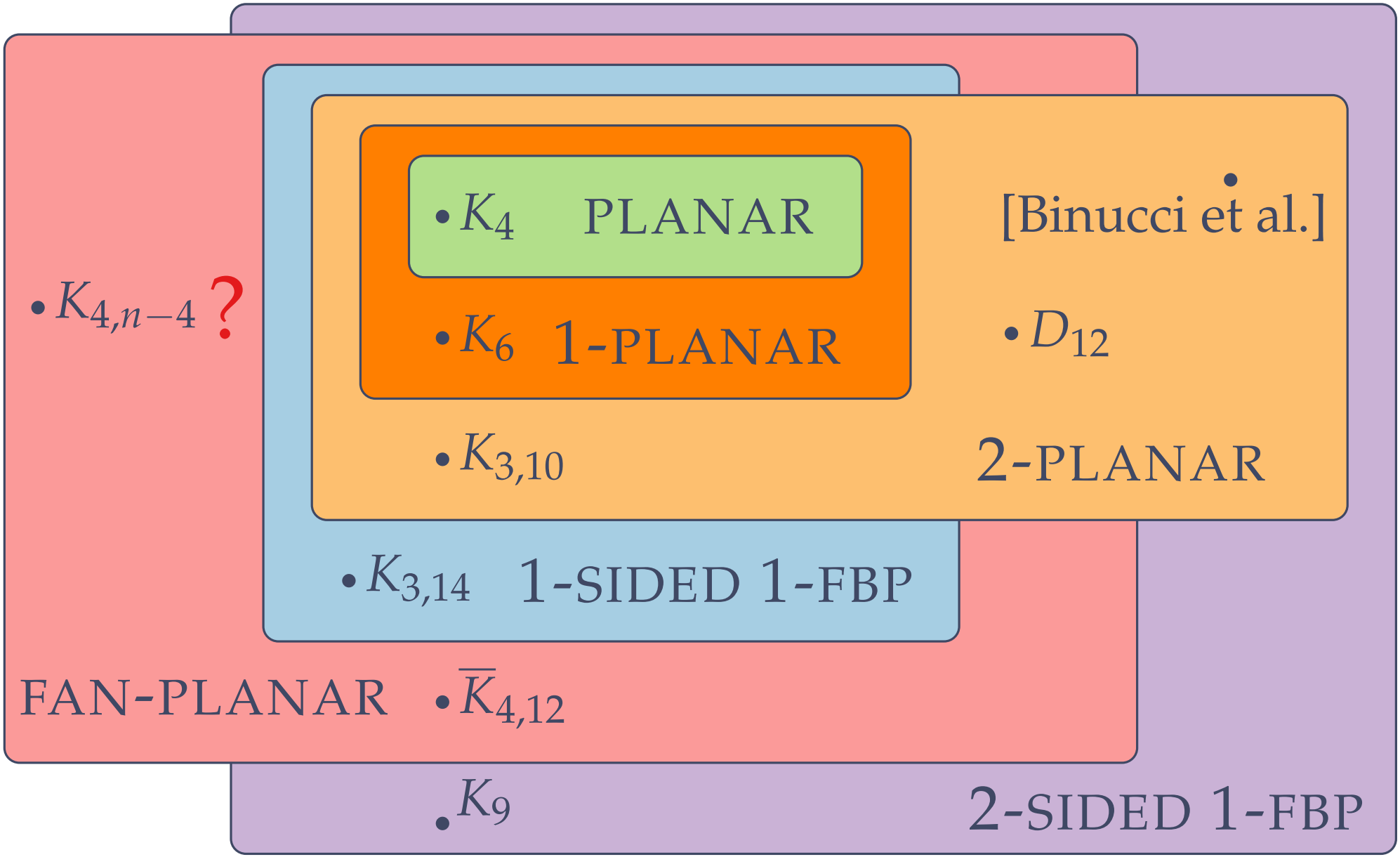
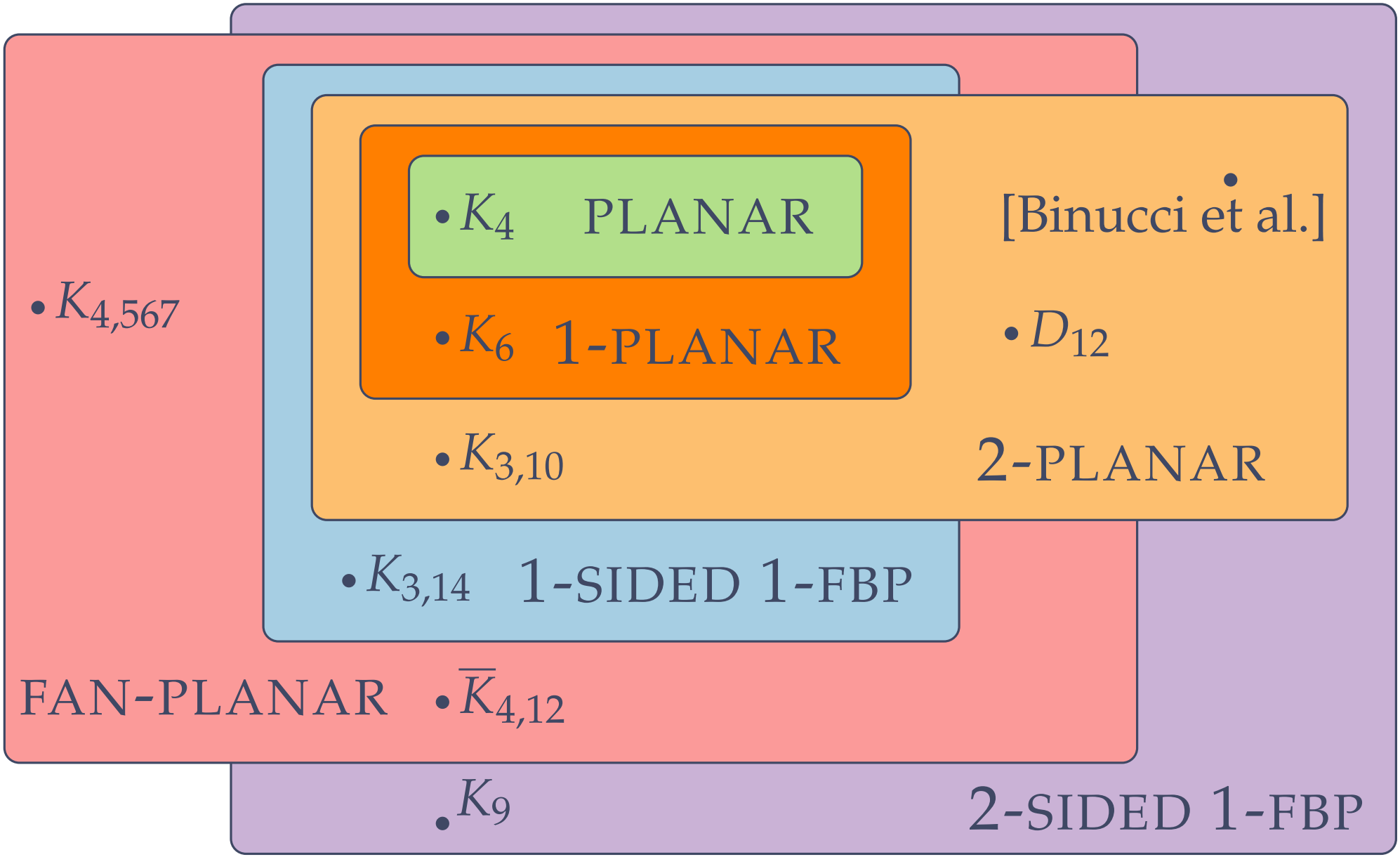
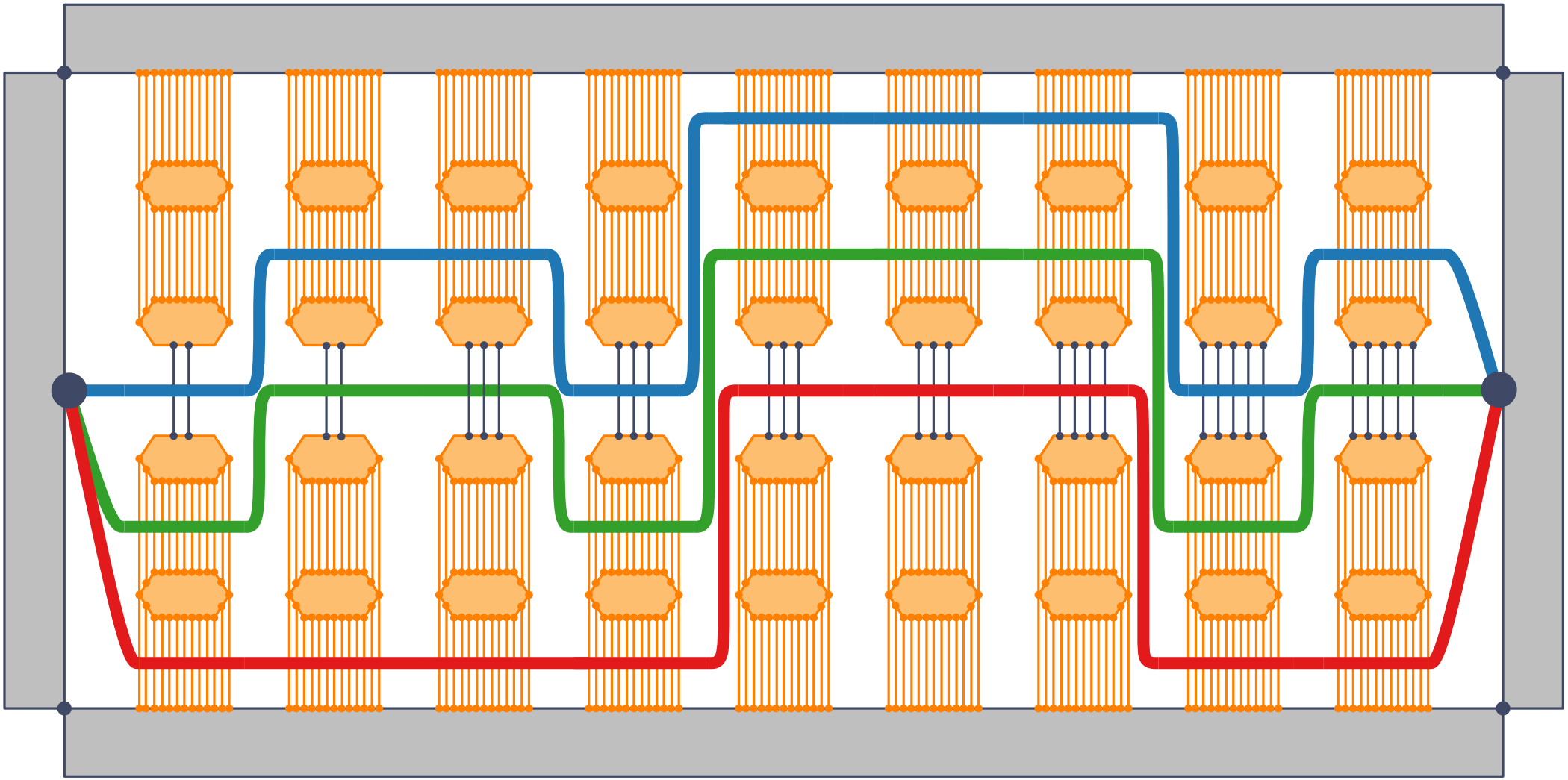


