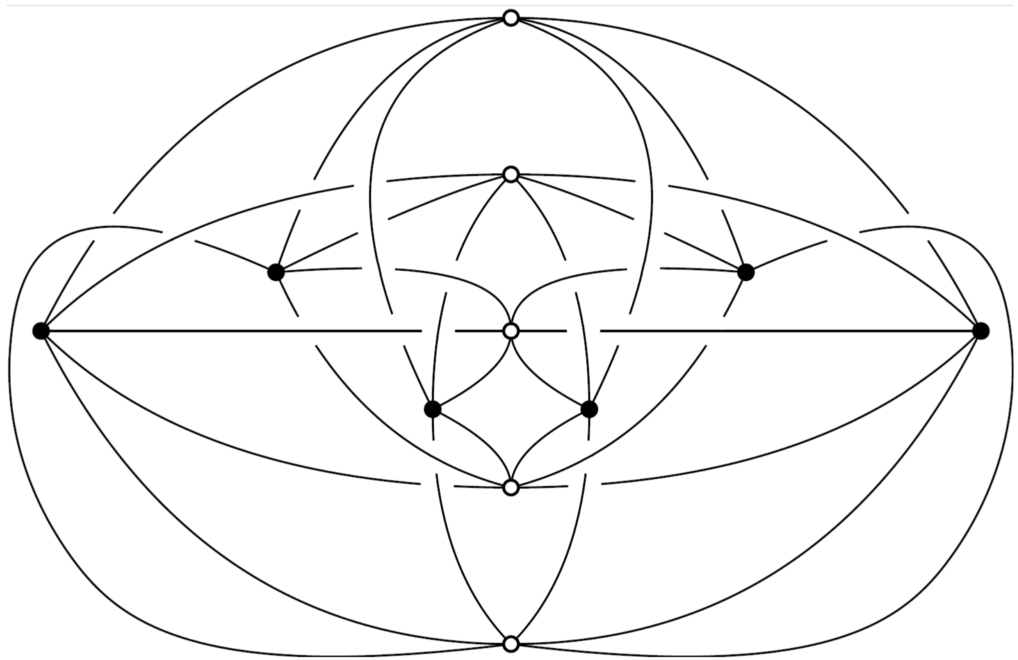
Gap-Planar Graphs

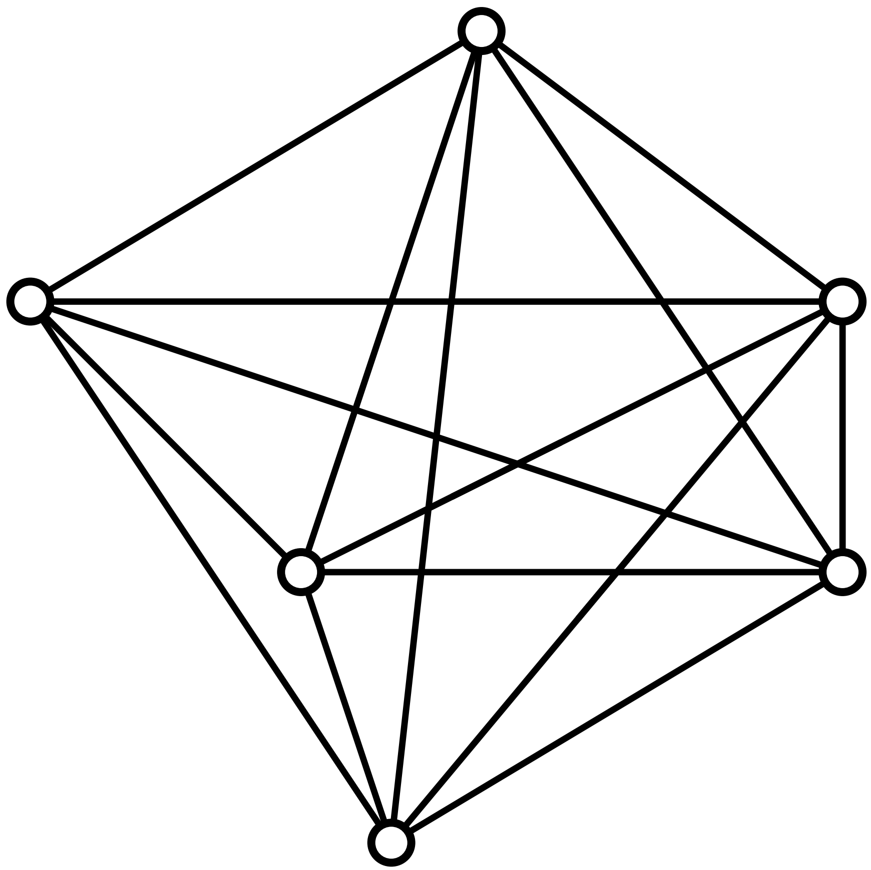
Sang Won Bae, Jean-Francois Baffier, Jinhee Chun, Peter Eades, Kord Eickmeyer, Luca Grilli, Seok-Hee Hong, Matias Korman, Fabrizio Montecchiani, Ignaz Rutter, Csaba D. Tóth



Graph Drawing · September 27, 2017

- Many graphs need crossings.

Crossings impede readability $\Rightarrow$ Make crossings nice.



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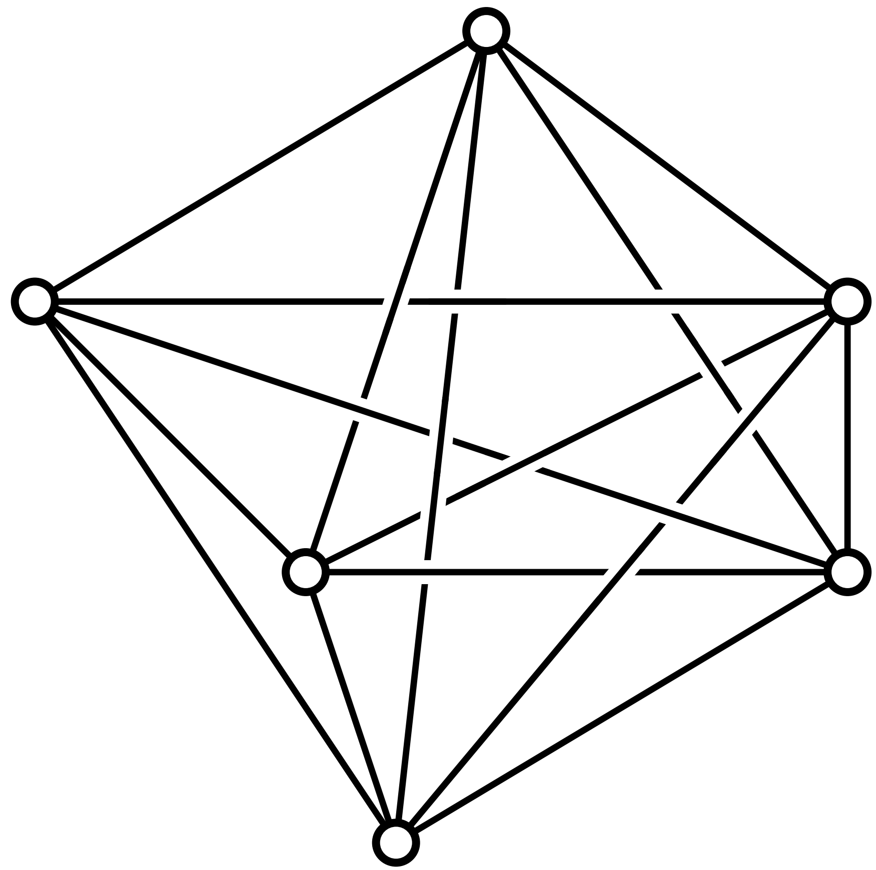
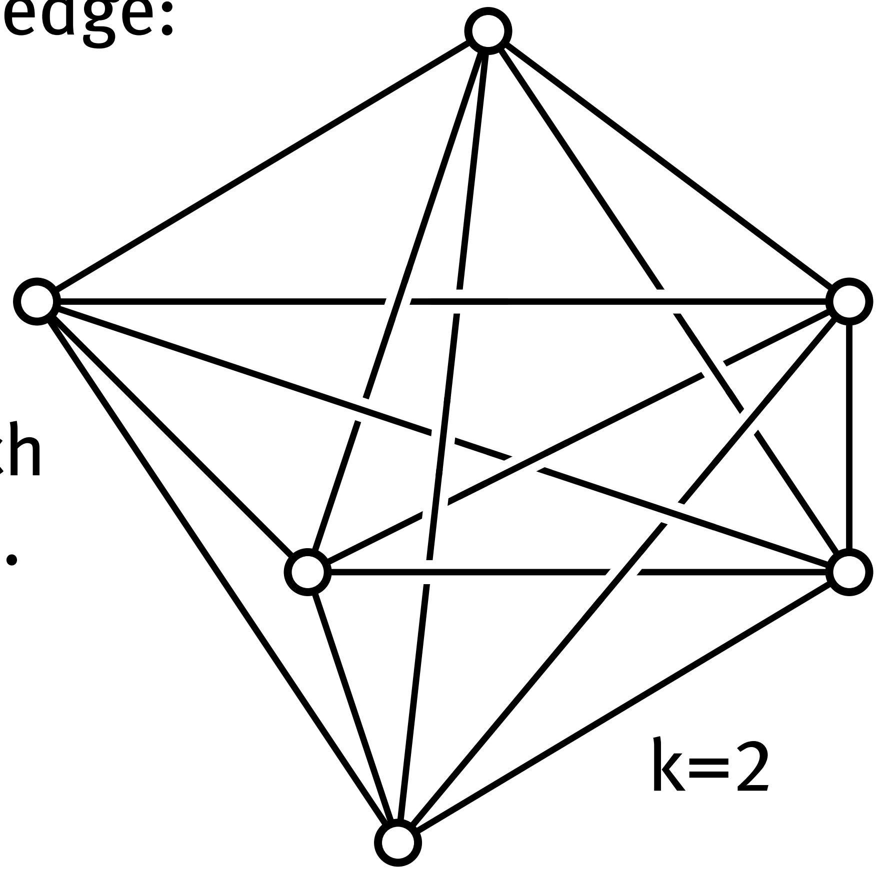


Edge casings:

at each crossing insert a small gap into one of the edges.

We restrict the number of gaps per edge:

- Gap assignment assigns each crossing to one of its edges.
- Drawing Γ is k-gap planar if each edge is assigned $\leq k$ crossings.



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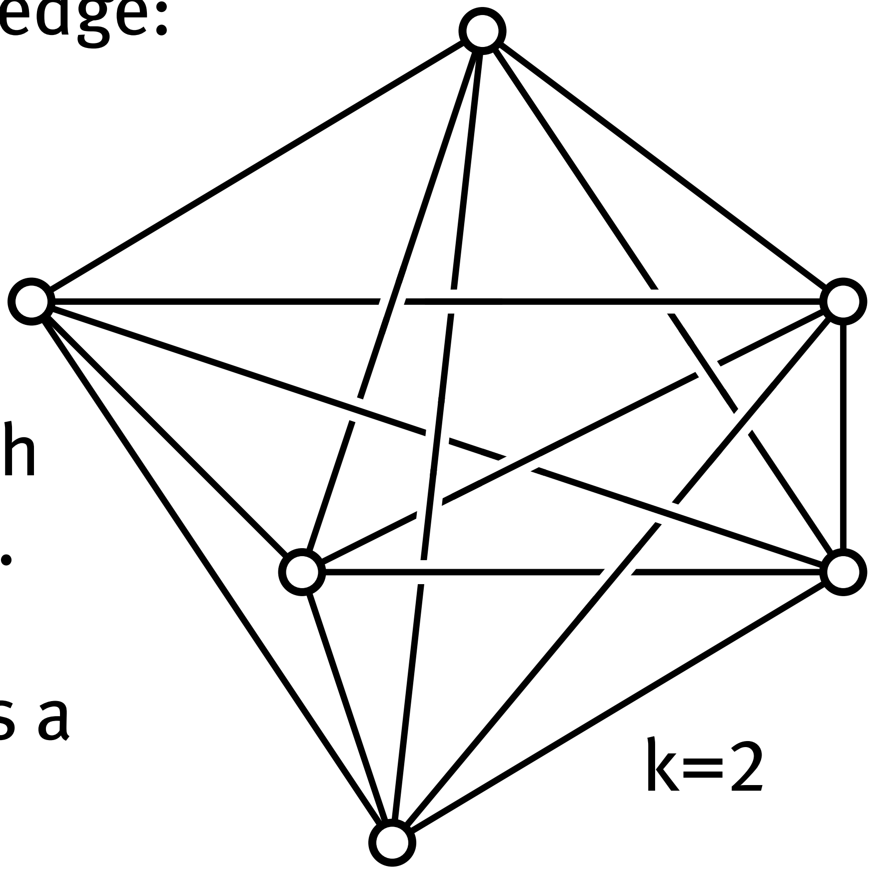

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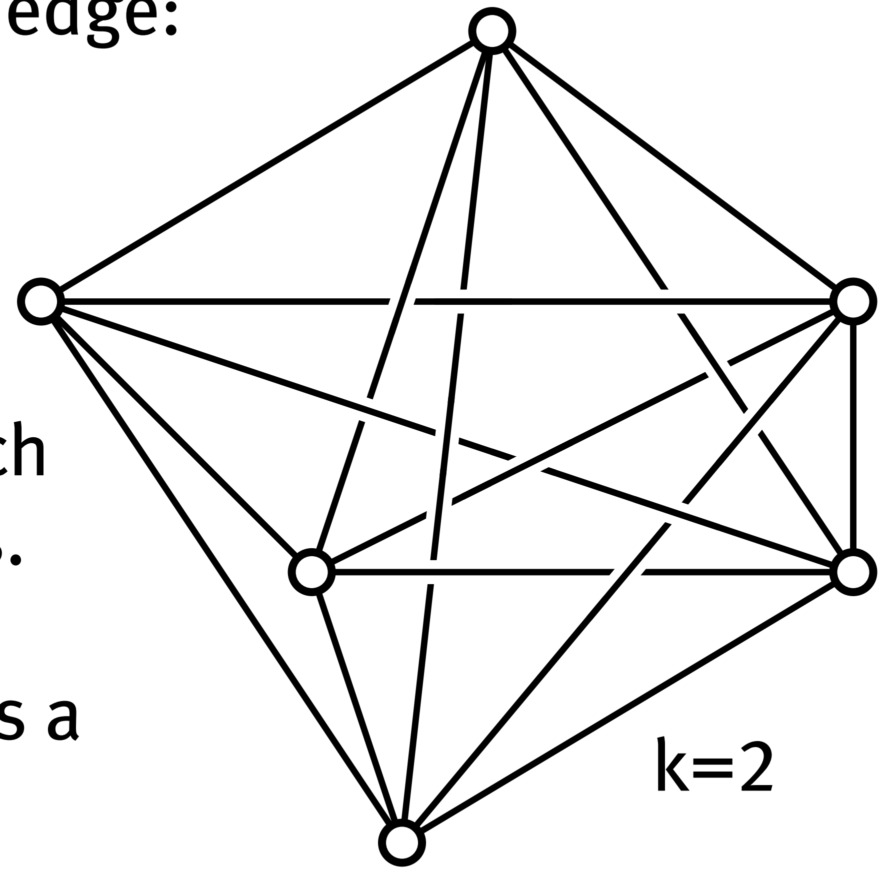

Questions:

- What is the maxium density of k-gap planar graphs?
- Which graphs are k-gap planar? Can we recognize them?
- What is the relation to k-planarity? To k-quasiplanarity?

Edge casings: [Eppstein, van Kreveld, Mumford, Speckmann, GD '07]
optimize casings in a given drawing:

- Optimize tunnels (= gaps) and switches
- Stacking and weaving model

$O(m^4)$-algorithm for minimizing maximum number of gaps.

Related Work

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Near-planarity:

- k-planar graphs [Pach, Tóth'97, Bekos, Kaufmann, Raftopoulou' 16, Kobourov, Liotta, Montecchiani'17]
- k-quasiplanar graphs [Agarwal, Aronov, Pach, Pollack, Sharir '97, Ackerman, Tardos '07, Fox, Pach, Suk, '13]
- fan-planarity [Kaufmann, Ueckerdt '14, Binucci, Di Giacomo, Didimo, Montecchiani, Patrignani, Symvonis, Tollis '15, Bekos, Cornelsen, Grilli, Hong, Kaufmann '16]

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1. Density of k-gap planar graphs
2. Complete (bipartite) graphs
3. Complexity of recognizing 1-gap planar graphs
4. Relation to other graph classes

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Crossing lemma: $\mathrm{cr}(G) \in \Omega(\frac{m^3}{n^2})$, i.e. $\mathrm{cr}(G) \geq c \cdot m^3/n^2$.
$\Rightarrow c \cdot m^3/n^2 \leq \mathrm{cr}(G) \leq k \cdot m$
$\Rightarrow m \leq \sqrt{c}/c \cdot \sqrt{k} \cdot n$ □

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This is asymptotically tight.