Relationships



Recognition

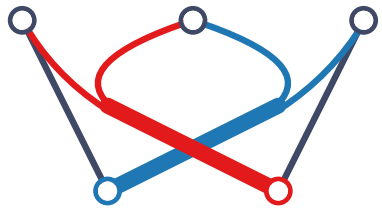
Recognition: general



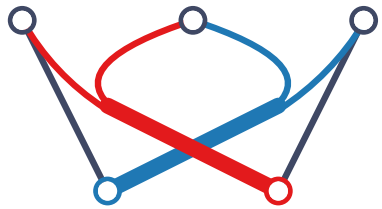
Recognition: 2-layer 1-sided

Recognition: 2-layer 1-sided

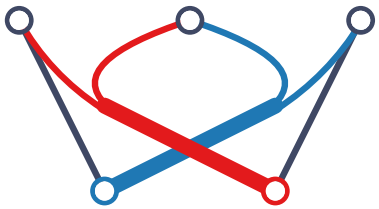
$K_{2,3}$ is drawable

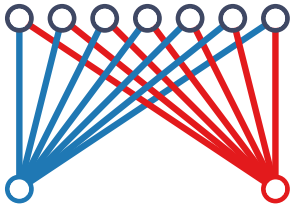


Recognition: 2-layer 1-sided

$K_{2,3}$ is drawable  ... but $K_{2,4}$ is not!

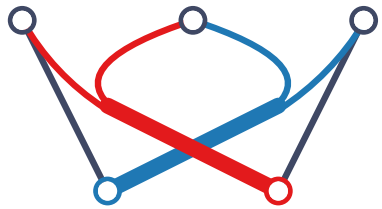
Recognition: 2-layer 1-sided

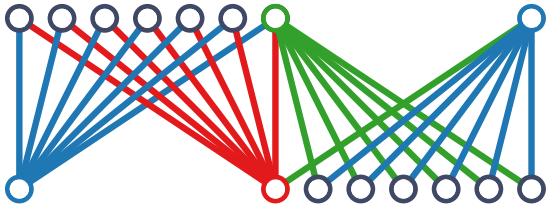
$K_{2,3}$ is drawable  ... but $K_{2,4}$ is not!



[Binucci et al.]

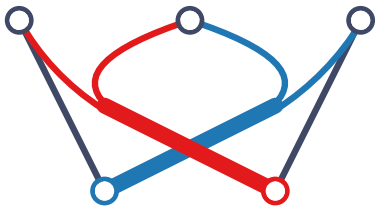
Recognition: 2-layer 1-sided

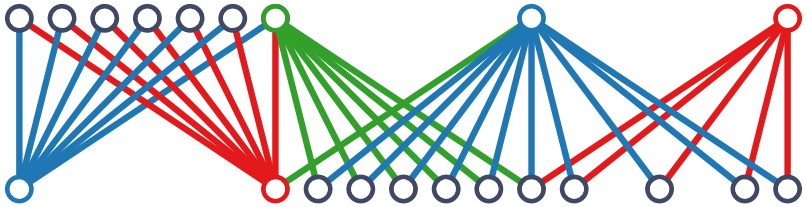
$K_{2,3}$ is drawable  ... but $K_{2,4}$ is not!



[Binucci et al.]

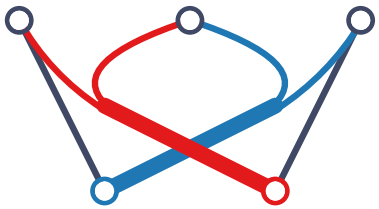
Recognition: 2-layer 1-sided

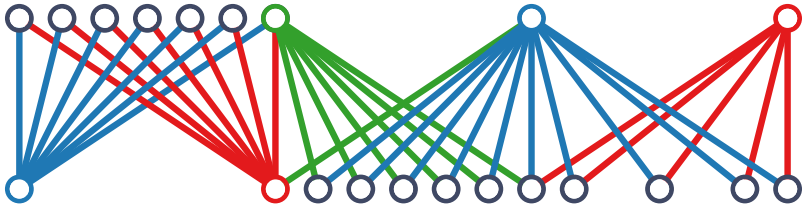
$K_{2,3}$ is drawable  ... but $K_{2,4}$ is not!



[Binucci et al.]

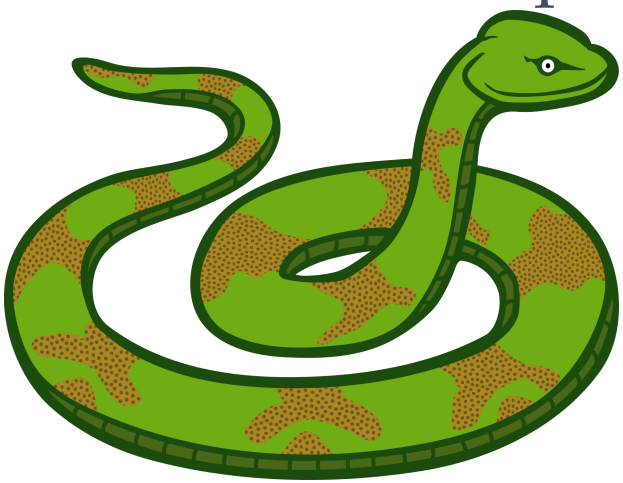
Recognition: 2-layer 1-sided

$K_{2,3}$ is drawable  ... but $K_{2,4}$ is not!

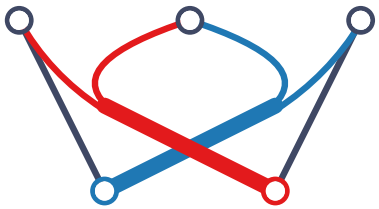


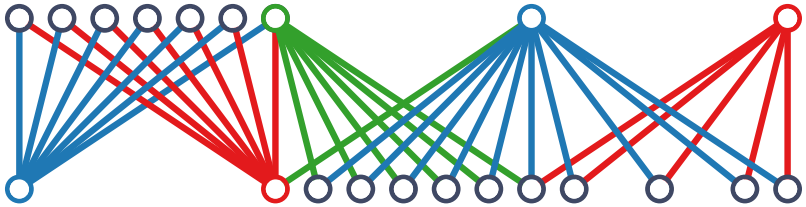
[Binucci et al.]

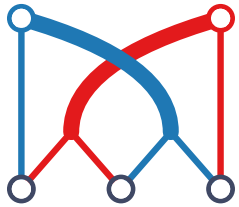
Max. bicon. fan-planar:



Recognition: 2-layer 1-sided

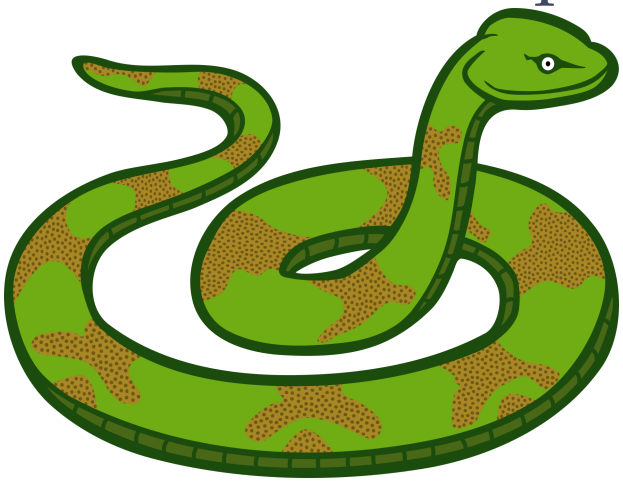
$K_{2,3}$ is drawable  ... but $K_{2,4}$ is not!



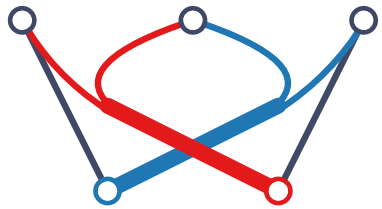


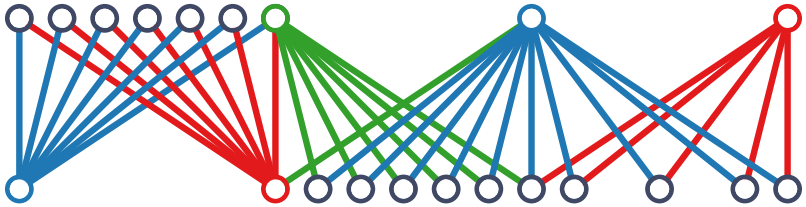
[Binucci et al.]

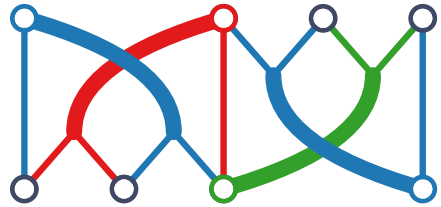
Max. bicon. fan-planar:



Recognition: 2-layer 1-sided

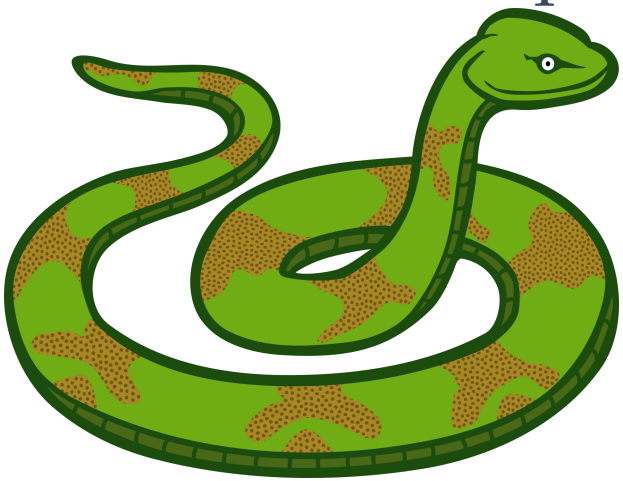
$K_{2,3}$ is drawable  ... but $K_{2,4}$ is not!