What are the constants for 1-gap planar?

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A 1-gap planar graph on n vertices has $\leq 5n - 10$ edges.
A 1-gap planar graph with $5n - 10$ edges exists for all $n \geq 20$.

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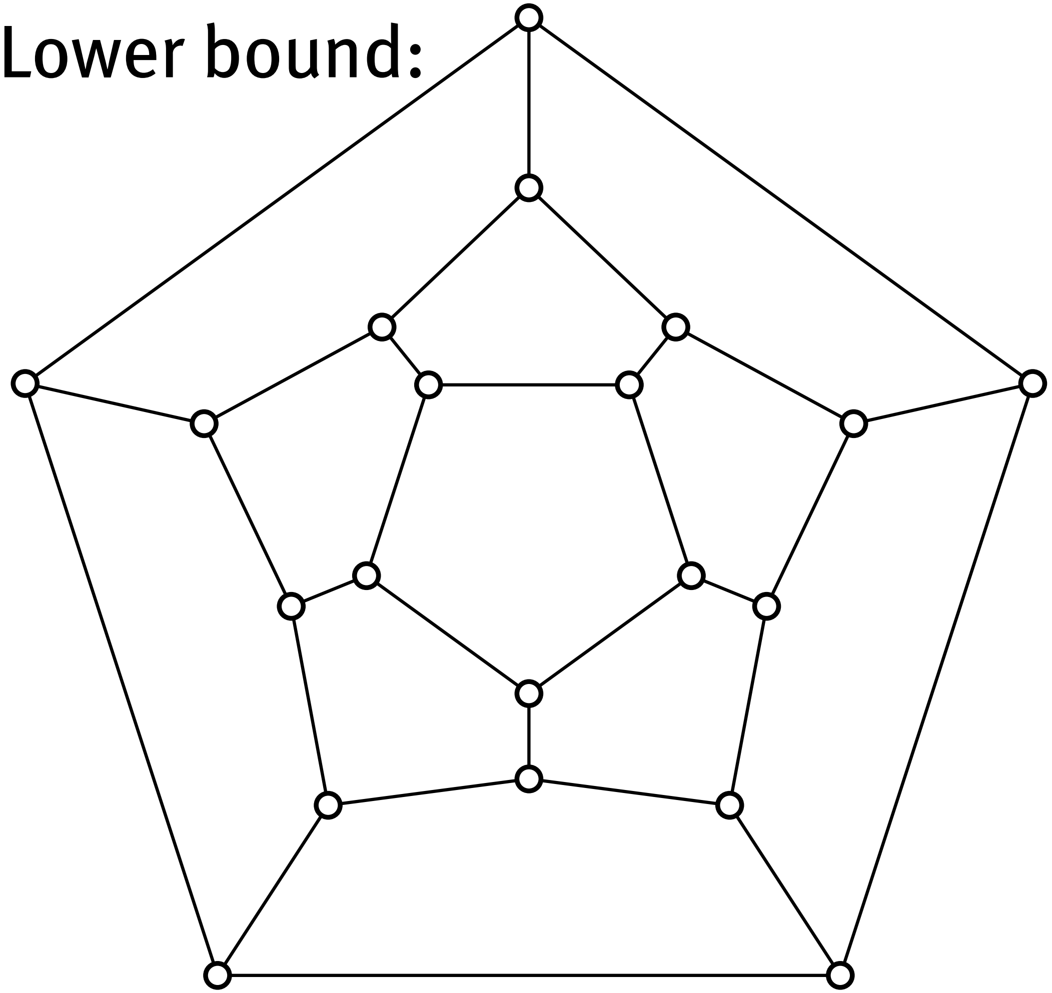

Density of 1-Gap Planar Graphs

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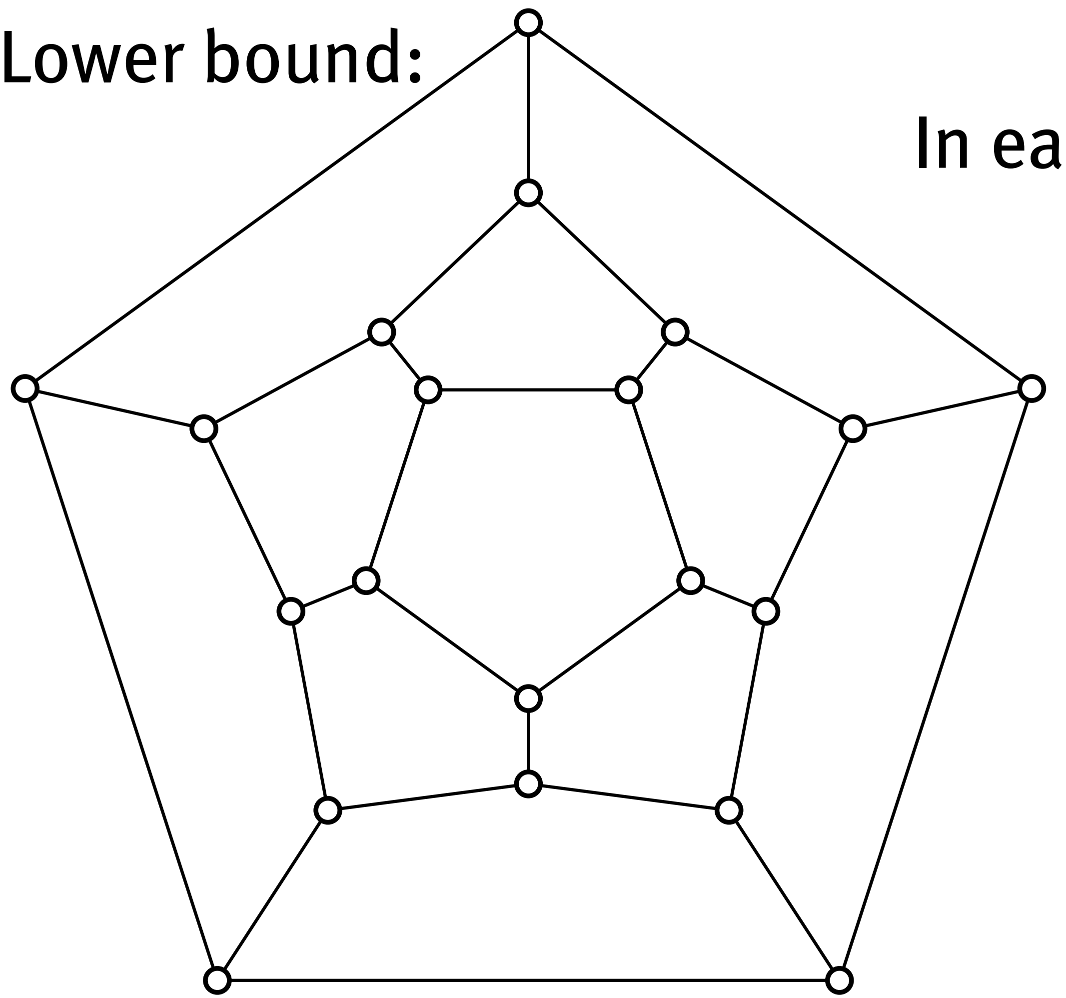
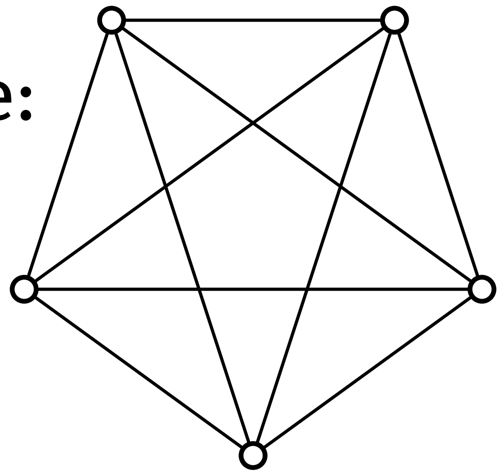


In each face:



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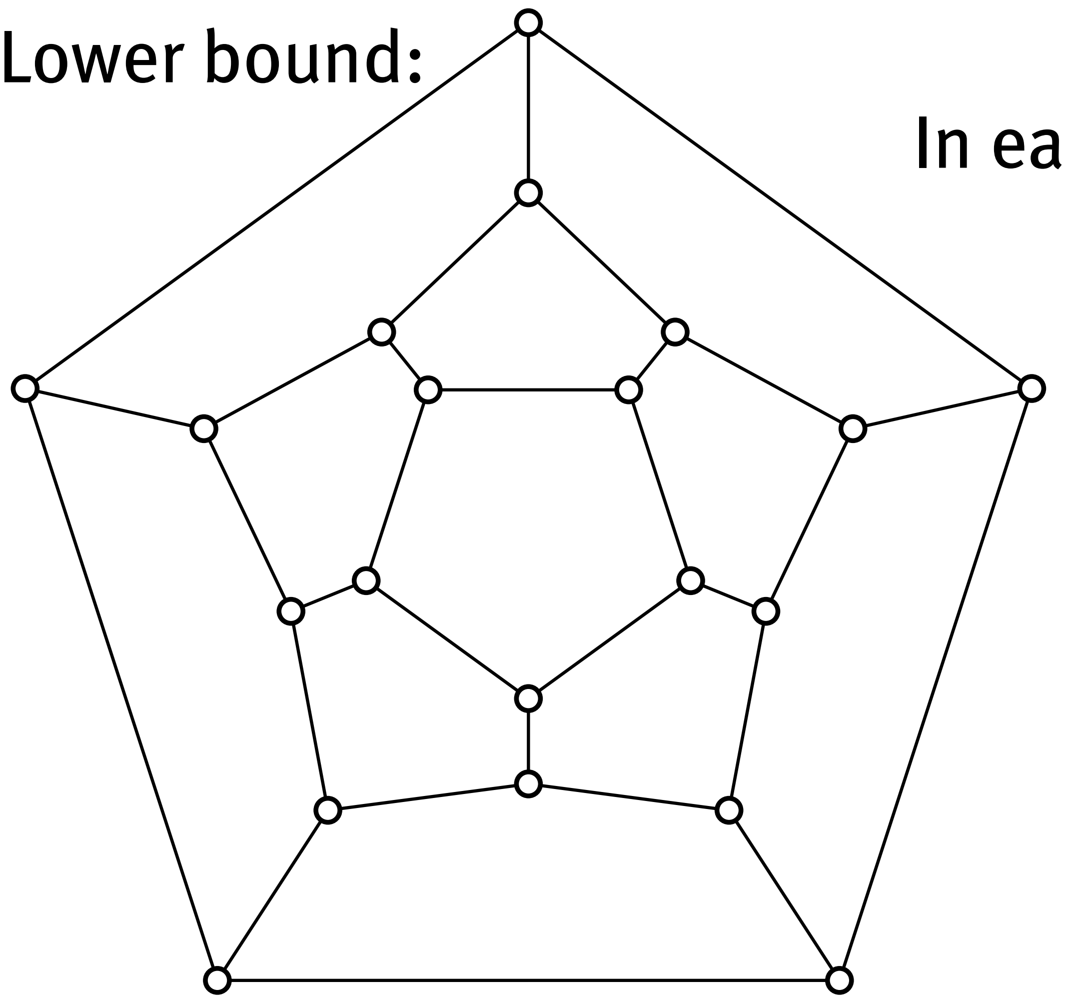

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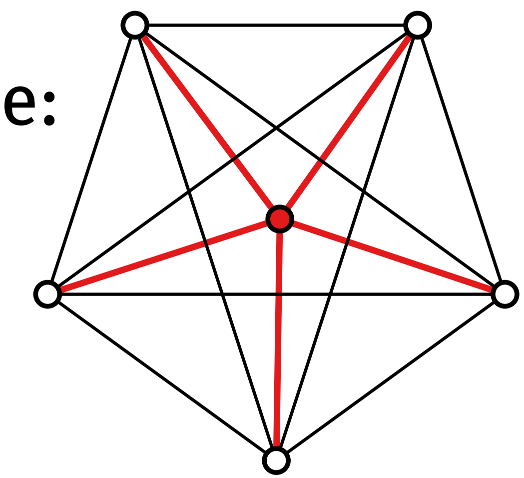

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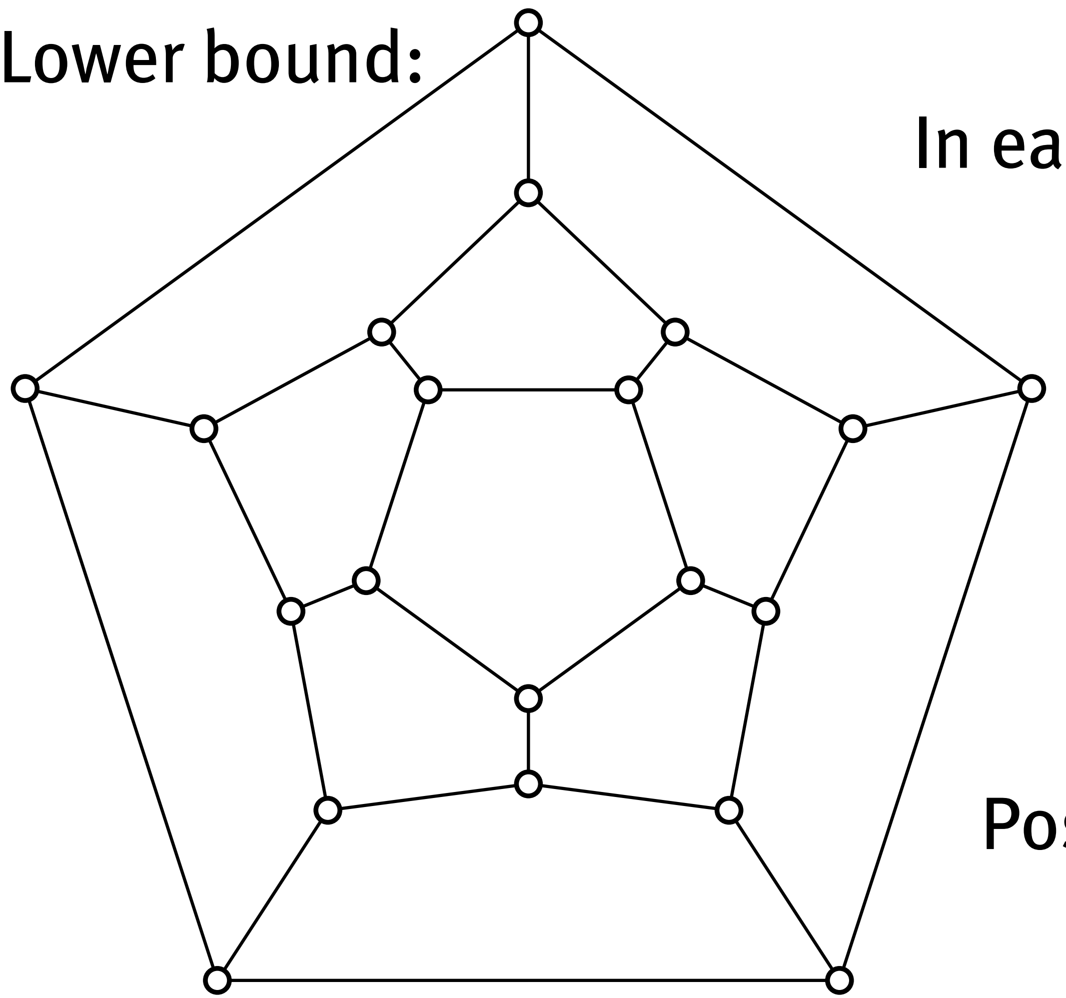

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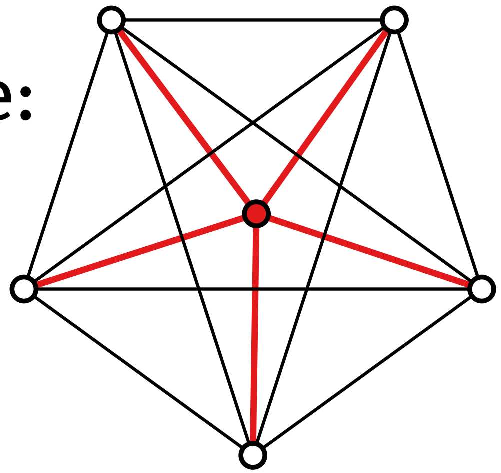

Possibly nest the construction.

Let G be a 1-gap-planar multigraph on $n \geq 3$ vertices without homotopic parallel edges that has maximum number of edges.

- Fix a 1-gap-planar drawing Γ minimizing crossings.
- Pick maximum $H \subseteq G$ with edges pairwise non-crossing.
- Tie breaker: Minimize connected components of H.