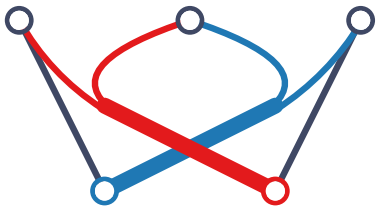


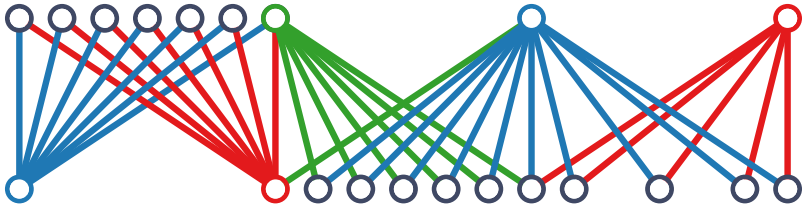
[Binucci et al.]

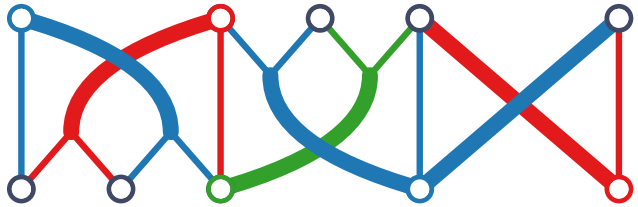
Max. bicon. fan-planar:



Recognition: 2-layer 1-sided

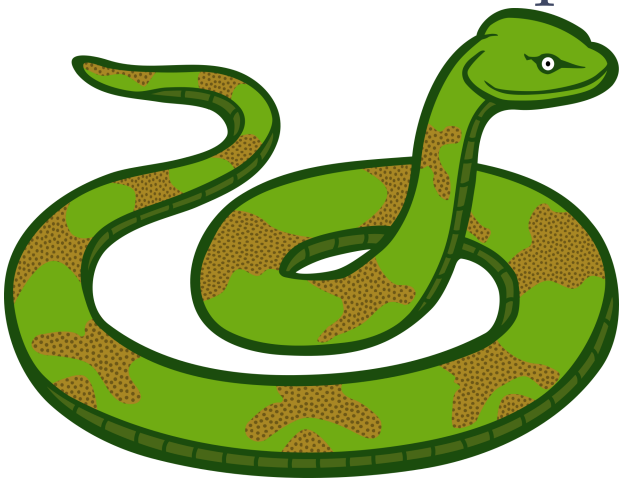
$K_{2,3}$ is drawable  ... but $K_{2,4}$ is not!



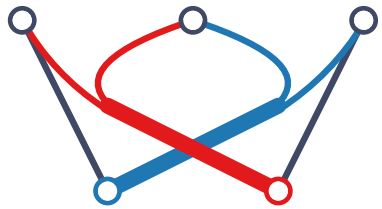


[Binucci et al.]

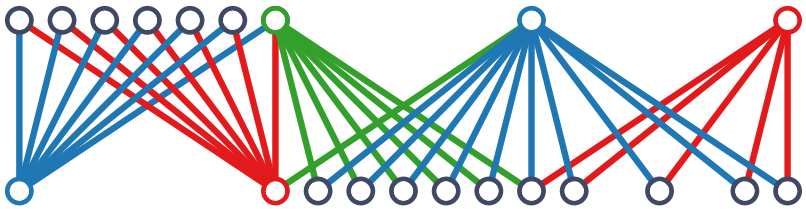
Max. bicon. fan-planar:

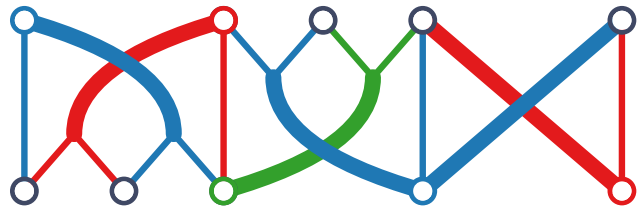


Recognition: 2-layer 1-sided

$K_{2,3}$ is drawable 

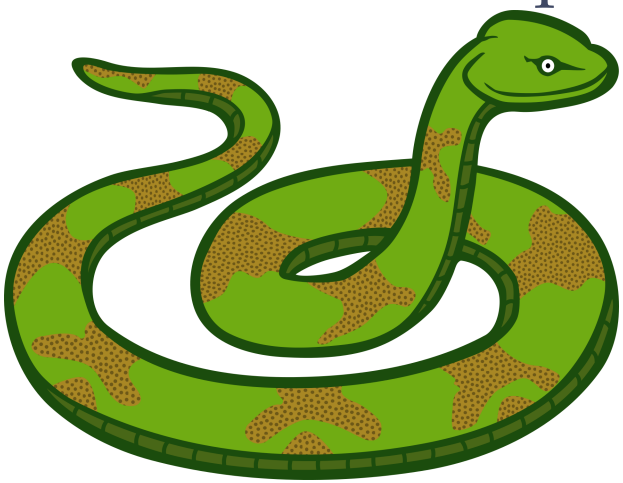
... but $K_{2,4}$ is not!



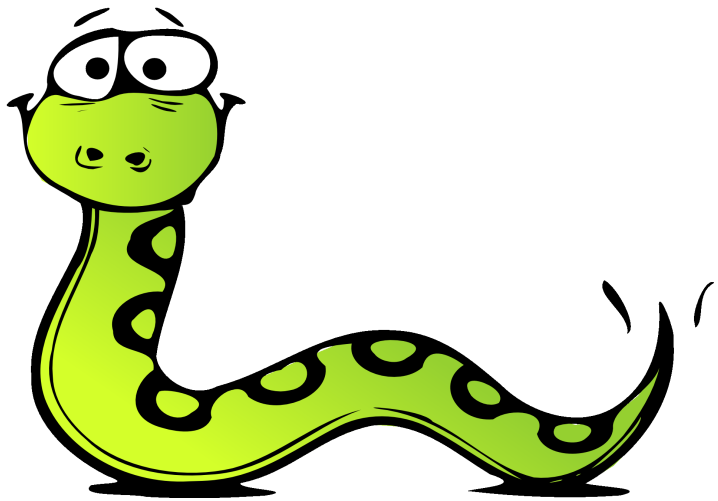


[Binucci et al.]

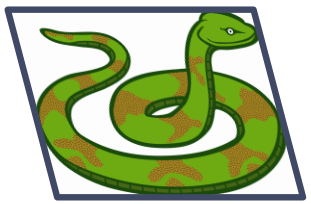
Max. bicon. fan-planar:



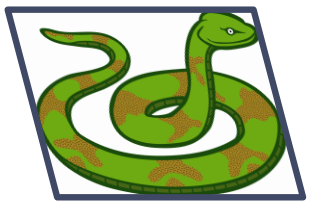
Max. bicon. 1-sided 1-fbp:



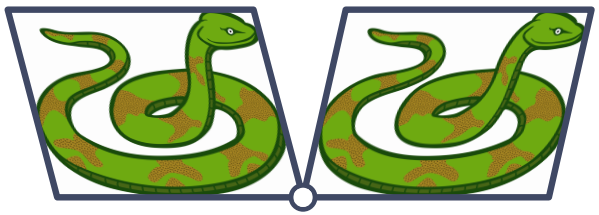
Recognition: 2-layer 1-sided



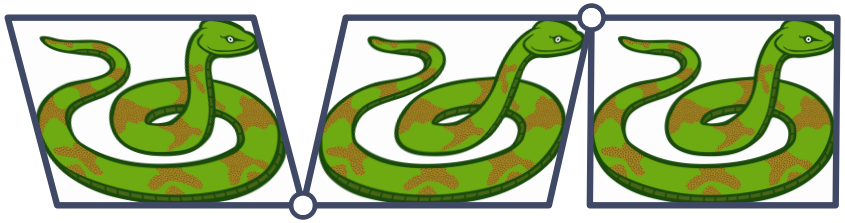
Recognition: 2-layer 1-sided



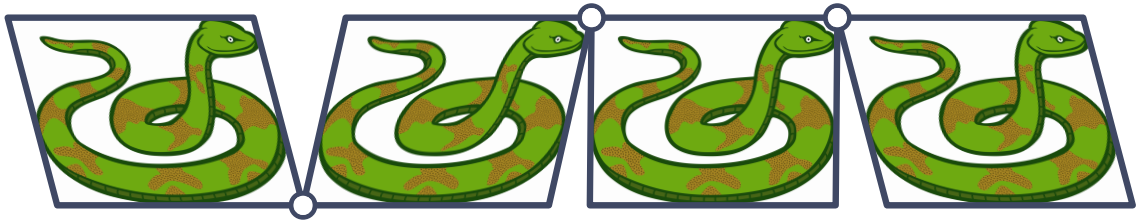
Recognition: 2-layer 1-sided



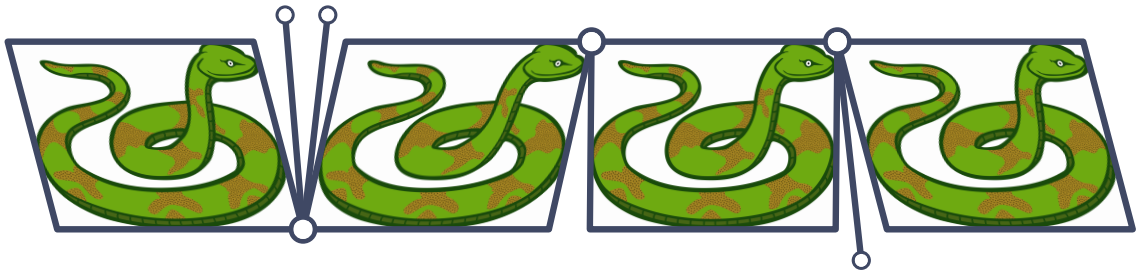
Recognition: 2-layer 1-sided



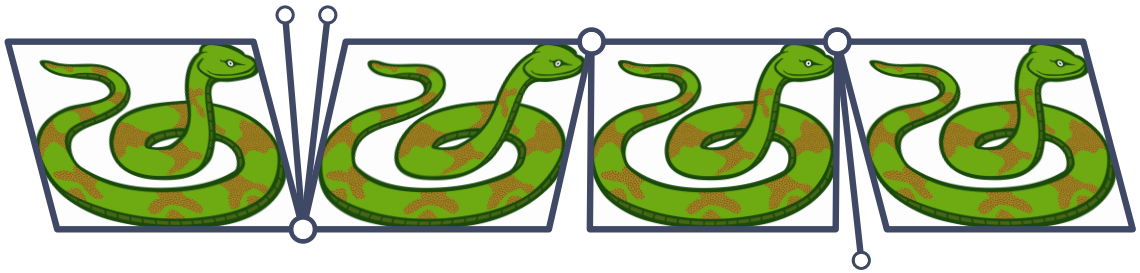
Recognition: 2-layer 1-sided



Recognition: 2-layer 1-sided



Recognition: 2-layer 1-sided



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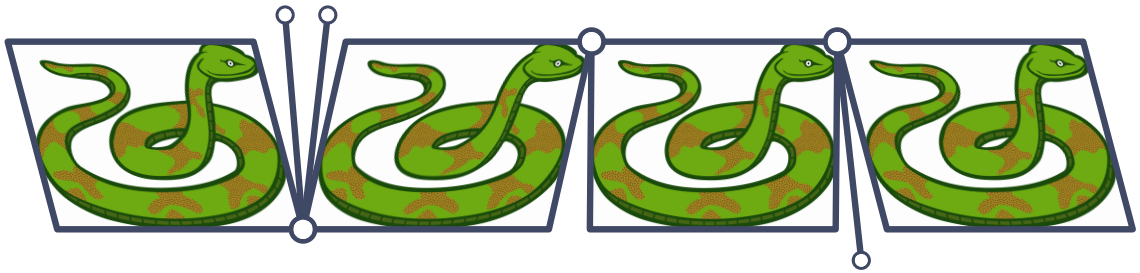
Recognition: 2-layer 1-sided



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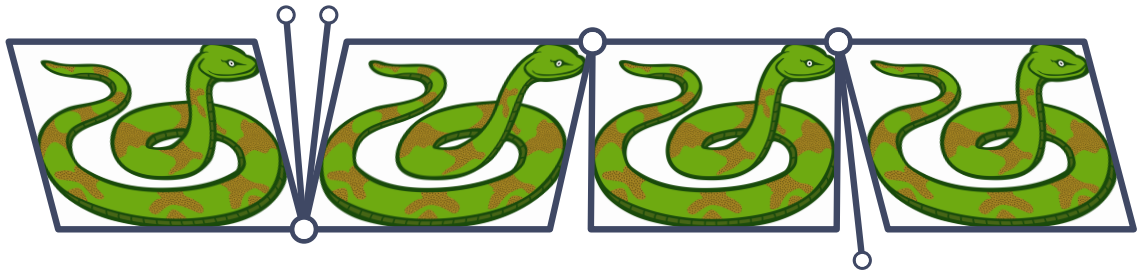
Max. fan-planar =

Recognition: 2-layer 1-sided



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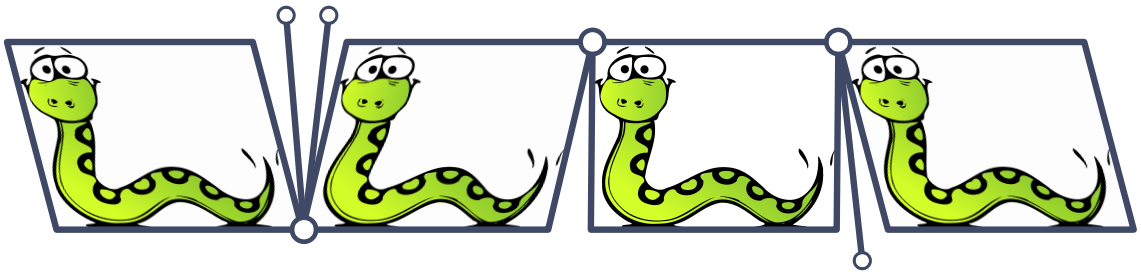
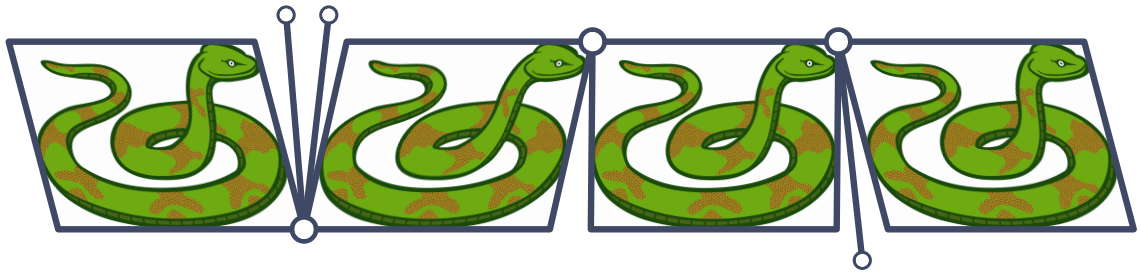
Max. fan-planar =



Recognition: 2-layer 1-sided



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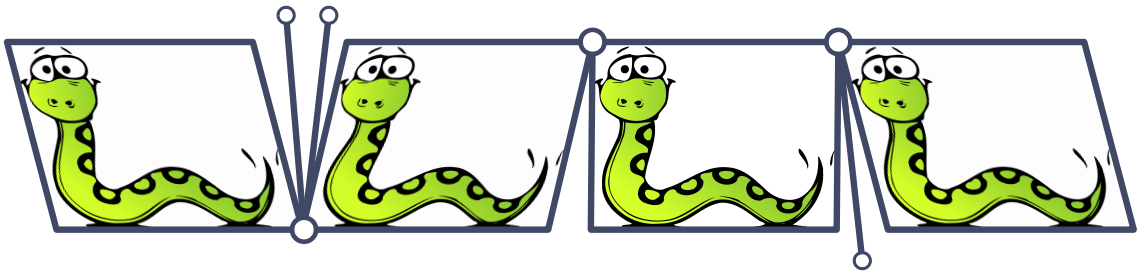
Max. fan-planar =



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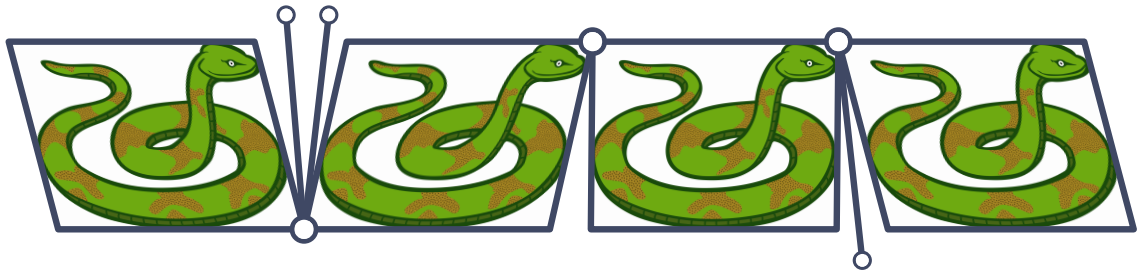
Recognition: 2-layer 1-sided



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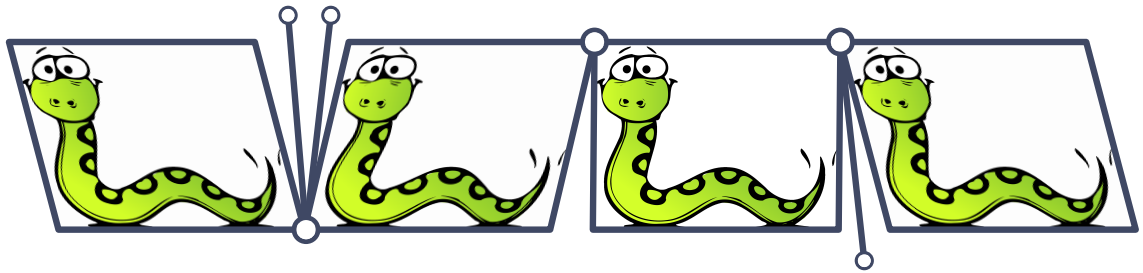
Max. fan-planar =



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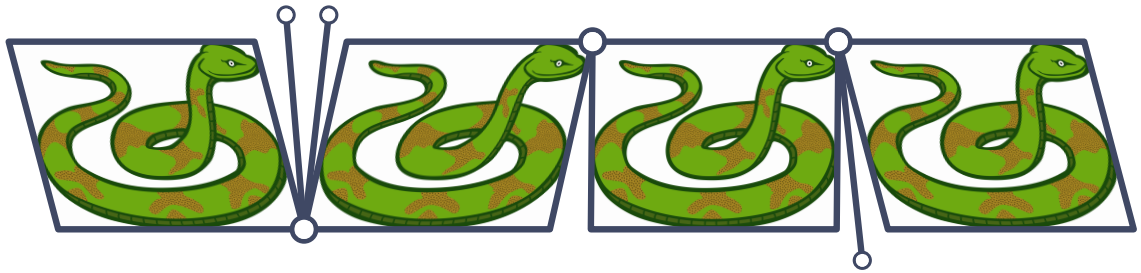
Max. 1-sided 1-fbp =

Recognition: 2-layer 1-sided



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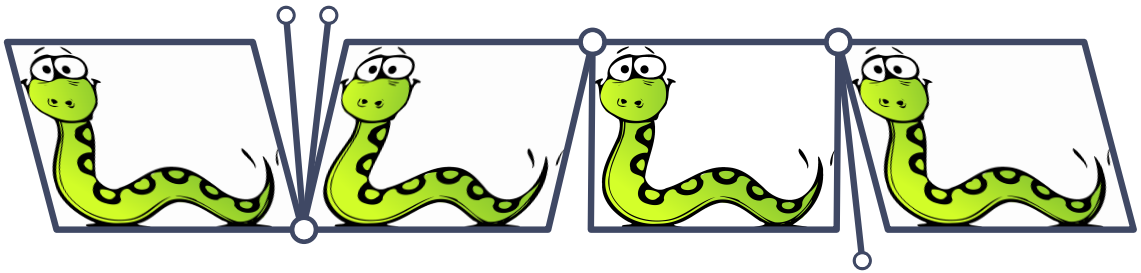
Max. fan-planar $=$



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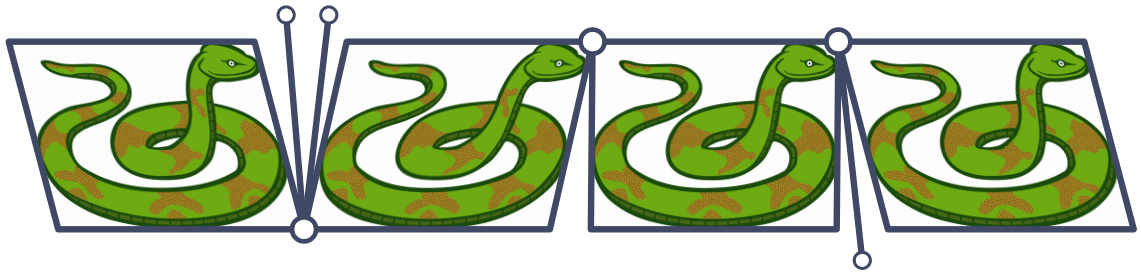
Max. 1-sided 1-fbp $\neq$

Recognition: 2-layer 1-sided



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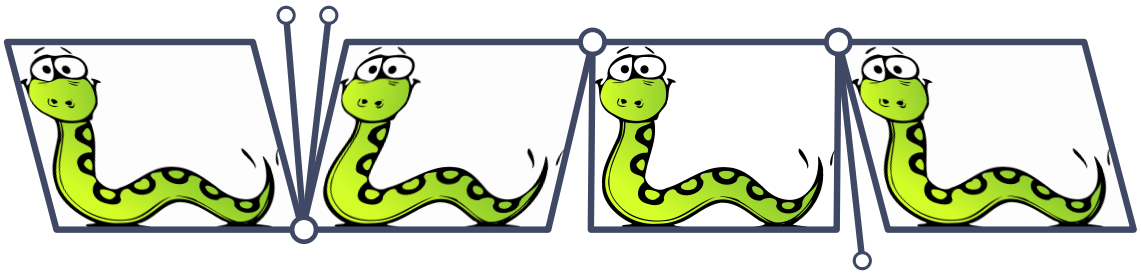
Max. fan-planar $=$



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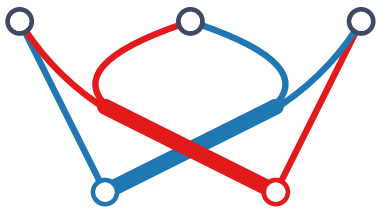
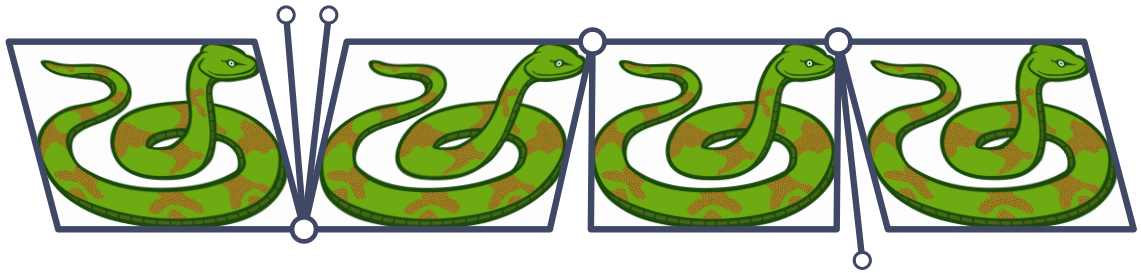
Max. 1-sided 1-fbp $\neq$



Recognition: 2-layer 1-sided



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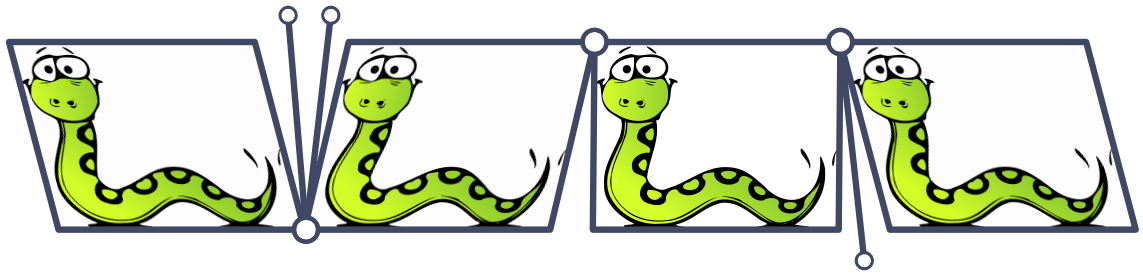
Max. fan-planar $=$



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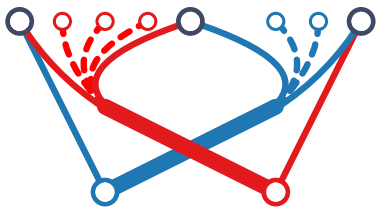
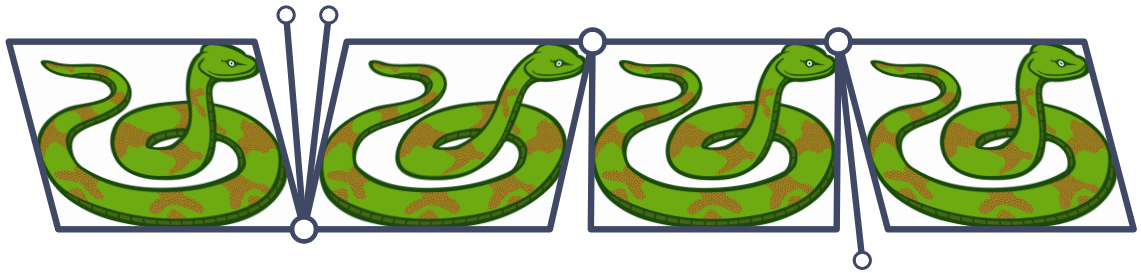
Max. 1-sided 1-fbp $\neq$



Recognition: 2-layer 1-sided



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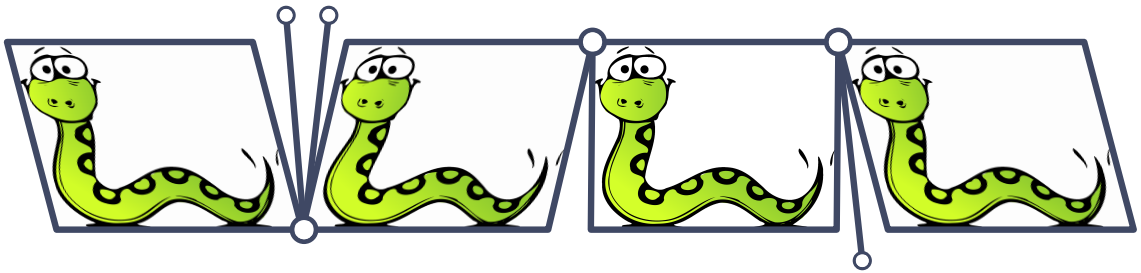
Max. fan-planar $=$



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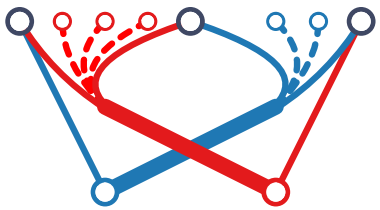
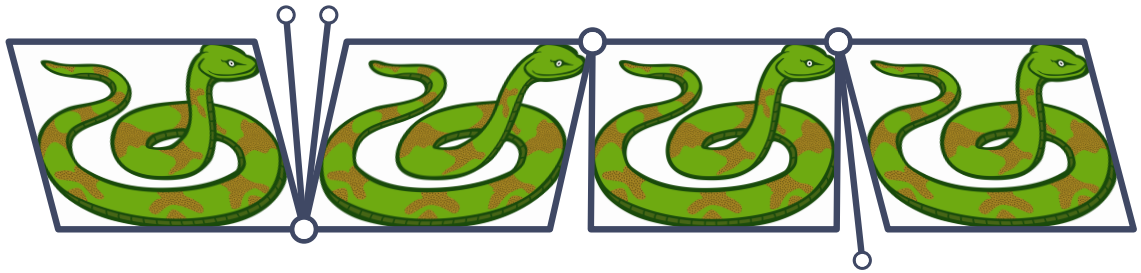
Max. 1-sided 1-fbp $\neq$



Big Legs

Recognition: 2-layer 1-sided



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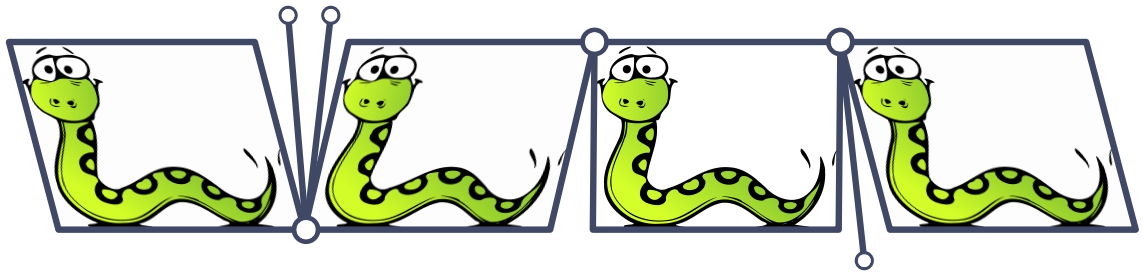
Max. fan-planar $=$