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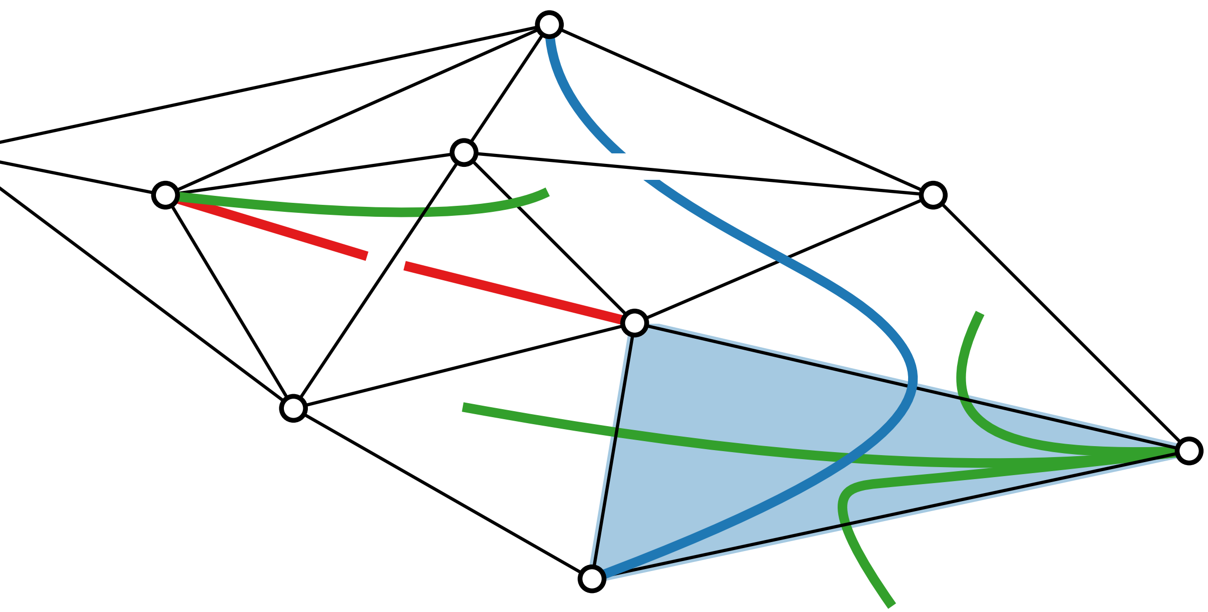

Charge edges $e \in E(G) \setminus E(H)$ to faces:

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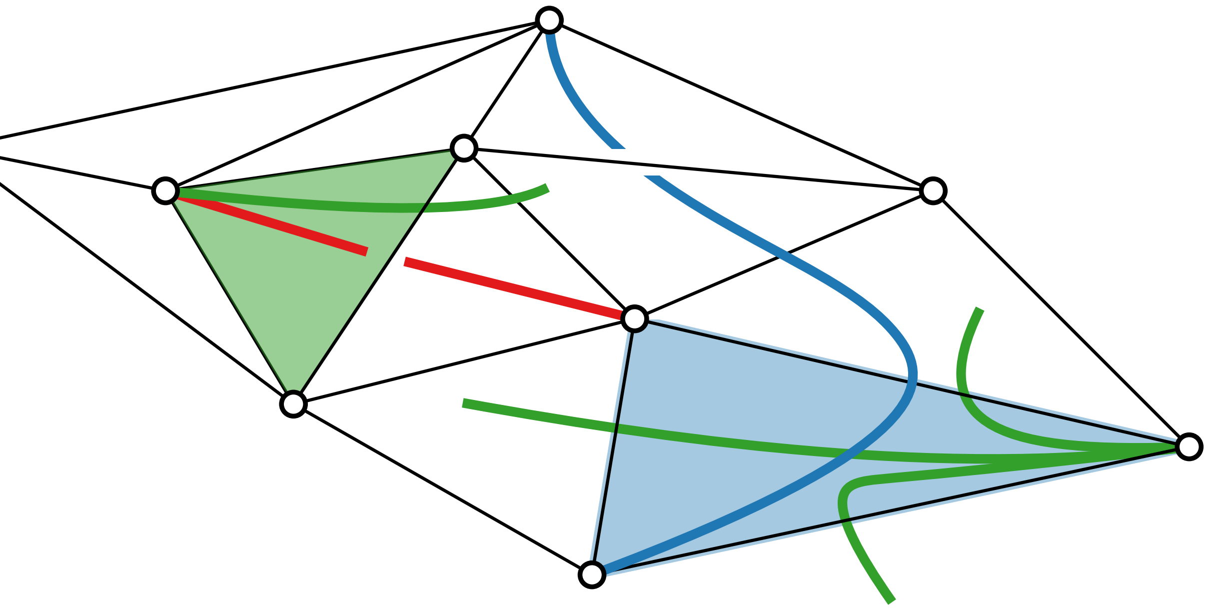

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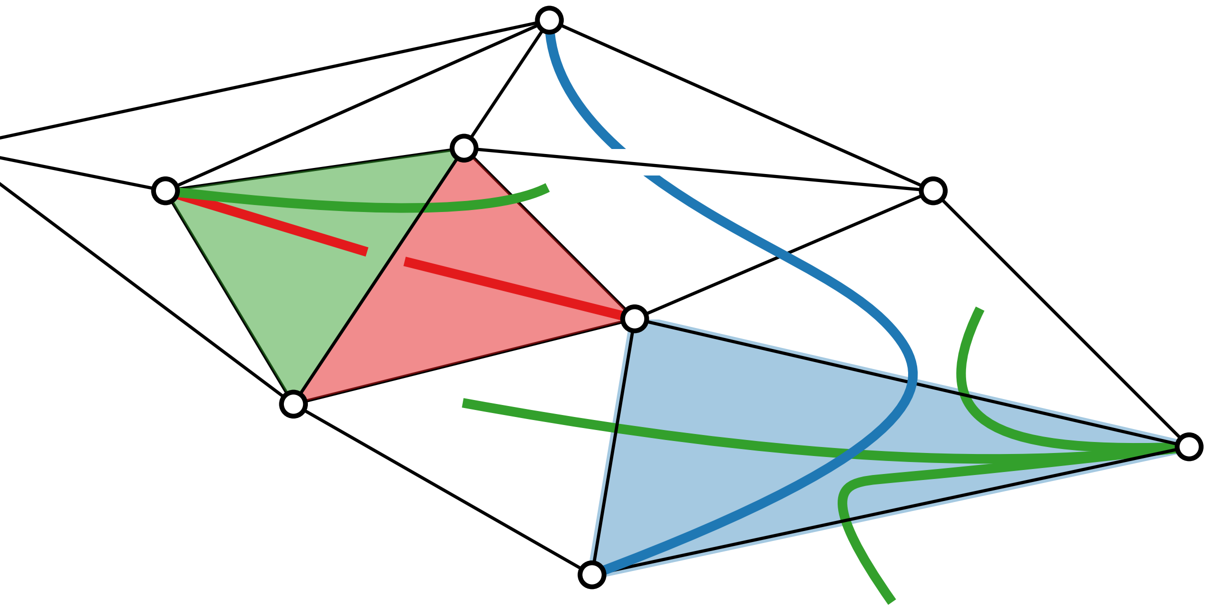

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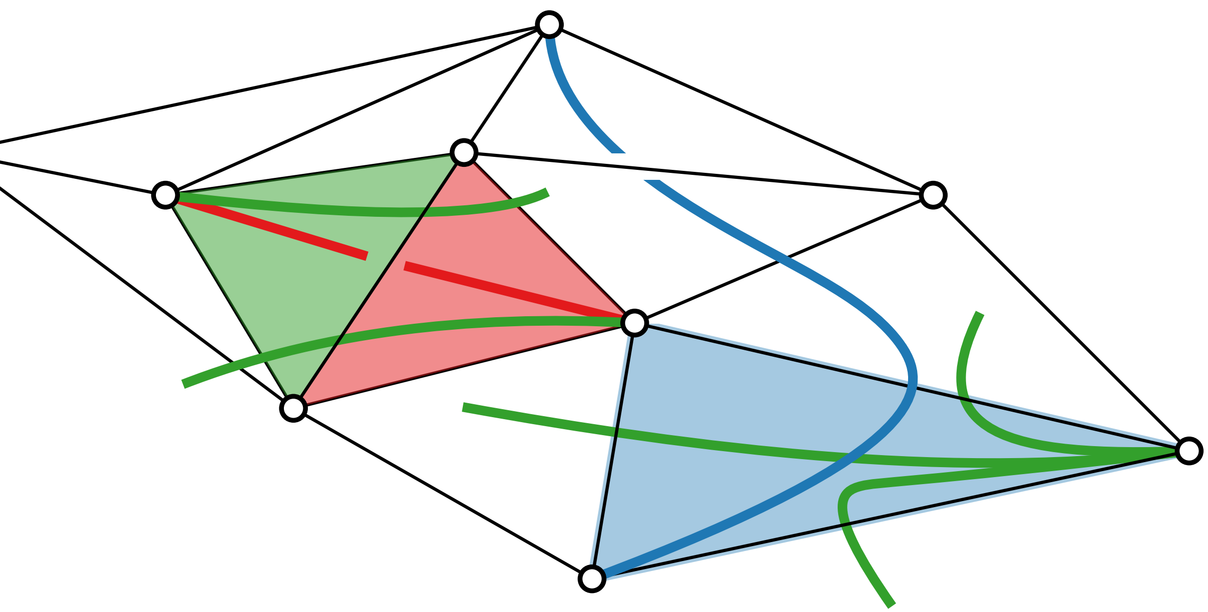

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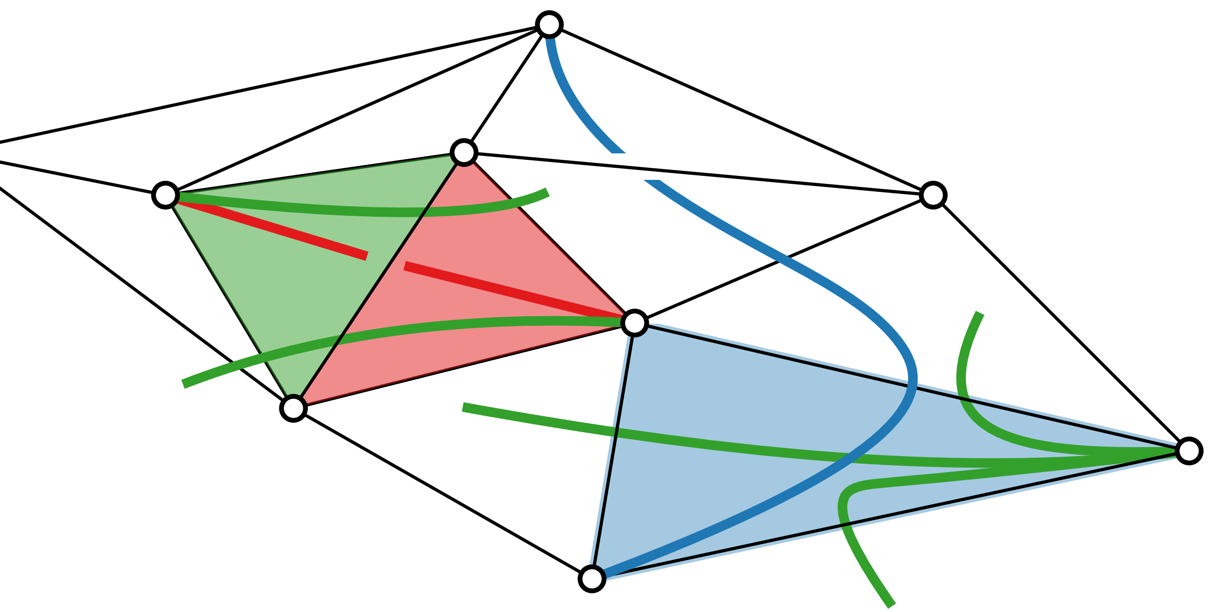

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$$\Rightarrow |E(H)| = 3n - 6,\ |E(G) \setminus E(H)| \leq 2n - 4$$

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Lemma 8. *Graph $G = (V, E)$ is connected.*

Proof. Suppose, to the contrary, that G is disconnected. Let $G_1 = (V_1, E_1)$ be one component, and let $G_2 = (V_2, E_2)$, where $V_2 = V \setminus V_1$ and $E_2 = E \setminus E_1$. For $i = 1, 2$, let Γ_i be the drawing of G_i inherited from G, and let Γ_i^* be its planarization.

Let f_2 be a face in Γ_2^* incident to some vertex $v_2 \in V_2$. Apply a projective transformation to Γ_1 so that the outer face is incident to some vertex $v_1 \in V_1$; followed by an affine transformation that maps Γ_1 into the interior of face f_2. Now we can add a new edge (v_1, v_2), contradicting the maximality of G. □

Since G is connected, every face in the planarization Γ^* of Γ has a connected boundary. The *boundary walk* of a face f is a closed walk $(a_1, a_2, \ldots, a_m)$ in Γ^* such that f lies on the left hand side of each edge (a_i, a_{i+1}) along the walk; and every two consecutive edges of the walk, (a_{i-1}, a_i) and (a_i, a_{i+1}), are also consecutive in the counterclockwise rotation of all edges incident to a_i. Let F_0 denote the set of faces in the planarization Γ^* that are not incident to any vertex in V.

Lemma 9. *If $f \in F_0$, then the boundary walk of f is*

1. *a simple cycle (i.e., has no repeated vertices) with at least 3 vertices;*
2. *disjoint from the boundary walk of any other face in F_0.*

Proof. 1. Let $f \in F_0$, and let $w = (a_1, a_2, \ldots, a_\ell)$ be its boundary walk for some $\ell \geq 3$. Let $C_f = \{a_1, \ldots, a_\ell\}$ be the set of vertices in w; and let $E_f \subset E$ be the set of edges in G that contain some edge of w. It suffices to show that $|C_f| = \ell$, and then w has no repeated vertices, hence it is a simple cycle.

Suppose, to the contrary, that the vertices in w are not distinct. Since $f \in F_0$, all vertices in w are crossings in the drawing Γ, consequently they all have degree 4 in the planarization Γ^*. If $a_i = a_j$, $i \neq j$, then a_i and a_j cannot be consecutive vertices in w, and two pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (a_{j-1}, a_j), (a_j, a_{j+1}) are part of the same edge in E. If $|C_f| = \ell - k$, for some $k \in \mathbb{N}$, then $|E_f| \leq \ell - 2k$. This implies $|E_f| < |C_f|$. That is, the edges in E_f are involved in more than $|E_f|$ crossings, contradicting the assumption that Γ is a 1-gap-planar drawing.

2. Let $f_1, f_2 \in F_0$ be two faces, with boundary walks $w_1 = (a_1, \ldots, a_\ell)$ and $w_2 = (b_1, \ldots, b_{\ell'})$. Both w_1 and w_2 are simple cycles by part 1. For $i = 1, 2$, let C_i be the set of vertices in w_i, and $E_i \subseteq E$ the set of edges of G that contain the edges of the walk w_i.

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Note that w_1 and w_2 cannot share two consecutive edges, say (a_{i-1}, a_i) and (a_i, a_{i+1}), since the middle vertex a_i has degree 4 in Γ^*. When w_1 and w_2 have a common edge, say $(a_i, a_{i+1}) = (b_{j+1}, b_j)$, then three pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (a_{i+1}, a_{i+2}), (b_{j-1}, b_j), (b_j, b_{j+1}) (b_{j+1}, b_{j+2}) are part of the same edge in E. When w_1 and w_2 have a common vertex $a_i = b_j$ but no common edge incident to $a_i = b_j$, then two pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (b_{j-1}, b_j), (b_j, b_{j+1}) are part of the same edge in E. This implies $|E_1 \cup E_2| < |C_1 \cup C_2|$. That is, the edges in $E_1 \cup E_2$ are involved in more than $|E_1 \cup E_2|$ crossings, contradicting the assumption that Γ is 1-gap-planar. □

Lemma 10. *Graph $H = (V, E')$ is connected.*

Proof. Suppose, to the contrary, that H is disconnected. Let $H_1 = (V_1, E'_1)$ be one component, and let $H_2 = (V_2, E'_2)$, where $V_2 = V \setminus V_1$ and $E'_2 = E' \setminus E'_1$.

Consider the faces in the planarization Γ^* of Γ. Notice that there is no face in Γ^* incident to a vertex $v_1 \in V_1$ and a vertex $v_2 \in V_2$, otherwise we could either add a new edge (v_1, v_2) (contradicting the maximality of G), or redraw an existing the edge (v_1, v_2) to pass through the interior of this face, contradicting the maximality of E'.

Consequently, we can partition the faces in Γ^* into three categories: For $i = 1, 2$, let F_i be the set of faces incident to a vertex in V_i; and let F_0 be the set of faces incident to neither V_1 nor V_2. By Lemma 9, the region obtained by removing all faces in F_0 (i.e., $\mathbb{R}^2 \setminus \bigcup_{f \in F_0}$) is connected. Consequently, there exist some faces $f_1 \in F_1$ and $f_2 \in F_2$ that have a common edge in Γ^*. Let $v_1 \in V_1$ and $v_2 \in V_2$ be incident to $f_1 \in F_1$ and $f_2 \in F_2$. Let $e \in E$ be the edge on the common boundary of f_1 and f_2, and denote its endpoints by $a, b \in V$.

We consider three possible edges (some of which may be homotopic to an existing edge in G): let $e_0 = (v_1, v_2)$ such that it lies in $f_1 \cup f_2$; let $e_1 = (v_1, a)$ (resp., $e_2 = (v_1, b)$) such that it starts in f_1 and follow edge e from f_1 to its endpoint a (resp., b).

- If $e \notin E'$, then replace edge $e = (a, b)$ by a new edge $e_0 = (v_1, v_2)$ in G, and add this new edge to H. This modification contradicts the assumption that H has the minimum number of components.
- Assume $e \in E'$. Note that e_1 and e_2 form a path between a and b, consequently at most one of these edges may be present in G (as a homotopic copy), otherwise we could modify E' by replacing e with these edges, contradicting the maximality of E'. Now we can increase E by replacing e with e_1 or e_2 (whichever is not already present), contradicting the choice of H.

Both cases lead to a contradiction. □

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Lemma 9. *If $f \in F_0$, then the boundary walk of f is*

1. *a simple cycle (i.e., has no repeated vertices) with at least 3 vertices;*
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Suppose, to the contrary, that the vertices in w are not distinct. Since $f \in F_0$, all vertices in w are crossings in the drawing Γ, consequently they all have degree 4 in the planarization Γ^*. If $a_i = a_j$, $i \neq j$, then a_i and a_j cannot be consecutive vertices in w, and two pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (a_{j-1}, a_j), (a_j, a_{j+1}) are part of the same edge in E. If $|C_f| = \ell - k$, for some $k \in \mathbb{N}$, then $|E_f| \leq \ell - 2k$. This implies $|E_f| < |C_f|$. That is, the edges in E_f are involved in more than $|E_f|$ crossings, contradicting the assumption that Γ is a 1-gap-planar drawing.

2. Let $f_1, f_2 \in F_0$ be two faces, with boundary walks $w_1 = (a_1, \ldots, a_\ell)$ and $w_2 = (b_1, \ldots, b_{\ell'})$. Both w_1 and w_2 are simple cycles by part 1. For $i = 1, 2$, let C_i be the set of vertices in w_i, and $E_i \subseteq E$ the set of edges of G that contain the edges of the walk w_i.

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Consequently, we can partition the faces in Γ^* into three categories: For $i = 1, 2$, let F_i be the set of faces incident to a vertex in V_i; and let F_0 be the set of faces incident to neither V_1 nor V_2. By Lemma 9, the region obtained by removing all faces in F_0 (i.e., $\mathbb{R}^2 \setminus \bigcup_{f \in F_0}$) is connected. Consequently, there exist some faces $f_1 \in F_1$ and $f_2 \in F_2$ that have a common edge in Γ^*. Let $v_1 \in V_1$ and $v_2 \in V_2$ be incident to $f_1 \in F_1$ and $f_2 \in F_2$. Let $e \in E$ be the edge on the common boundary of f_1 and f_2, and denote its endpoints by $a, b \in V$.

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We are now ready to prove Lemma 1.

Lemma 1. *The multigraph H is a triangulation. That is, a plane multi-graph in which every face is bounded by a walk with three vertices and three edges.*

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By Sperner's Lemma, K has a triangle whose nodes have all three different colors, say $f_1 \in F_i$, $f_2 \in F_j$, and $f_3 \in F_k$. Without loss of generality, assume that $j \notin \{i-1, i+1\}$. We add a new edge (v_i, v_j), as follows. There are three cases depending on how the edge $f_i f_j$ in K was created:

- Faces f_1 and f_2 are adjacent in Γ^*. Then we can add a new edge (v_i, v_j) to G such that (v_i, v_j) lies in $f_i \cup f_j$ and uses a gap to cross the boundary between these faces. This contradicts the maximality of G.
- A vertex $c \in V_f$ is incident to both f_1 and f_2. Then two edges $e_1, e_2 \in E \setminus E'$ cross at c. We can replace edge e_1 with a new edge (v_i, v_j) that lies in $f_1 \cup f_2$ that crosses edge e_2 at c. The new edge can be inserted into both G and H, contradicting the maximality of H.
- A face $f_0 \in F_0$ is adjacent to both f_1 and f_2. Then two edges $e_1, e_2 \in E \setminus E'$ are on the common boundary of the adjacent pairs f_1, f_0 and f_0, f_2. We can replace edge e_1 with a new edge (v_i, v_j) that lies in $f_1 \cup f_0 \cup f_2$ that crosses edge e_2. The new edge can be inserted into both G and H, contradicting the maximality of H.

Case 2. There is an index $i \in \{1, \ldots, m\}$, such that (v_i, v_{i+1}) is incident to a face in F_j for some $j \neq 0, i, i+1$. Without loss of generality, we may assume that edge (v_1, v_m) is incident to a face in F_j for some $1 < j < m$. Note that edge (v_1, v_m) must be incident to some face in F_j for *all* $1 \leq j \leq m$; otherwise $v_1 v_m$ would be incident to two faces, $f_i \in F_i$ and $f_j \in F_j$, $j \notin \{i-1, i, i+1\}$, that are either adjacent to each other or both adjacent to some face $f_0 \in F_0$; and then we could add a new edge (v_i, v_j) lying in $f_i \cup f_j$ or $f_i \cup f_0 \cup f_j$.

It follows that there are faces $f_2 \in F_2$ and $f_3 \in F_3$ that are incident to some point $c \in (v_1, v_2)$; or both are adjacent to some common face $f_0 \in F_0$ that is incident to $v_1 v_2$.

Consider the face f' of H on the opposite side of (v_1, v_m), and let F' be the set of faces in the planarization Γ^* contained in f'. Let $f'' \in F'$ be a face incident to $c \in (v_1, v_m)$ or adjacent to face f_0. By Lemma 9(2), we may assume that f'' is incident to a vertex v_k on the boundary of the face f'. It is possible that $v_k = v_1$ or $v_k = v_m$.

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- If $v_k = v_m$ and $v_{m-1} = v_3$, we make similar changes replacing edge e with (v_2, v_m).
- Otherwise we similarly modify G, Γ, and H as follows: first replace the edge (v_1, v_m) with two new edges (v_2, v_k) and (v_3, v_k), that lie in $f_2 \cup f''$ and $f_3 \cup f''$, respectively, and one of them may cross some edge at c. Both (v_2, v_k) and (v_3, v_k) can be added to E', contradicting the maximality of E'.

All cases lead to a contradiction. Therefore, our initial assumption must be dropped, consequently the multigraph H is a triangulation, as claimed. □

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Let f_2 be a face in Γ_2^* incident to some vertex $v_2 \in V_2$. Apply a projective transformation to Γ_1 so that the outer face is incident to some vertex $v_1 \in V_1$; followed by an affine transformation that maps Γ_1 into the interior of face f_2. Now we can add a new edge (v_1, v_2), contradicting the maximality of G. □

Since G is connected, every face in the planarization Γ^* of Γ has a connected boundary. The *boundary walk* of a face f is a closed walk $(a_1, a_2, \ldots, a_m)$ in Γ^* such that f lies on the left hand side of each edge (a_i, a_{i+1}) along the walk; and every two consecutive edges of the walk, (a_{i-1}, a_i) and (a_i, a_{i+1}), are also consecutive in the counterclockwise rotation of all edges incident to a_i. Let F_0 denote the set of faces in the planarization Γ^* that are not incident to any vertex in V.

Lemma 9. *If $f \in F_0$, then the boundary walk of f is*

1. *a simple cycle (i.e., has no repeated vertices) with at least 3 vertices;*
2. *disjoint from the boundary walk of any other face in F_0.*

Proof. 1. Let $f \in F_0$, and let $w = (a_1, a_2, \ldots, a_\ell)$ be its boundary walk for some $\ell \geq 3$. Let $C_f = \{a_1, \ldots, a_\ell\}$ be the set of vertices in w; and let $E_f \subset E$ be the set of edges in G that contain some edge of w. It suffices to show that $|C_f| = \ell$, and then w has no repeated vertices, hence it is a simple cycle.

Suppose, to the contrary, that the vertices in w are not distinct. Since $f \in F_0$, all vertices in w are crossings in the drawing Γ, consequently they all have degree 4 in the planarization Γ^*. If $a_i = a_j$, $i \neq j$, then a_i and a_j cannot be consecutive vertices in w, and two pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (a_{j-1}, a_j), (a_j, a_{j+1}) are part of the same edge in E. If $|C_f| = \ell - k$, for some $k \in \mathbb{N}$, then $|E_f| \leq \ell - 2k$. This implies $|E_f| < |C_f|$. That is, the edges in E_f are involved in more than $|E_f|$ crossings, contradicting the assumption that Γ is a 1-gap-planar drawing.

2. Let $f_1, f_2 \in F_0$ be two faces, with boundary walks $w_1 = (a_1, \ldots, a_\ell)$ and $w_2 = (b_1, \ldots, b_{\ell'})$. Both w_1 and w_2 are simple cycles by part 1. For $i = 1, 2$, let C_i be the set of vertices in w_i, and $E_i \subseteq E$ the set of edges of G that contain the edges of the walk w_i.

Note that w_1 and w_2 cannot share two consecutive edges, say (a_{i-1}, a_i) and (a_i, a_{i+1}), since the middle vertex a_i has degree 4 in Γ^*. When w_1 and w_2 have a common edge, say $(a_i, a_{i+1}) = (b_{j+1}, b_j)$, then three pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (a_{i+1}, a_{i+2}), (b_{j-1}, b_j), (b_j, b_{j+1}) (b_{j+1}, b_{j+2}) are part of the same edge in E. When w_1 and w_2 have a common vertex $a_i = b_j$ but no common edge incident to $a_i = b_j$, then two pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (b_{j-1}, b_j), (b_j, b_{j+1}) are part of the same edge in E. This implies $|E_1 \cup E_2| < |C_1 \cup C_2|$. That is, the edges in $E_1 \cup E_2$ are involved in more than $|E_1 \cup E_2|$ crossings, contradicting the assumption that Γ is 1-gap-planar. □

Lemma 10. *Graph $H = (V, E')$ is connected.*

Proof. Suppose, to the contrary, that H is disconnected. Let $H_1 = (V_1, E_1')$ be one component, and let $H_2 = (V_2, E_2')$, where $V_2 = V \setminus V_1$ and $E_2' = E' \setminus E_1'$.

Consider the faces in the planarization Γ^* of Γ. Notice that there is no face in Γ^* incident to a vertex $v_1 \in V_1$ and a vertex $v_2 \in V_2$, otherwise we could either add a new edge (v_1, v_2) (contradicting the maximality of G), or redraw an existing the edge (v_1, v_2) to pass through the interior of this face, contradicting the maximality of E'.

Consequently, we can partition the faces in Γ^* into three categories: For $i = 1, 2$, let F_i be the set of faces incident to a vertex in V_i; and let F_0 be the set of faces incident to neither V_1 nor V_2. By Lemma 9, the region obtained by removing all faces in F_0 (i.e., $\mathbb{R}^2 \setminus \bigcup_{f \in F_0}$) is connected. Consequently, there exist some faces $f_1 \in F_1$ and $f_2 \in F_2$ that have a common edge in Γ^*. Let $v_1 \in V_1$ and $v_2 \in V_2$ be incident to $f_1 \in F_1$ and $f_2 \in F_2$. Let $e \in E$ be the edge on the common boundary of f_1 and f_2, and denote its endpoints by $a, b \in V$.

We consider three possible edges (some of which may be homotopic to an existing edge in G): let $e_0 = (v_1, v_2)$ such that it lies in $f_1 \cup f_2$; let $e_1 = (v_1, a)$ (resp., $e_2 = (v_1, b)$) such that it starts in f_1 and follow edge e from f_1 to its endpoint a (resp., b).

- If $e \notin E'$, then replace edge $e = (a, b)$ by a new edge $e_0 = (v_1, v_2)$ in G, and add this new edge to H. This modification contradicts the assumption that H has the minimum number of components.
- Assume $e \in E'$. Note that e_1 and e_2 form a path between a and b, consequently at most one of these edges may be present in G (as a homotopic copy), otherwise we could modify E' by replacing e with these edges, contradicting the maximality of E'. Now we can increase E by replacing e with e_1 or e_2 (whichever is not already present), contradicting the choice of H.

Both cases lead to a contradiction. □

In the proof of Lemma 1, we shall use Sperner's Lemma [37], a well-known discrete analogue of Brouwer's fixed point theorem.

Lemma 11. (Sperner [37]) *Let K be a geometric simplicial complex in the plane, where the union of faces is homeomorphic to a disk. Assume that each vertex is assigned a color from the set $\{1, 2, 3\}$ such that three vertices $v_1, v_2, v_3 \in \partial K$ are colored 1, 2, and 3, respectively, and for any pair $i, j \in \{1, 2, 3\}$, the vertices on the path between v_i and v_j along ∂K that does not contain the 3rd vertex are colored with $\{i, j\}$. Then K contains a triangle whose vertices have all three different colors.*

We are now ready to prove Lemma 1.

Lemma 1. *The multigraph H is a triangulation. That is, a plane multi-graph in which every face is bounded by a walk with three vertices and three edges.*

Proof. We need to show that the multigraph H is a triangulation. Suppose, to the contrary, that H is not a triangulation. Then H has a face f whose boundary walk $w = (v_1, v_2, \ldots, v_m)$ has more than three vertices ($m \geq 4$). To simplify notation, we assume that w is a simple cycle; this assumption is not essential for the proof.

Let P_f be the subgraph of Γ^* formed by all edges and vertices lying in the interior or on the boundary of f; let V_f denote the set of vertices of P_f (it consists of $v_1, \ldots, v_m$ and all crossings in the interior or on the boundary of f); and let F denote the set of faces of Γ^* that lie in f. Let $F_0 \subseteq F$ be the set of faces that are not incident to any vertex; and for $i = 1, \ldots, m$, let $F_i \subseteq F$ be the set of faces incident to v_i.

Note that a face in F cannot be incident to two nonconsecutive vertices v_i and v_j, $j \notin \{i-1, i, i+1\}$, otherwise we could add a new edge $v_i v_j$, contradicting the maximality of G. A vertex $c \in V_f$ cannot be incident to two faces $f_1 \in F_i$ and $f_2 \in F_j$ such that $j \notin \{i-1, i, i+1\}$, otherwise two edges $e_1, e_2 \in E \setminus E'$ cross at c, and we can replace edge e_1 with a new edge $v_i v_j$ that lies in $f_1 \cup f_2$ that uses one gap to cross edge e_2—the new edge can be inserted into E', contradicting the maximality of H.

We distinguish two cases.

Case 1. For every $i \in \{1, \ldots, m\}$, the edge (v_i, v_{i+1}) is incident to faces in $F_0 \cup F_i \cup F_{i+1}$ only. We use Sperner's Lemma [37] for a triangulation K of the *dual graph* on the faces $F_1 \cup \ldots \cup F_m$, that we define here. We first create the standard dual graph of all faces in F: The nodes correspond to the faces in F; and two nodes are adjacent iff the corresponding faces are adjacent in Γ^*. We then triangulate the standard dual graph as follows. If a crossing $c \in V_f$ is incident to four faces in F, then the adjacency graph forms a 4-cycle in the standard dual. By Lemma 9(2), at least three of those faces are in $F \setminus F_0$, and we triangulate the 4-cycle by an arbitrary diagonal between two faces in $F \setminus F_0$. Note that the faces in F_0 still form an independent set by Lemma 9(2)). Finally, remove all nodes corresponding to F_0, and triangulate the chain of adjacent nodes arbitrarily to obtain a triangulation K. The condition in Case 1 implies that K is a geometric simplicial complex, where the union of faces is homeomorphic to a disk.

We now define a 3-coloring of K (the coloring need not be proper). Assign color 1 to all faces in F_1. For $i = 2, \ldots, m$, assign color 2 to all faces in $F_i \setminus \bigcup_{j<i} F_j$ if i is even, and color 3 if i is odd.

By Sperner's Lemma, K has a triangle whose nodes have all three different colors, say $f_1 \in F_i$, $f_2 \in F_j$, and $f_3 \in F_k$. Without loss of generality, assume that $j \notin \{i-1, i+1\}$. We add a new edge (v_i, v_j), as follows. There are three cases depending on how the edge $f_i f_j$ in K was created:

- Faces f_1 and f_2 are adjacent in Γ^*. Then we can add a new edge (v_i, v_j) to G such that (v_i, v_j) lies in $f_i \cup f_j$ and uses a gap to cross the boundary between these faces. This contradicts the maximality of G.
- A vertex $c \in V_f$ is incident to both f_1 and f_2. Then two edges $e_1, e_2 \in E \setminus E'$ cross at c. We can replace edge e_1 with a new edge (v_i, v_j) that lies in $f_1 \cup f_2$ that crosses edge e_2 at c. The new edge can be inserted into both G and H, contradicting the maximality of H.
- A face $f_0 \in F_0$ is adjacent to both f_1 and f_2. Then two edges $e_1, e_2 \in E \setminus E'$ are on the common boundary of the adjacent pairs f_1, f_0 and f_0, f_2. We can replace edge e_1 with a new edge (v_i, v_j) that lies in $f_1 \cup f_0 \cup f_2$ that crosses edge e_2. The new edge can be inserted into both G and H, contradicting the maximality of H.

Case 2. There is an index $i \in \{1, \ldots, m\}$, such that (v_i, v_{i+1}) is incident to a face in F_j for some $j \neq 0, i, i+1$. Without loss of generality, we may assume that edge (v_1, v_m) is incident to a face in F_j for some $1 < j < m$. Note that edge (v_1, v_m) must be incident to some face in F_j for *all* $1 \leq j \leq m$; otherwise $v_1 v_m$ would be incident to two faces, $f_i \in F_i$ and $f_j \in F_j$, $j \notin \{i-1, i, i+1\}$, that are either adjacent to each other or both adjacent to some face $f_0 \in F_0$; and then we could add a new edge (v_i, v_j) lying in $f_i \cup f_j$ or $f_i \cup f_0 \cup f_j$.

It follows that there are faces $f_2 \in F_2$ and $f_3 \in F_3$ that are incident to some point $c \in (v_1, v_2)$; or both are adjacent to some common face $f_0 \in F_0$ that is incident to $v_1 v_2$.

Consider the face f' of H on the opposite side of (v_1, v_m), and let F' be the set of faces in the planarization Γ^* contained in f'. Let $f'' \in F'$ be a face incident to $c \in (v_1, v_m)$ or adjacent to face f_0. By Lemma 9(2), we may assume that f'' is incident to a vertex v_k on the boundary of the face f'. It is possible that $v_k = v_1$ or $v_k = v_m$.

- If $v_k = v_1$, then we modify G, Γ, and H as follows: remove the edge that crosses (v_1, v_m) at c, and add a new edge (v_3, v_1) that lies in $f_3 \cup f''$ or $f_3 \cup f_0 \cup f''$, and crosses (v_1, v_m) at a point c. Then redraw the edges (v_1, v_m) and (v_1, v_3) by exchanging their initial arcs between v_1 and c, and eliminating the crossing at c. Both (v_1, v_m) and (v_1, v_3) can be added to E', contradicting the maximality of E'.
- If $v_k = v_m$ and $v_{m-1} = v_3$, we make similar changes replacing edge e with (v_2, v_m).
- Otherwise we similarly modify G, Γ, and H as follows: first replace the edge (v_1, v_m) with two new edges (v_2, v_k) and (v_3, v_k), that lie in $f_2 \cup f''$ and $f_3 \cup f''$, respectively, and one of them may cross some edge at c. Both (v_2, v_k) and (v_3, v_k) can be added to E', contradicting the maximality of E'.

All cases lead to a contradiction. Therefore, our initial assumption must be dropped, consequently the multigraph H is a triangulation, as claimed. □

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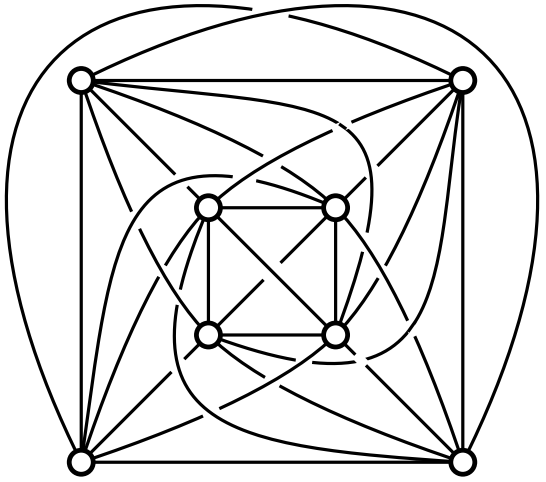
Outline

1. Density of k-gap planar graphs
2. Complete (bipartite) graphs
3. Complexity of recognizing 1-gap planar graphs
4. Relation to other graph classes

Theorem.
The complete graph K_n is 1-gap planar if and only if $n \leq 8$.

Proof. $\text{cr}(K_{10}) > 45 \Rightarrow$ not 1-gap planar

But $\text{cr}(K_9) = 36 = |E(K_9)| \ldots$



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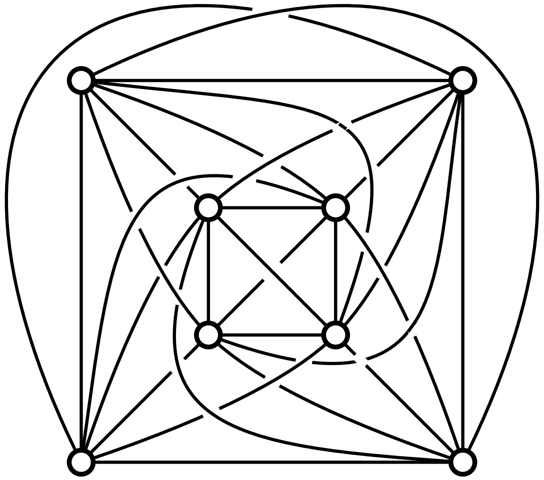

Assume Γ is a 1-gap planar drawing of K_9.

Consider planarization $\Gamma^\star$ of Γ:

- $|V(\Gamma^\star)| = 9 + 36 = 45$, $|E(\Gamma^\star)| = (9 \cdot 8 + 36 \cdot 4)/2 = 108$
- $\Rightarrow \Gamma^\star$ has 65 faces.

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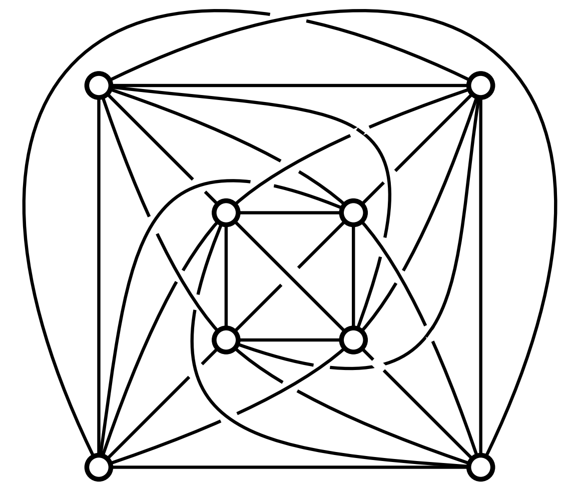

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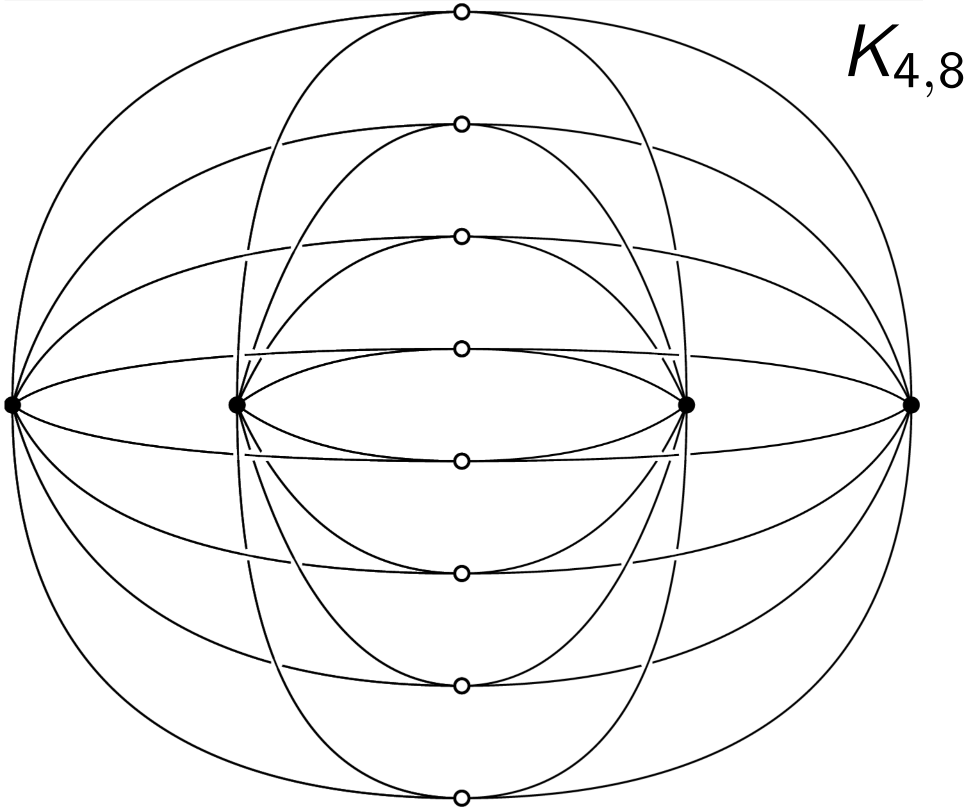
Two real vertices u and v share a face in $\Gamma^\star$:

- Each real vertex is incident to 8 faces, but there are less than $9 \cdot 8 = 72$ faces.
- Can redraw edge uv without crossings. □

Complete Bipartite Graphs



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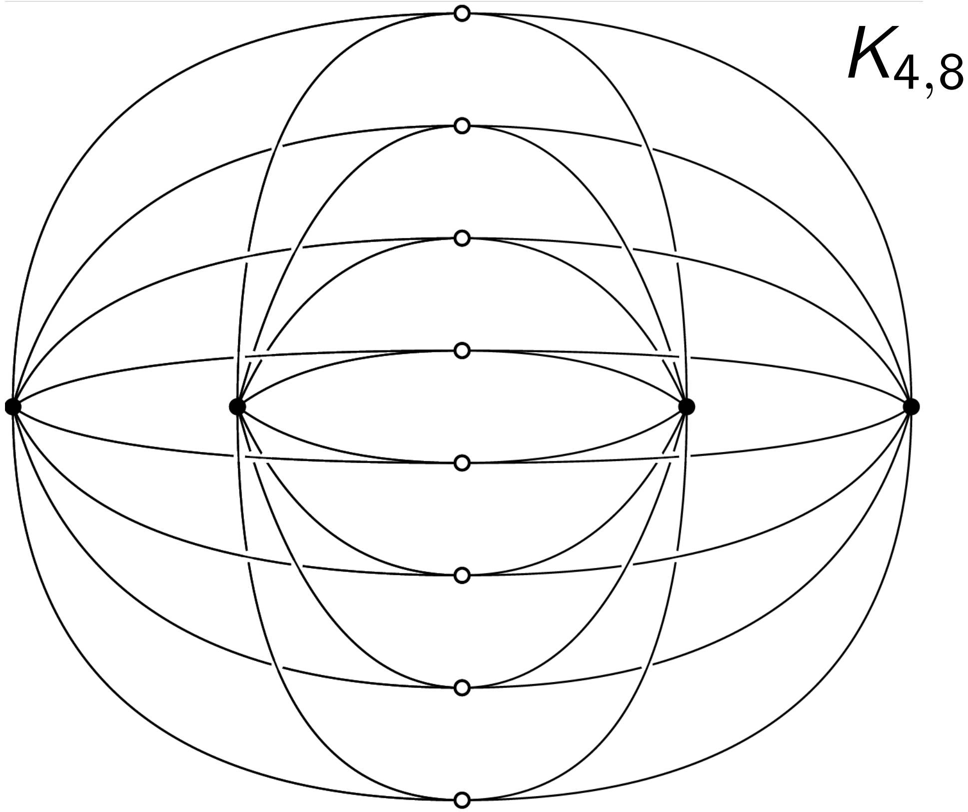

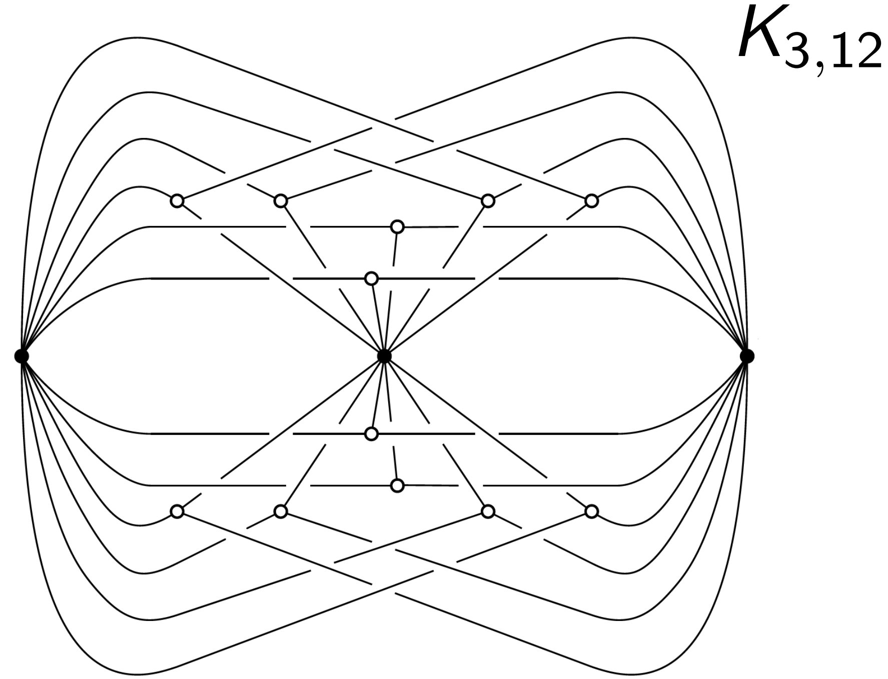

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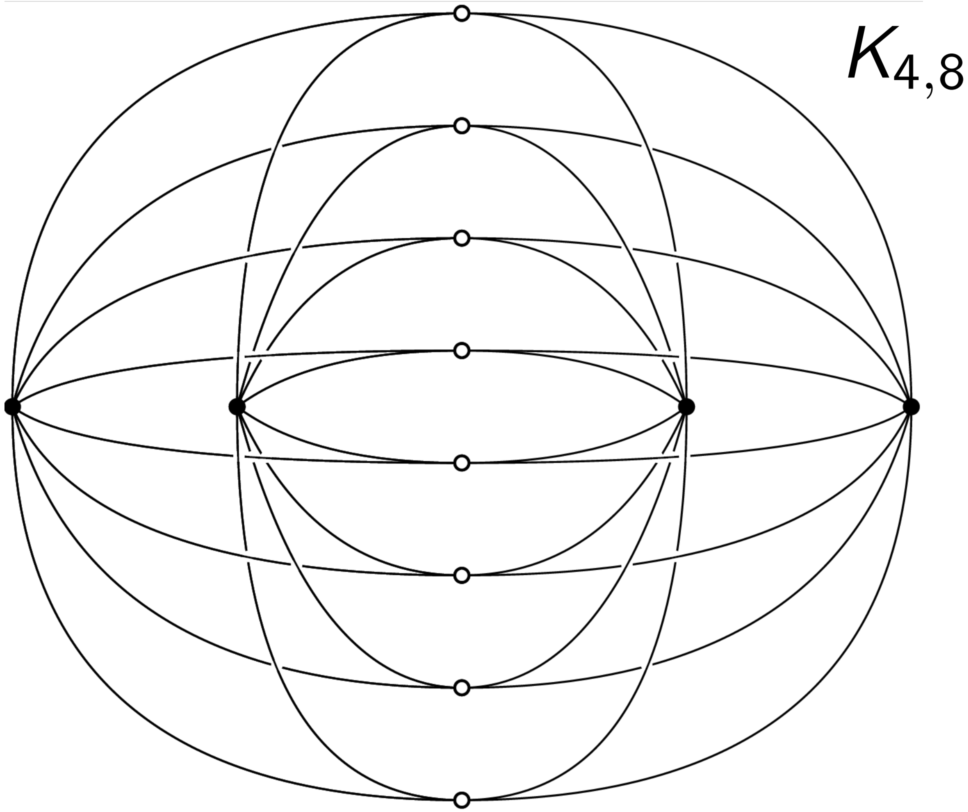

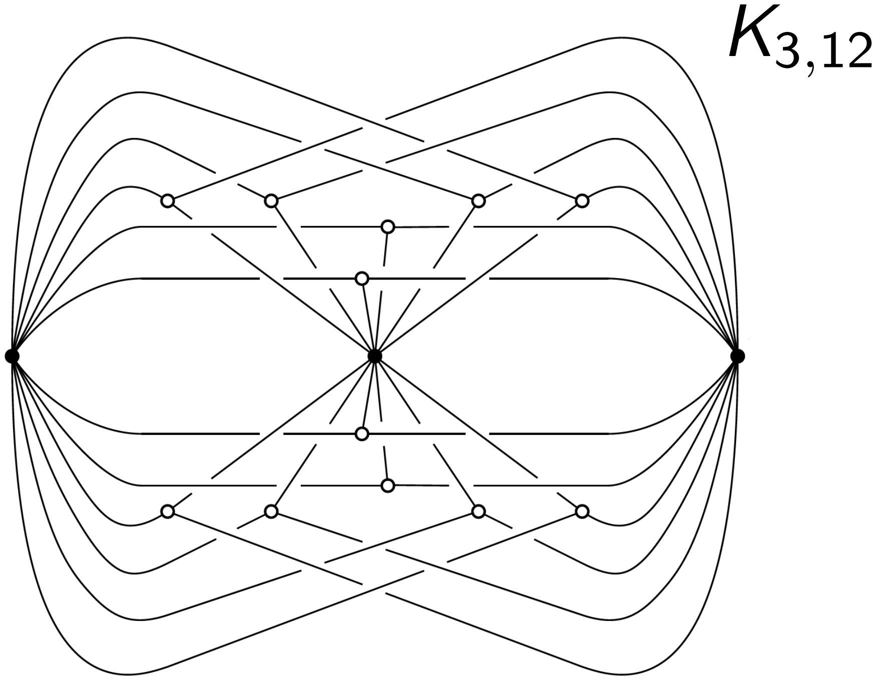

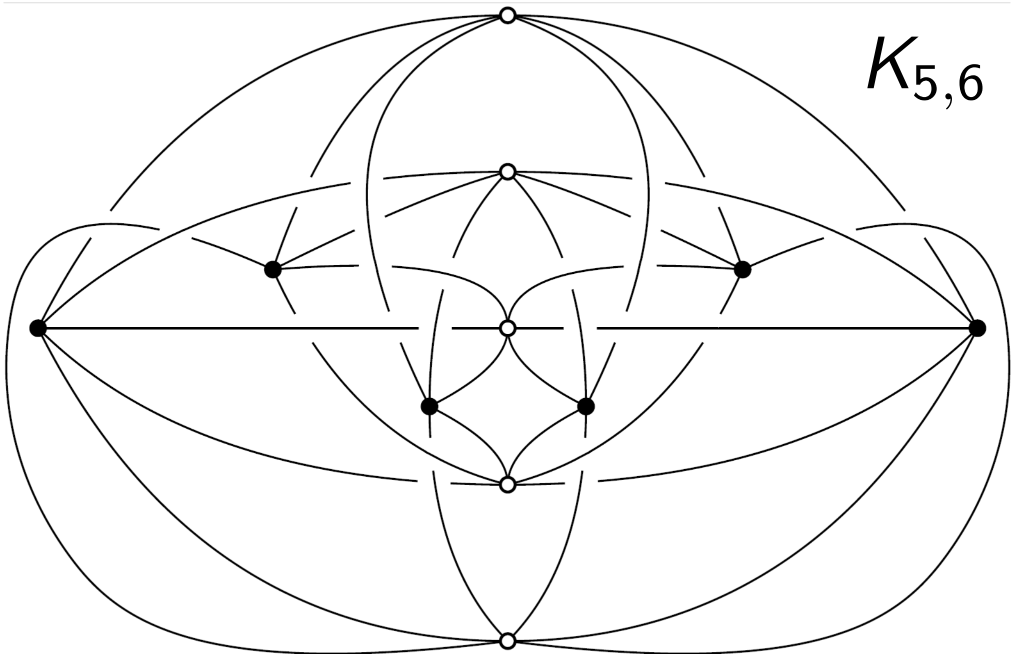

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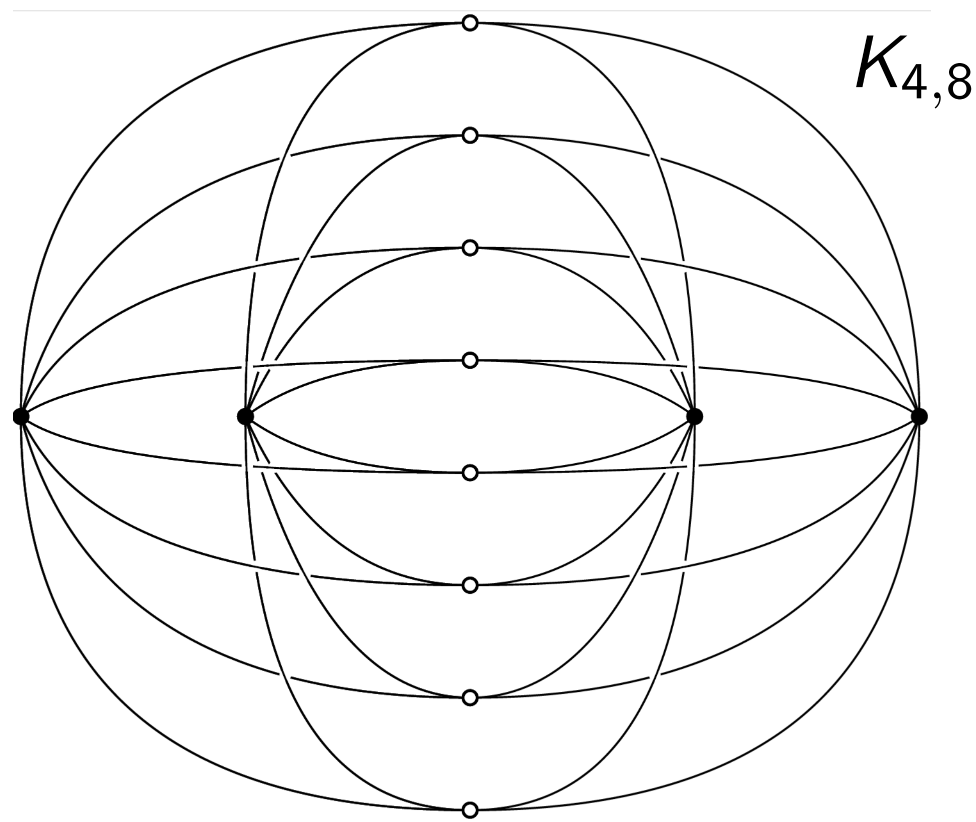

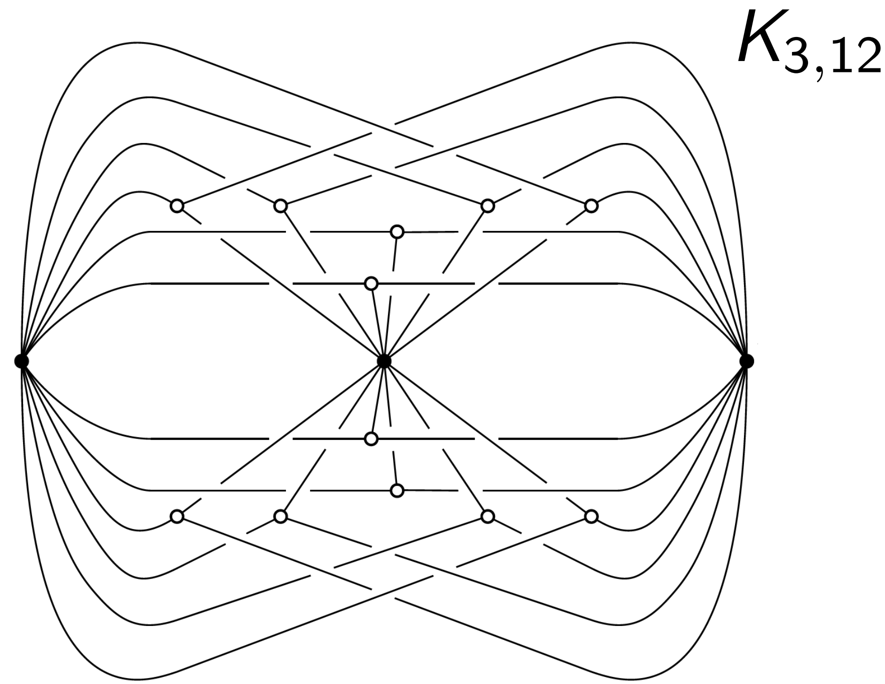

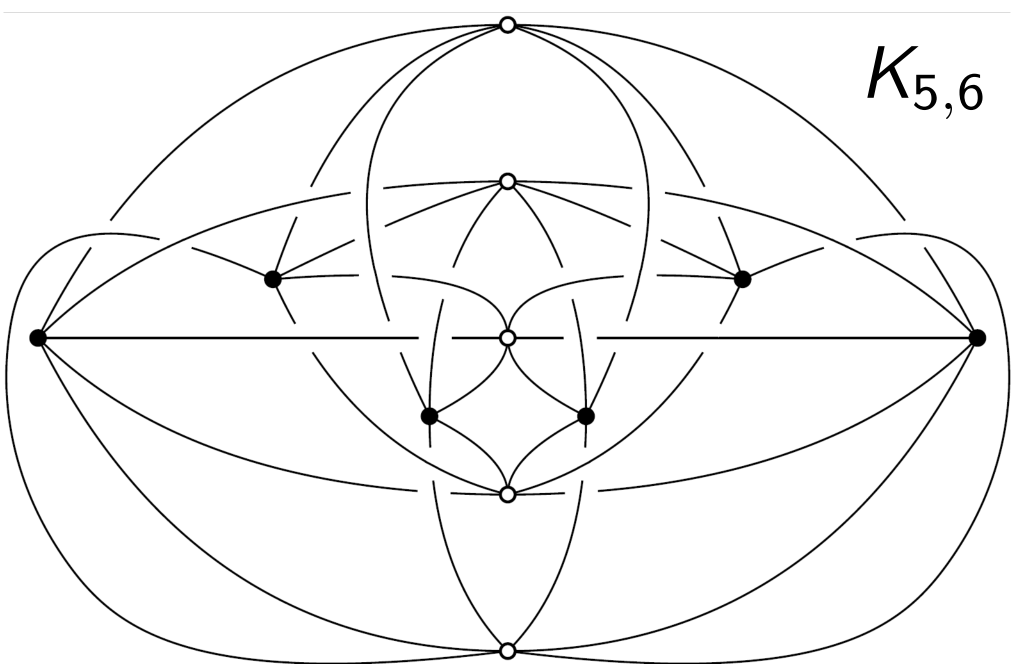

- $K_{3,13}$?
- $K_{4,9}$?
- $K_{6,6}$?

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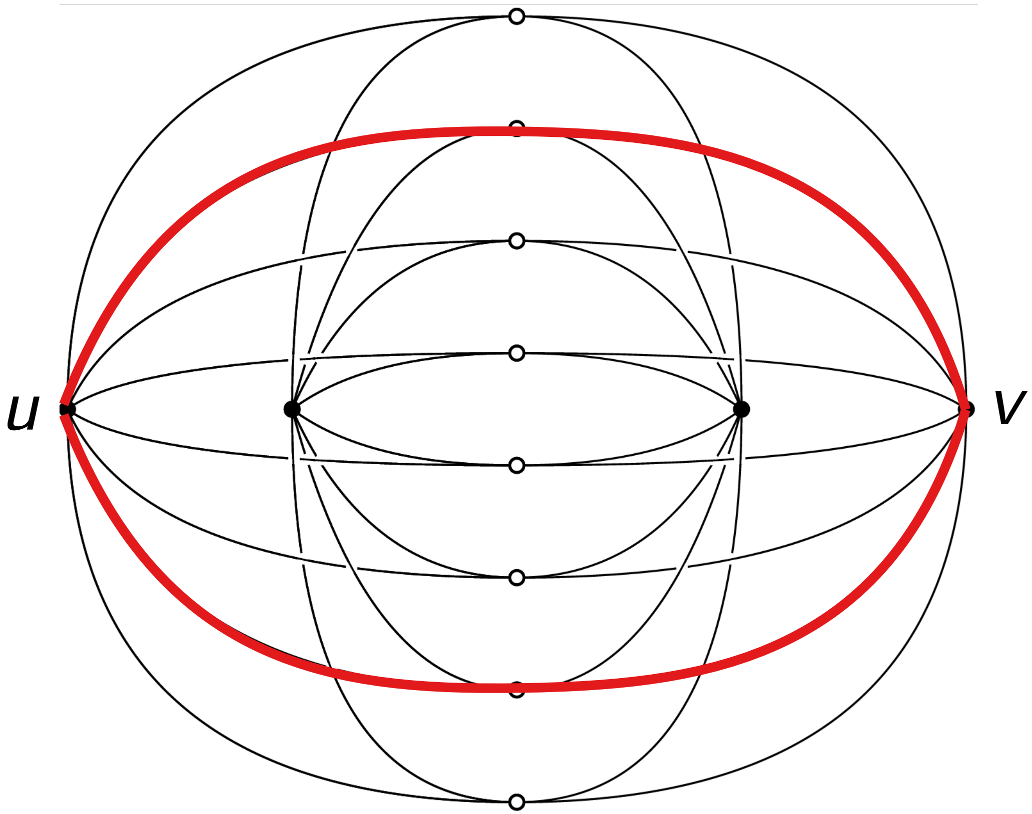

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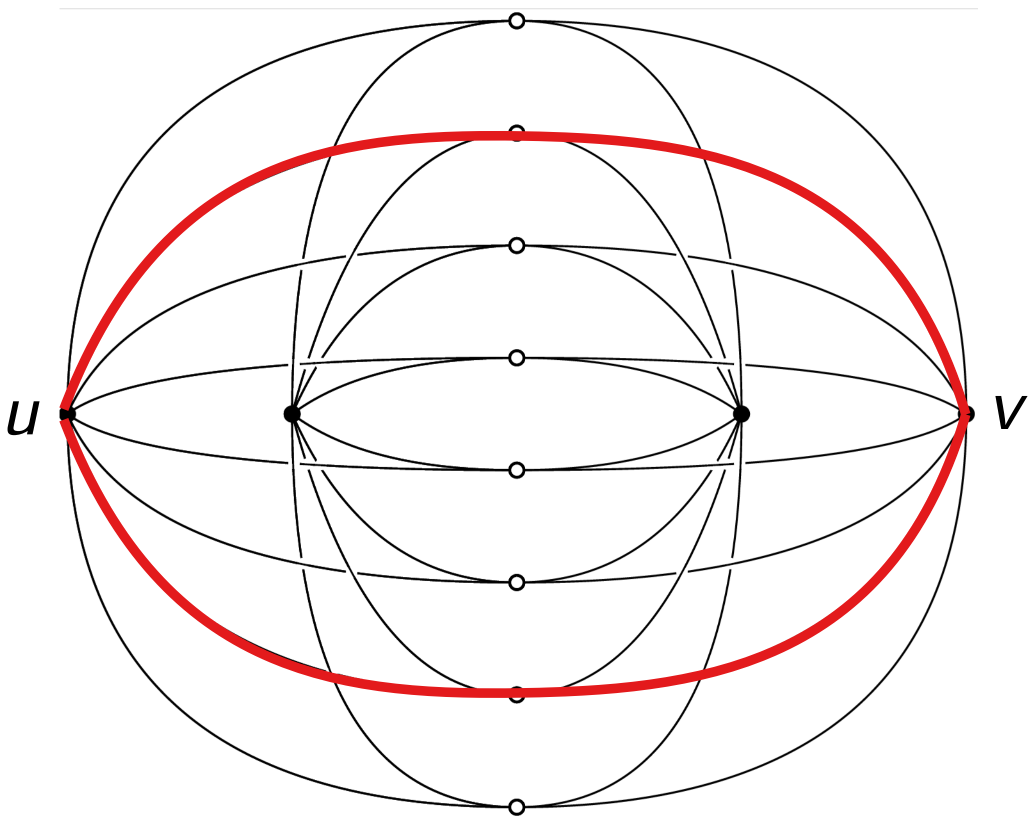

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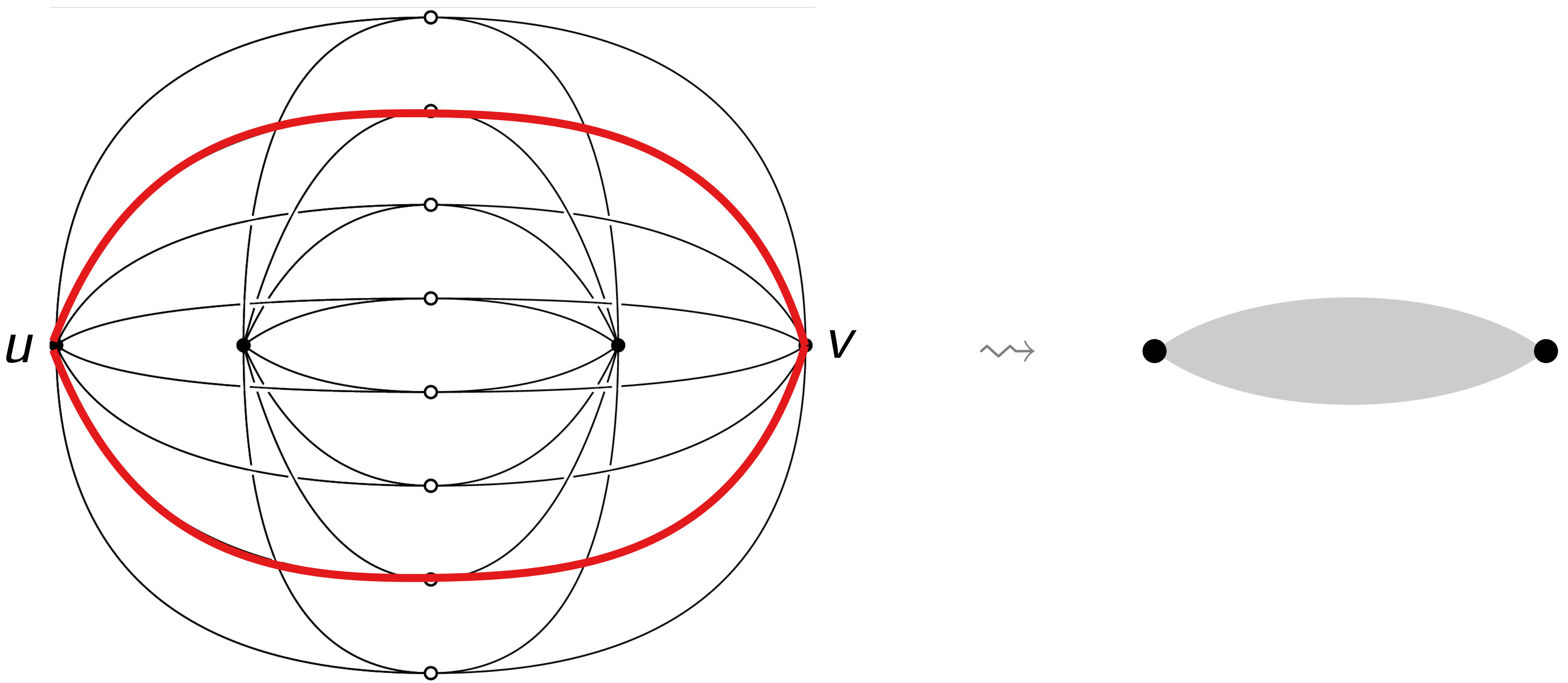

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Recognition

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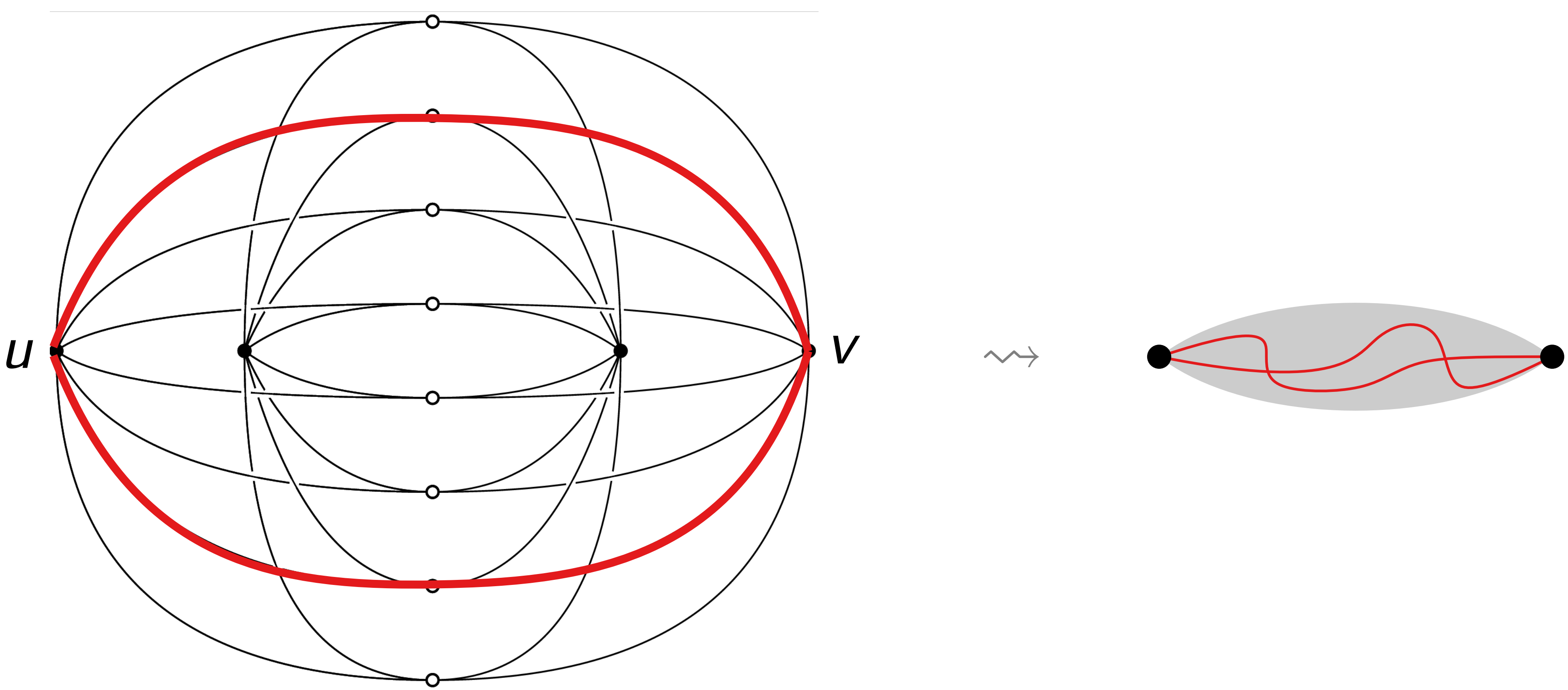

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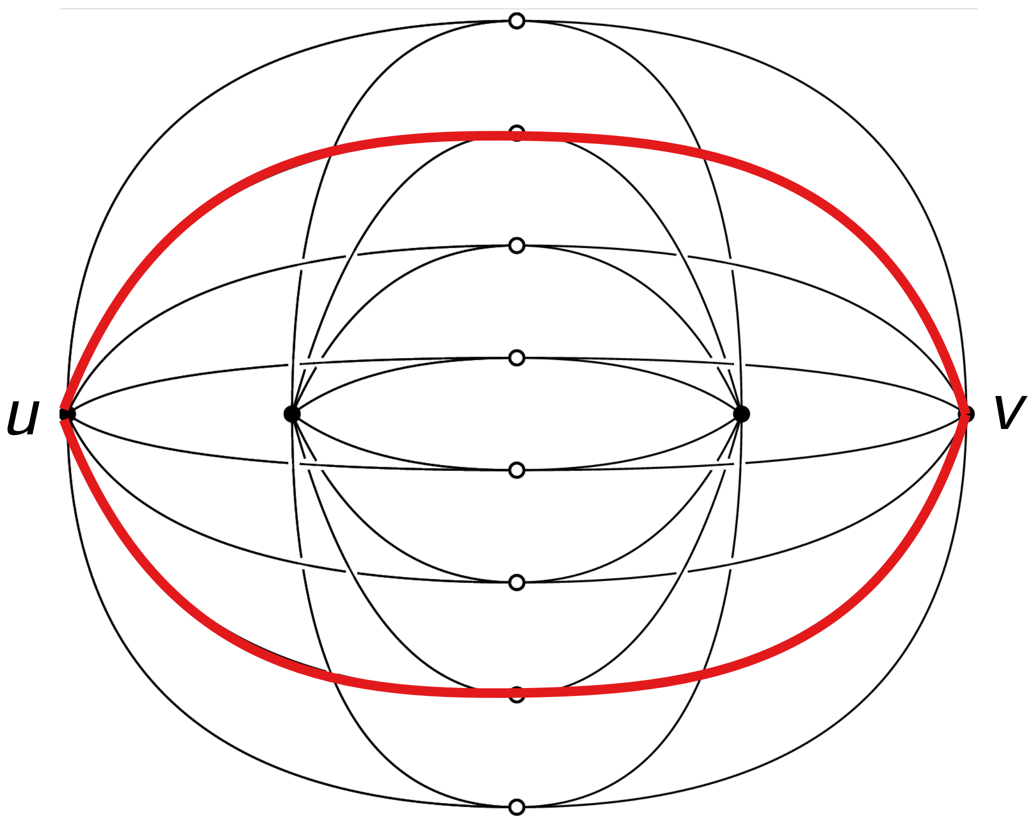

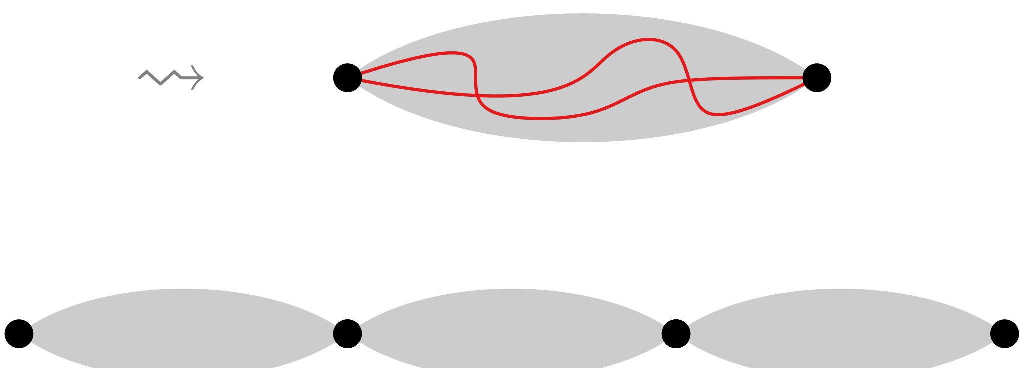

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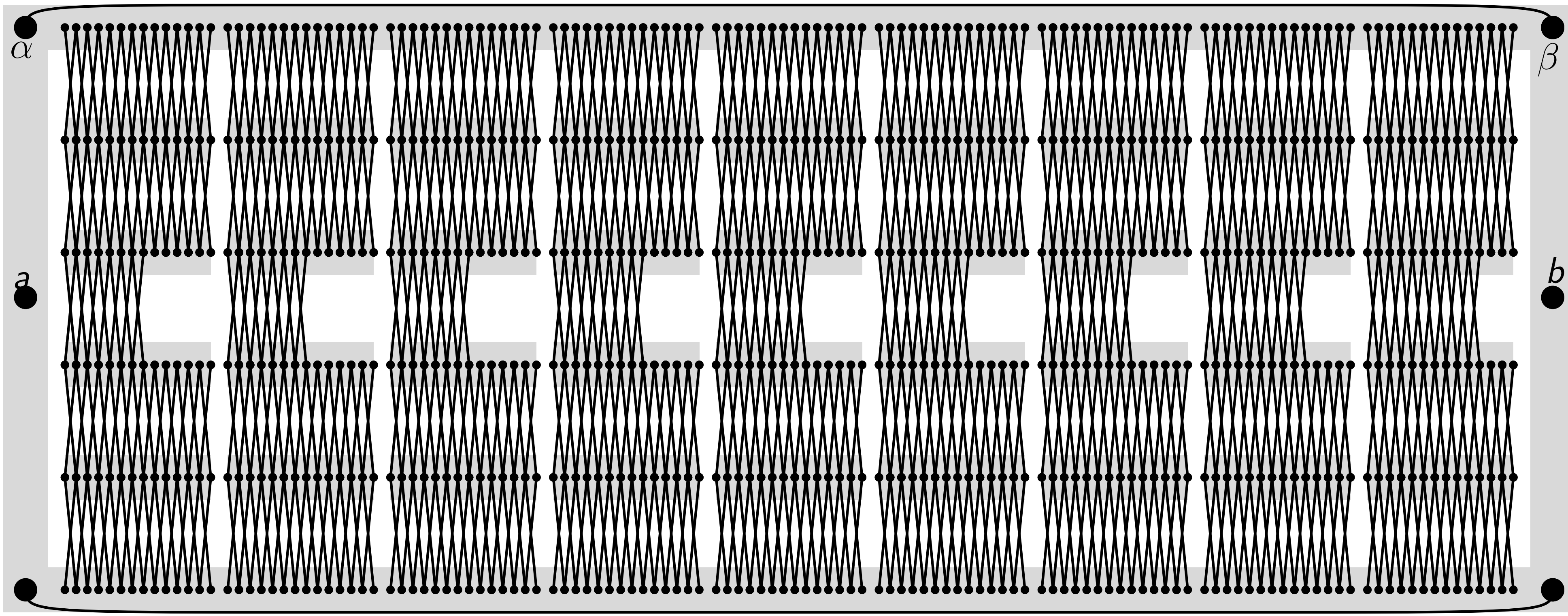

Reduction from 3-PARTITION:
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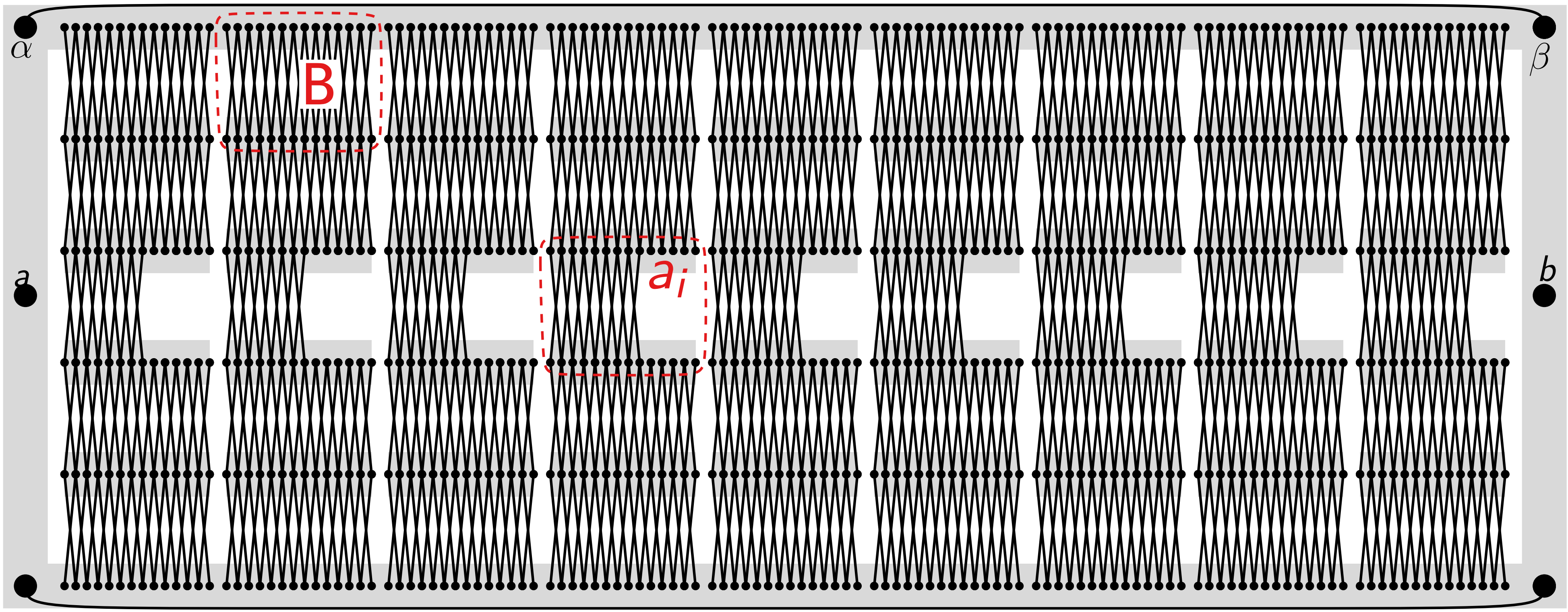
B crossing pairs



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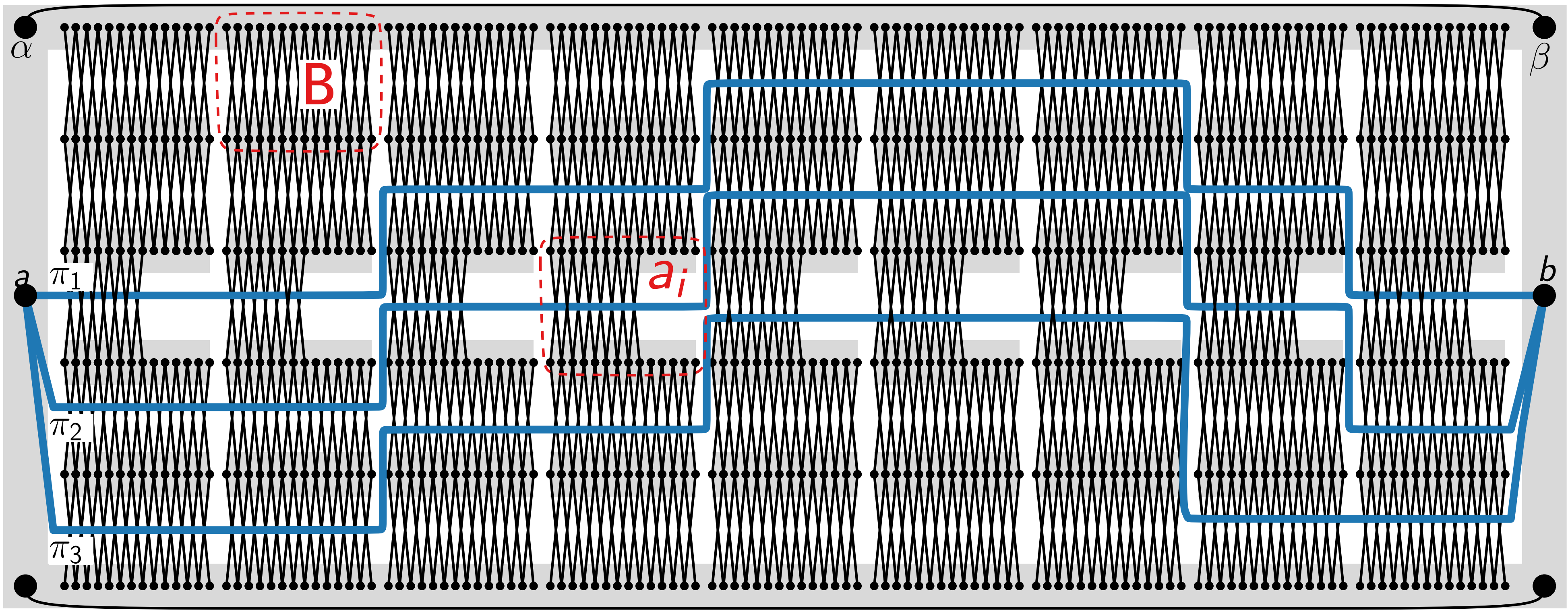

- Route m paths of length $(3m-3) \cdot B + B$
- Each path has to pick up a_i's summing to (at most) B.

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- drawing Γ is q-quasiplanar $\Leftrightarrow$ no subset of q edges has $\binom{q}{2} = q \cdot (q-1)/2$ crossings
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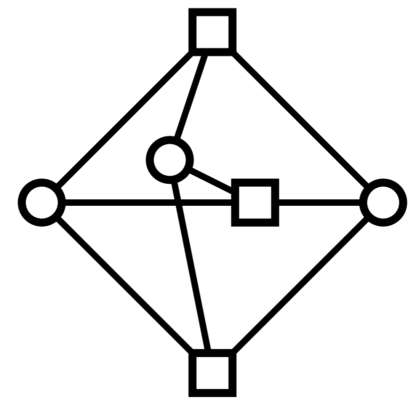
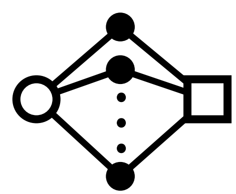
$\Rightarrow$ k-gap planar drawing is q-quasiplanar if $(q-1)/2 > k$, i.e., $q > 2k+1$.

Relation to Quasiplanarity

Lemma.
For every $k \geq 1$ there exists a graph that is (3-)quasiplanar but not k-gap planar.

Start with $K_{3,3}$:

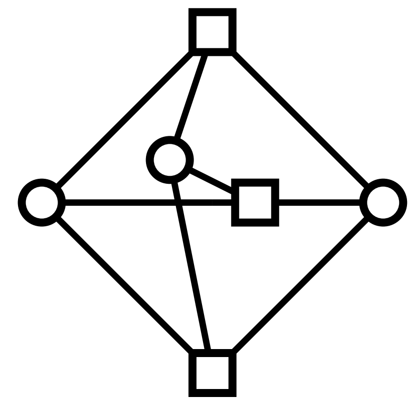


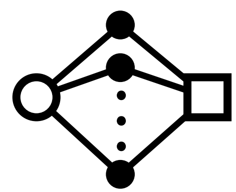
- Replace each  by  } $19k$ times
- Resulting graph G is quasiplanar

Relation to Quasiplanarity

Lemma.
For every $k \geq 1$ there exists a graph that is (3-)quasiplanar but not k-gap planar.

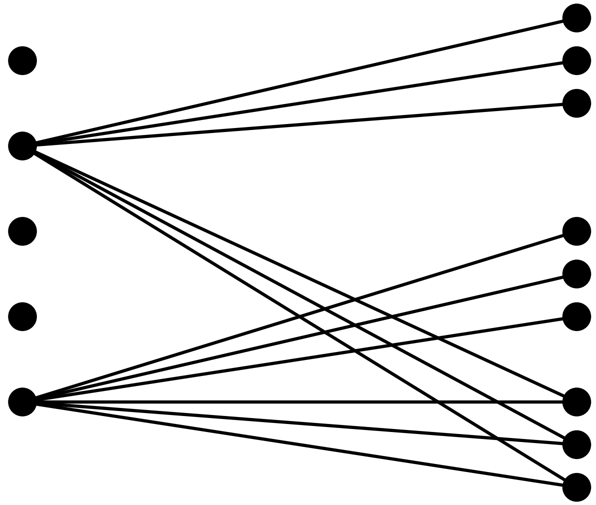
Start with $K_{3,3}$:



- Replace each  by  } $19k$ times
- Resulting graph G is quasiplanar
- But not k-gap planar:
  - choose one path per edge of $K_{3,3} \rightsquigarrow (19k)^9$ crossings.
  - each crossing counted $(19k)^7$ times.
  - $\Rightarrow \text{cr}(G) > (19k)^2 > 9 \cdot 19 \cdot 2k = |E(G)| \cdot k$

Lemma.
Every $2k$-planar drawing is k-gap planar.

A: crossings B: edges (k vertices per edge)



$\exists$ k-gap assignment
$\Leftrightarrow$
H has a matching of A into B.

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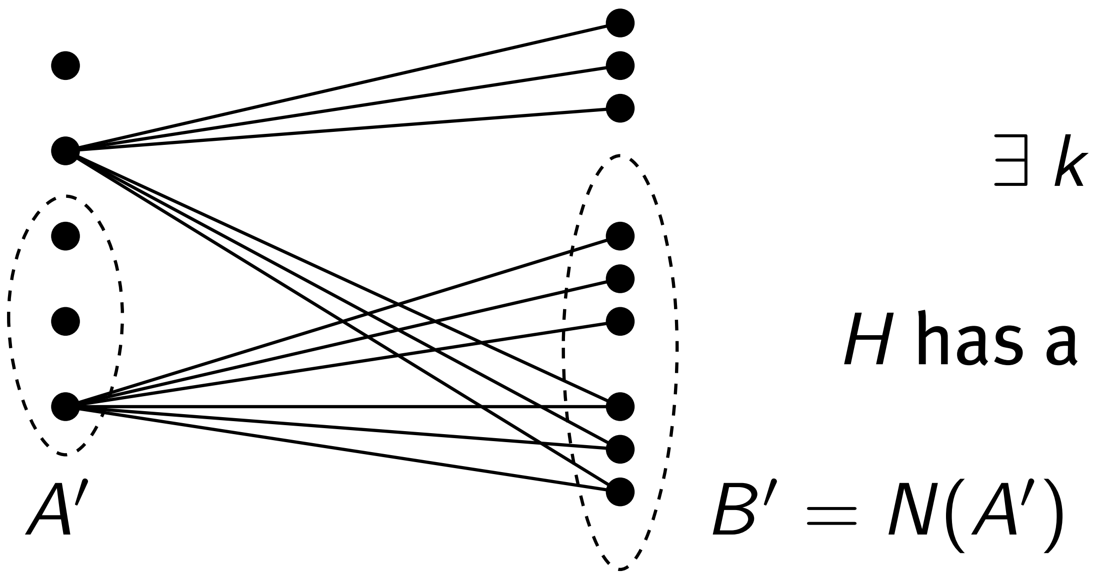

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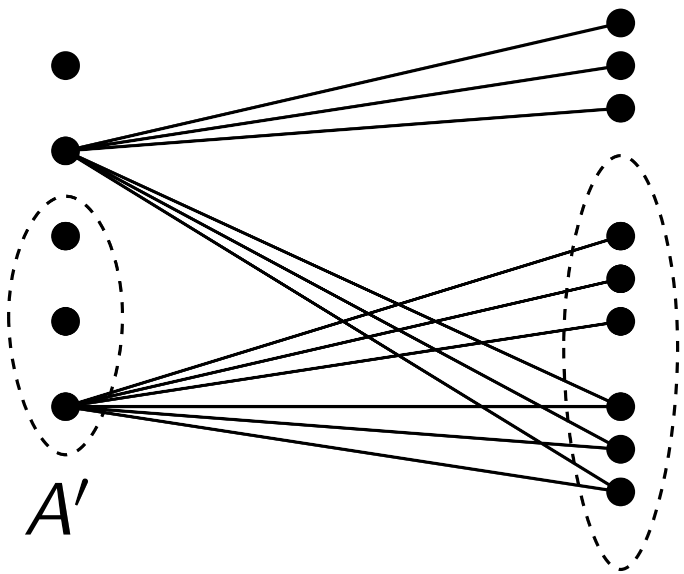

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- Vertices in B have degree $\leq 2k$.

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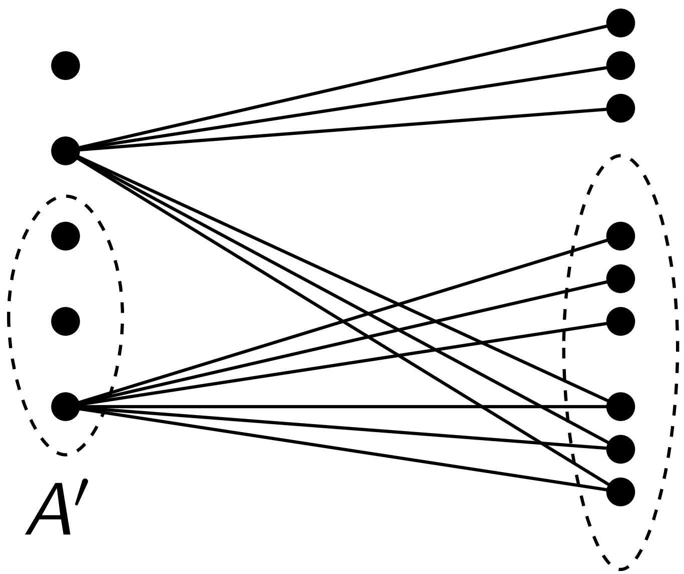

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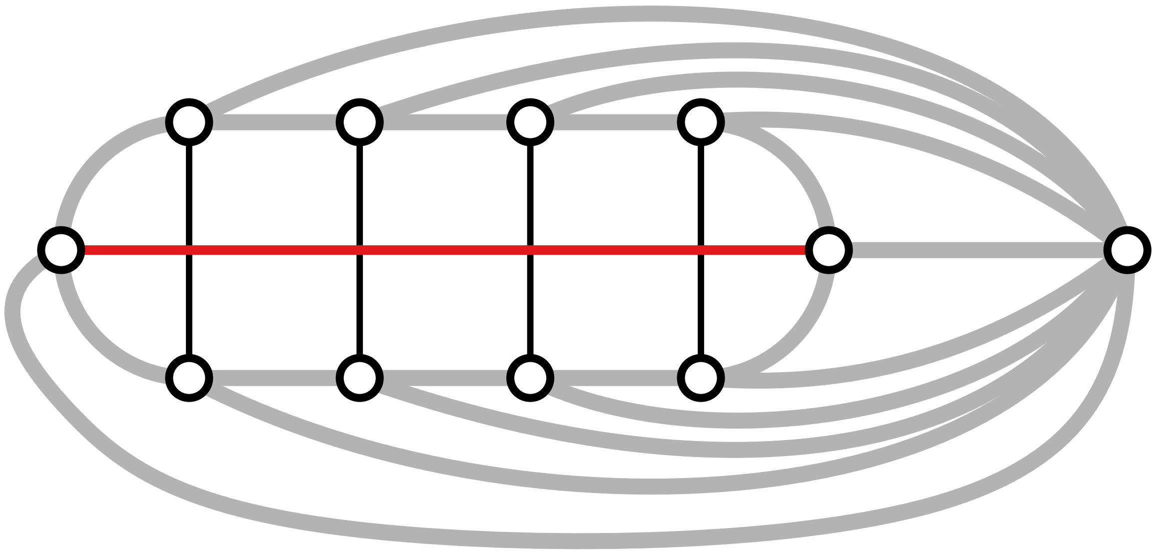
Hence a k-gap assignment exists by Hall's theorem.

Lemma.
For every $k \geq 1$ there exists a 1-gap planar graph that is not k-planar.



Replace each gray edge by $t = 5(k+1)^4$ parallel paths of length 2.

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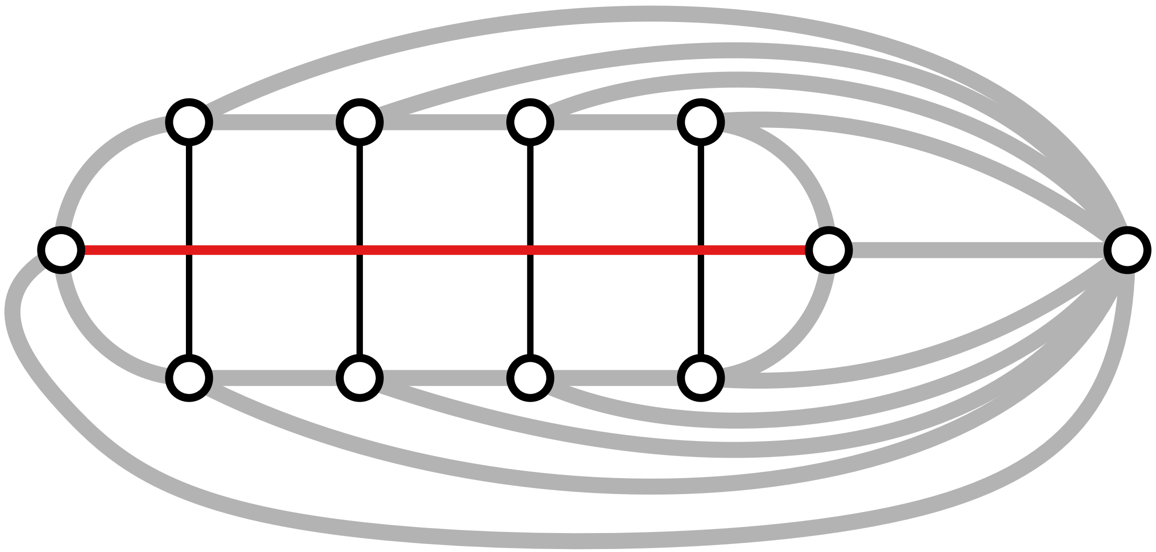
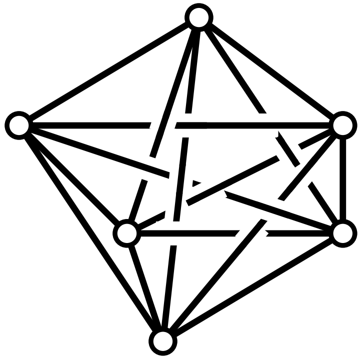


Replace each gray edge by $t = 5(k+1)^4$ parallel paths of length 2.

- In a k-planar drawing, we can pick $(k+1)$ paths for each edge such that paths of different edges do not cross.
- Wheel must be drawn as in the picture.
- Red edge has $\geq k+1$ crossings.

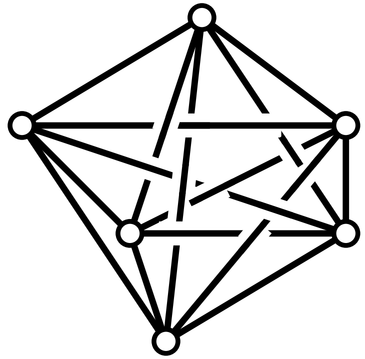
Conclusion



Gap planarity is a new beyond planarity concept:

- Density: linear, $5n - 10$ for 1-gap planar
- Complete graphs: up to $n - 8$
- Complexity: NP-hard (even with fixed rotation scheme)
- Interesting relation with k-planar graphs:
  - $2k$-planar graphs are k-gap planar
  - 1-gap planar graphs are not k-planar for any k

Conclusion



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Questions:

- Which complete bipartite graphs are 1-gap planar?
- Complexity of outer-k-gap-planarity?
- Do 1-gap planar drawings have RAC drawings with few bends?