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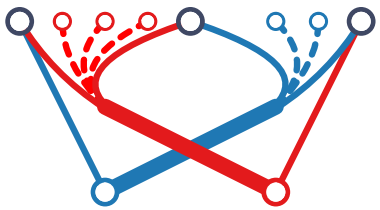
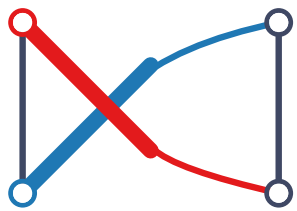
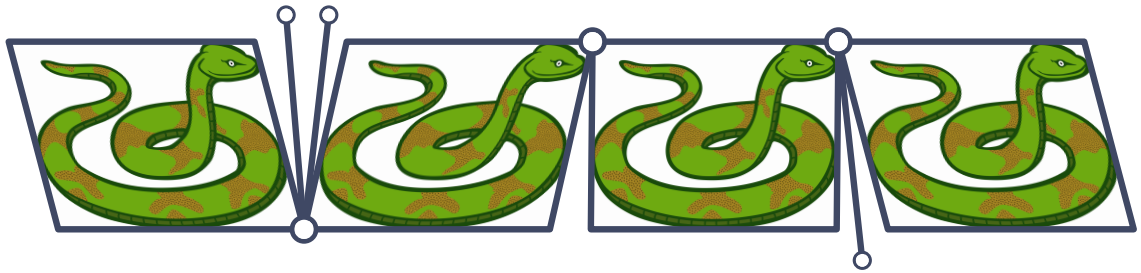
Max. 1-sided 1-fbp $\neq$



Big Legs



Recognition: 2-layer 1-sided



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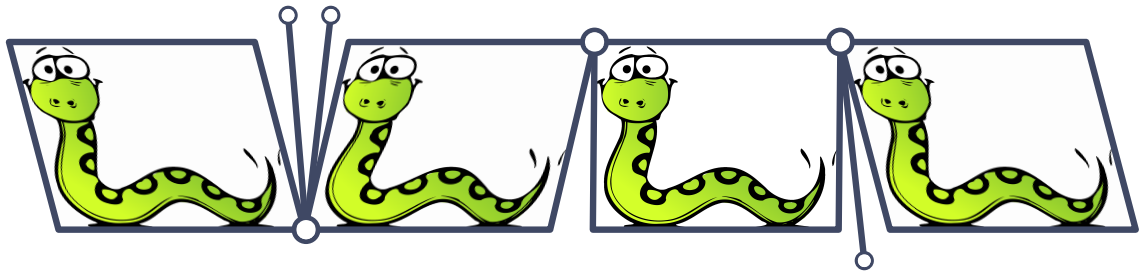
Max. fan-planar $=$



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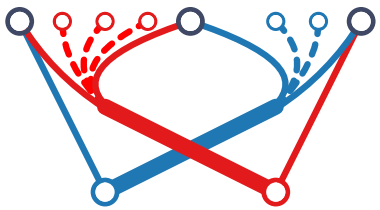
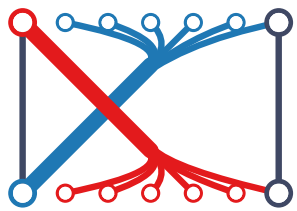
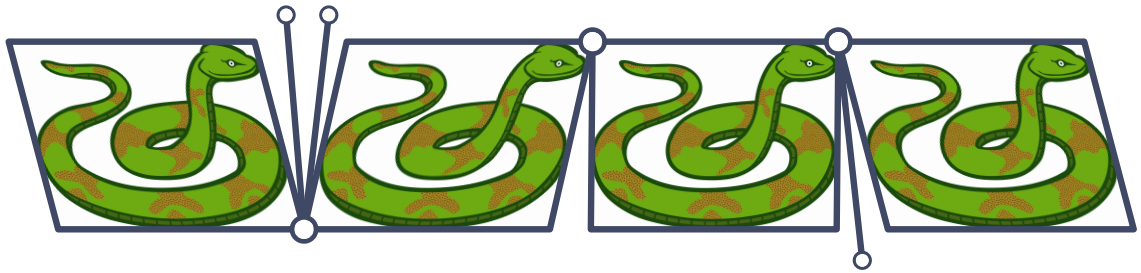
Max. 1-sided 1-fbp $\neq$



Big Legs



Recognition: 2-layer 1-sided



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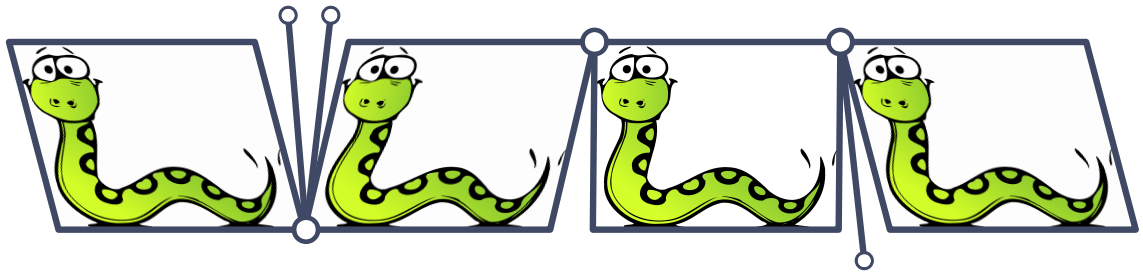
Max. fan-planar $=$



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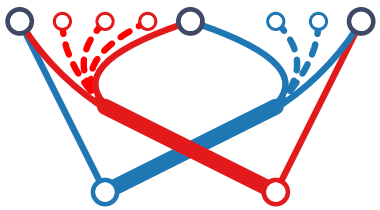
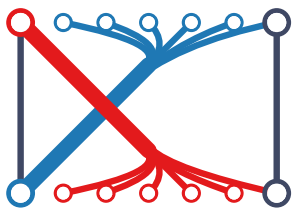
Max. 1-sided 1-fbp $\neq$

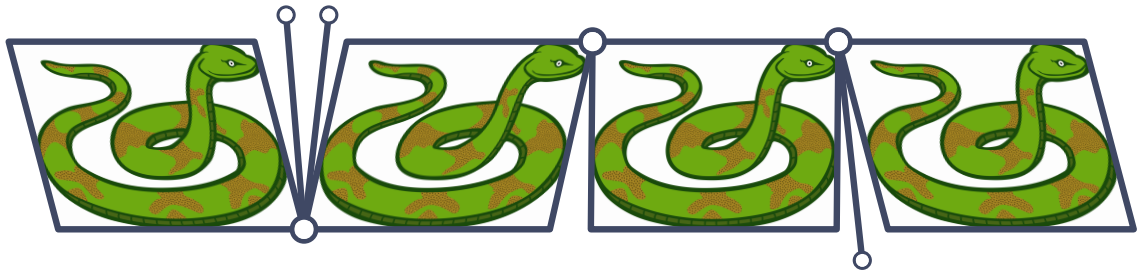


Big Legs



Small Legs

Recognition: 2-layer 1-sided



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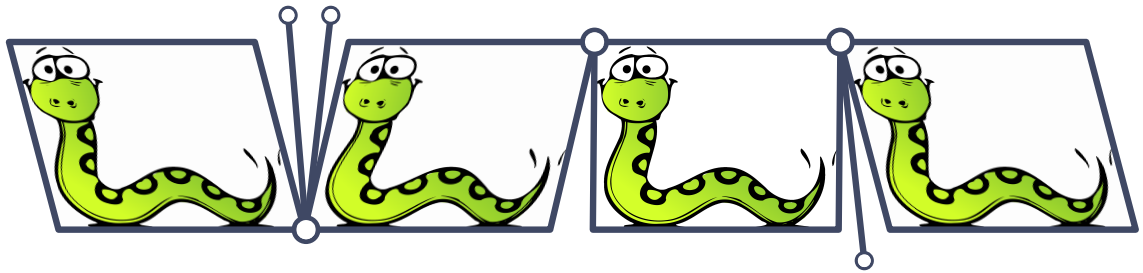
Max. fan-planar $=$



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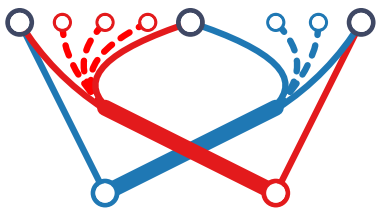
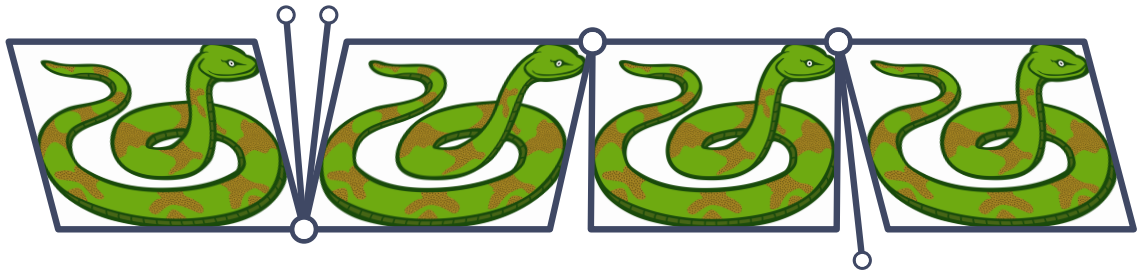
Max. 1-sided 1-fbp $\neq$



Big Legs

Small Legs

Recognition: 2-layer 1-sided



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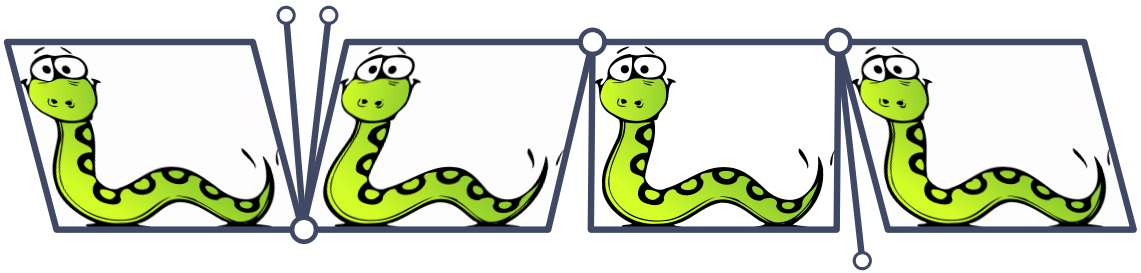
Max. fan-planar $=$



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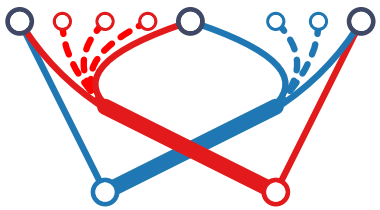
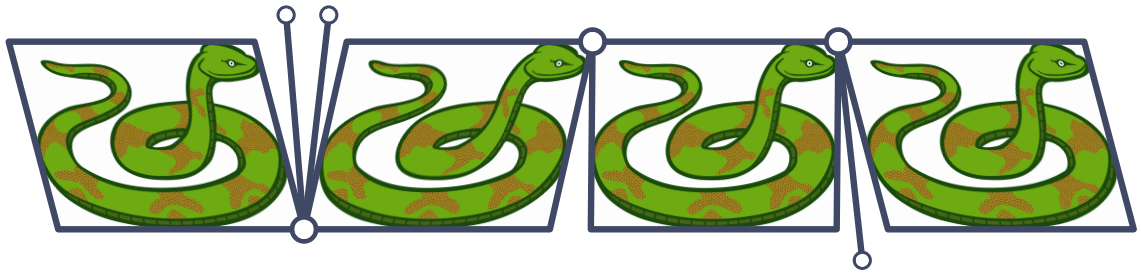
Max. 1-sided 1-fbp $\neq$



Big Legs

Recognition: 2-layer 1-sided



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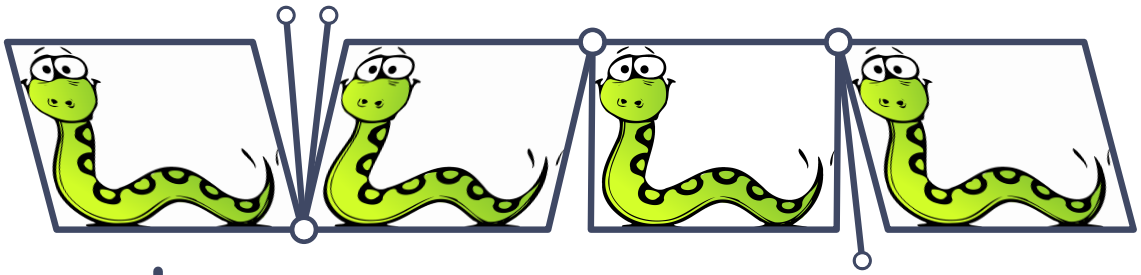
Max. fan-planar $=$



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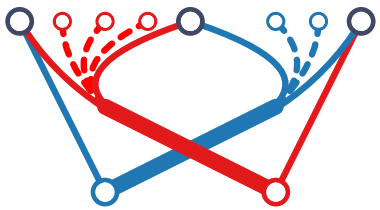
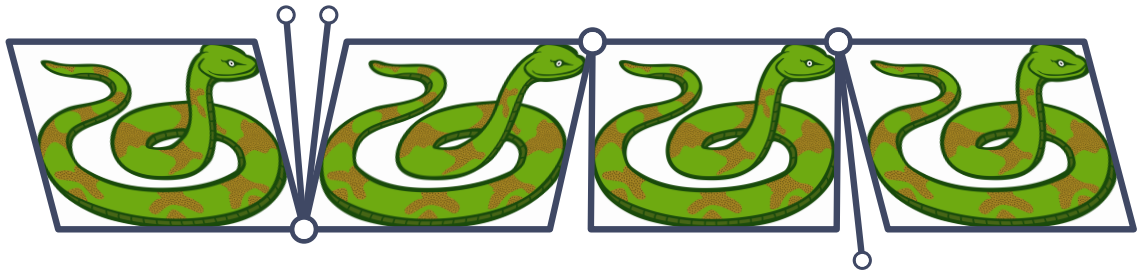
Max. 1-sided 1-fbp $\neq$



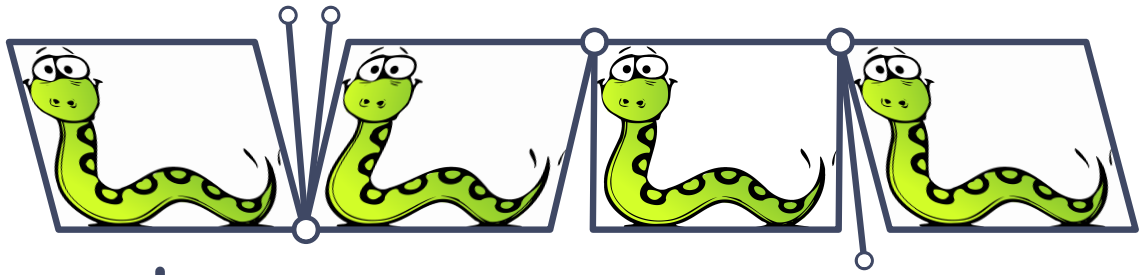
Big Legs

Recognition: 2-layer 1-sided



$=$

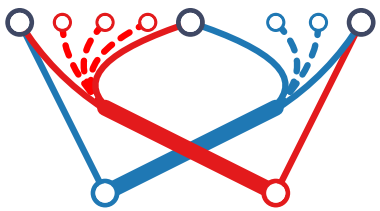
Max. fan-planar $=$



$=$

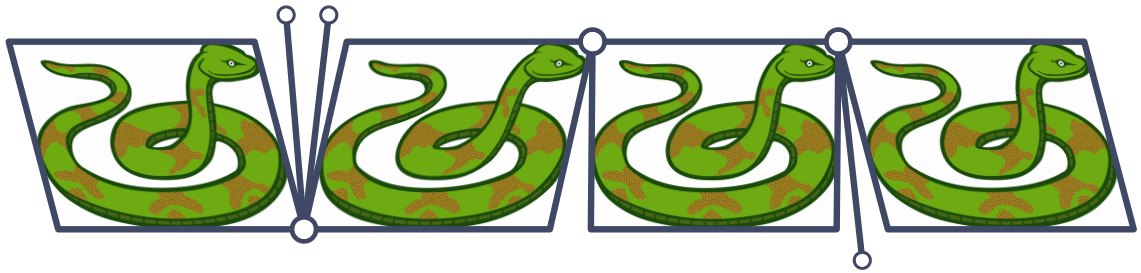
$+$

Max. 1-sided 1-fbp $\neq$



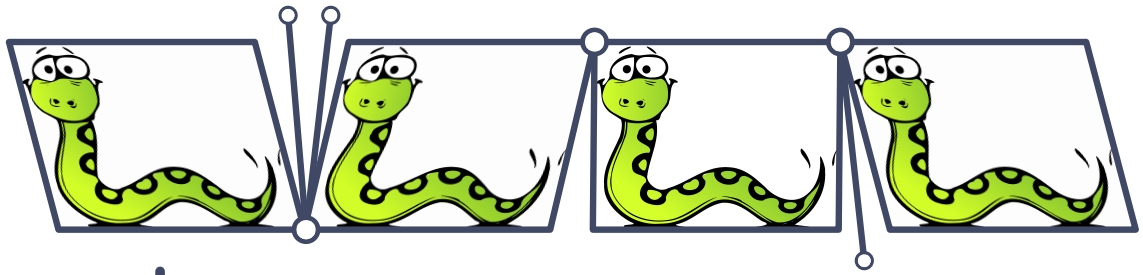
Big Legs $=$

Recognition: 2-layer 1-sided



=

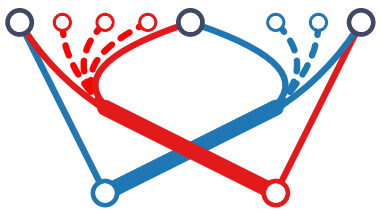
Max. fan-planar =



=

+

Max. 1-sided 1-fbp ≠



Big Legs =

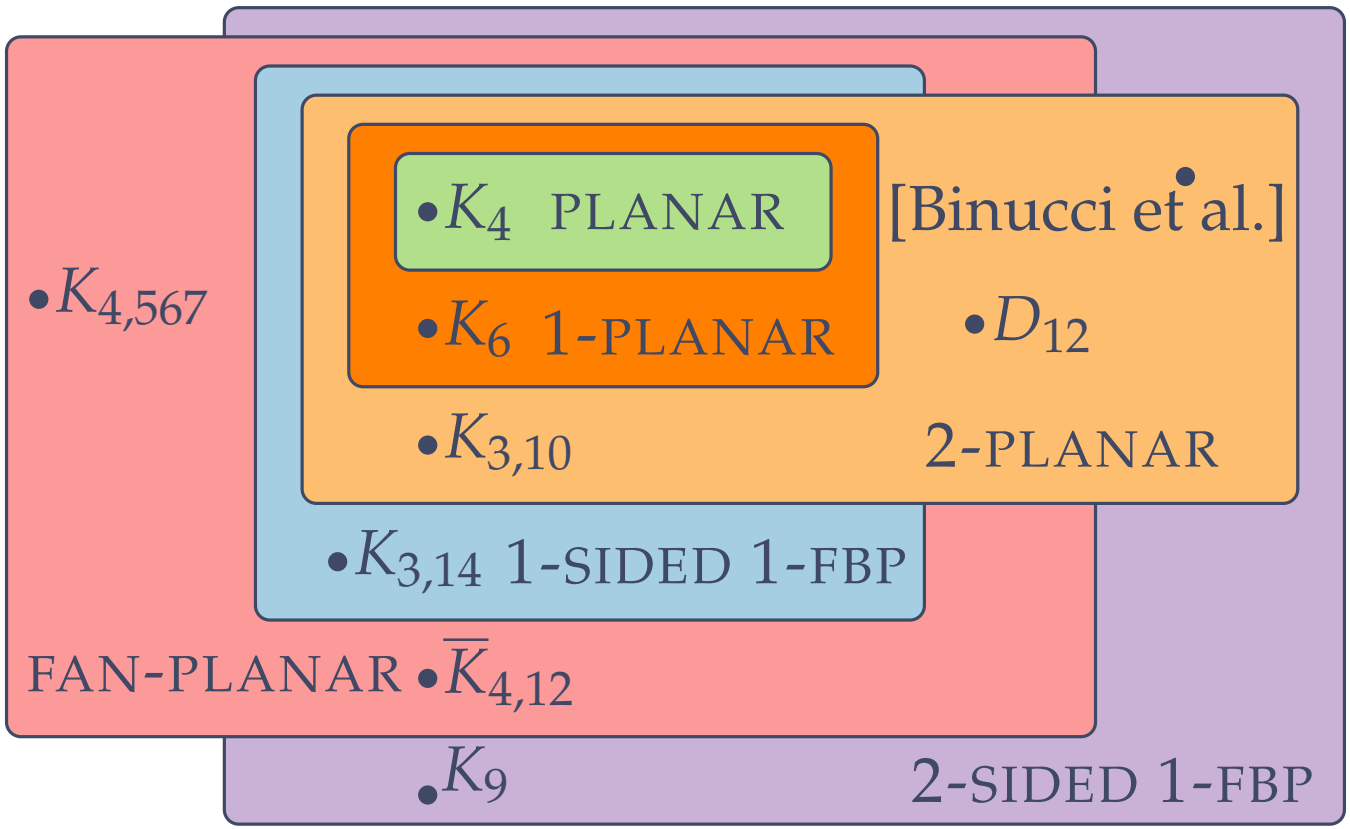
= Max. 1-sided 1-fbp

Summary

	2-layer		outer		general	
	LB	UB	LB	UB	LB	UB
1-sided	$\frac{5n-7}{3}$	$\frac{5n-7}{3}$	$\frac{8n-13}{3}$	$\frac{8n-13}{3}$	$\frac{13n-26}{3}$	$\frac{13n-26}{3}$
2-sided	$2n-4$	$3n-7$	$4n-9$	$4n-9$	$6n-18$	$8.6n-15.6$

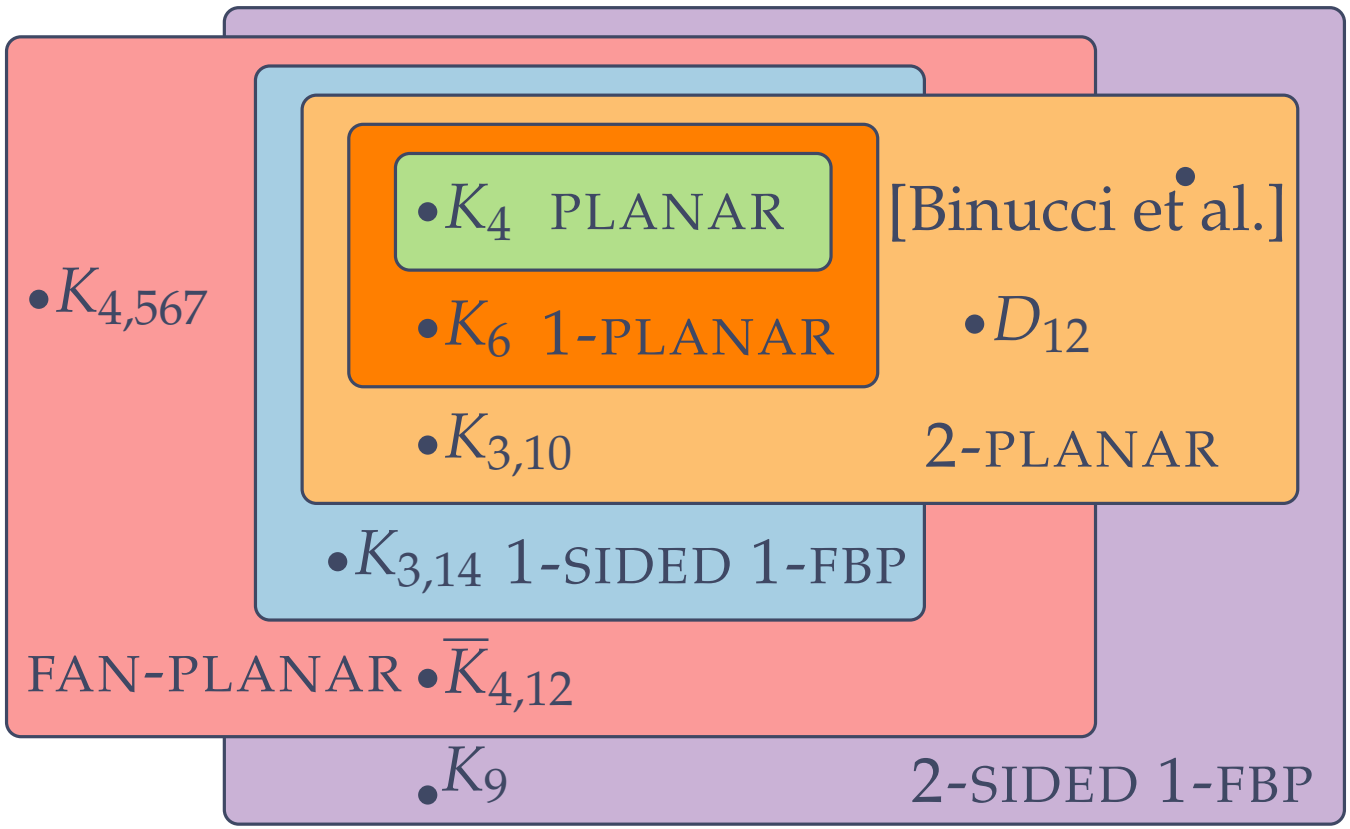
Summary

	2-layer		outer		general	
	LB	UB	LB	UB	LB	UB
1-sided	$\frac{5n-7}{3}$	$\frac{5n-7}{3}$	$\frac{8n-13}{3}$	$\frac{8n-13}{3}$	$\frac{13n-26}{3}$	$\frac{13n-26}{3}$
2-sided	$2n-4$	$3n-7$	$4n-9$	$4n-9$	$6n-18$	$8.6n-15.6$

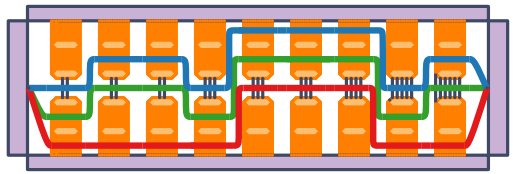


Summary

	2-layer		outer		general	
	LB	UB	LB	UB	LB	UB
1-sided	$\frac{5n-7}{3}$	$\frac{5n-7}{3}$	$\frac{8n-13}{3}$	$\frac{8n-13}{3}$	$\frac{13n-26}{3}$	$\frac{13n-26}{3}$
2-sided	$2n-4$	$3n-7$	$4n-9$	$4n-9$	$6n-18$	$8.6n-15.6$

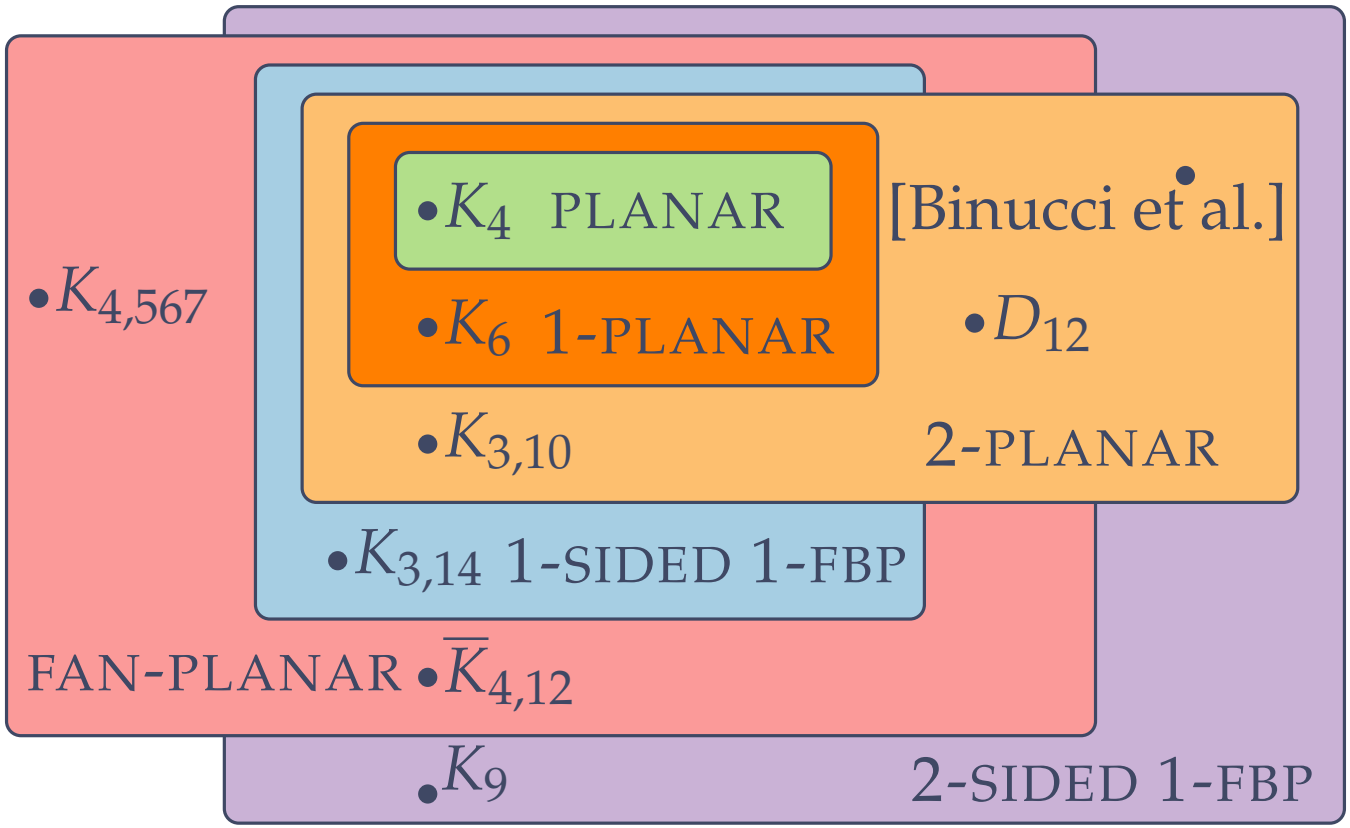


General:

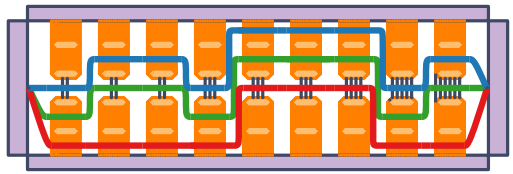


Summary

	2-layer		outer		general	
	LB	UB	LB	UB	LB	UB
1-sided	$\frac{5n-7}{3}$	$\frac{5n-7}{3}$	$\frac{8n-13}{3}$	$\frac{8n-13}{3}$	$\frac{13n-26}{3}$	$\frac{13n-26}{3}$
2-sided	$2n-4$	$3n-7$	$4n-9$	$4n-9$	$6n-18$	$8.6n-15.6$



General:



2-layer 1-sided: